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Degenerate and dihedral Heun functions with parameters

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Abstract. Just as with the Gauss hypergeometric function, particular cases of the local Heun function can be Liouvillian (that is, "elementary") functions. One way to obtain these functions is by pull-back transformations of Gauss hypergeometric equations with Liouvillian solutions. This paper presents the Liouvillian solutions of Heun's equations that are pull-backs of the parametric hypergeometric equations with cyclic or dihedral monodromy groups.

Key words: Heun equation, Liouvillian solutions, monodromy, Belyi maps.

1. Introduction

This paper presents all pull-back transformations to Heun equations from Gauss hypergeometric equations with *Liouvillian solutions* and a continuous parameter. This gives interesting series of examples of Liouvillian Heun functions. A Liouvillian function lies in a *Liouvillian extension* of $\mathbb{C}(x)$, that is [3], an extension of differential fields generated by sequentially adjoining a finite number of integrals, exponentials of integrals and algebraic functions. The Heun equation

$$\frac{\mathrm{d}^2Y(x)}{\mathrm{d}x^2}+\left(\frac{c}{x}+\frac{d}{x-1}+\frac{a+b-c-d+1}{x-t}\right)\frac{\mathrm{d}Y(x)}{\mathrm{d}x}+\frac{ab\,x-q}{x(x-1)(x-t)}Y(x)$$
$$=0 \tag{1}$$

is a canonical second-order Fuchsian differential equation on the Riemann sphere $\mathbb{P}^1$ with 4 regular singularities. The singular points and the local exponents are usefully encoded by the Riemann P-symbol

$$P\left\{\begin{array}{cccc|c} 0 & 1 & t & \infty & x \\ \hline 0 & 0 & 0 & a & \\ 1-c & 1-d & c+d-a-b & b & \end{array}\right\}. \tag{2}$$

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The local solution at $x = 0$ with the local exponent 0 and the value 1 of Heun's equation is denoted by

$$\mathrm{Hn}\left(\begin{array}{c|c|c} t & a,\, b & \\ q & c;\, d & x \end{array} \right). \tag{3}$$

Generally, the Heun functions are transcendental, and their monodromy is not known. But they can be Liouvillian for special values of the parameters a, b, c, d, q, t. The Kovacic algorithm [3] implies that Heun's equation has Liouvillian solutions if and only if its monodromy is either reducible, or infinite dihedral, or finite. The most straightforward case is a *Heun polynomial*; then necessarily a or b equals zero or a negative integer, and the monodromy representation of the Heun equation is reducible.

The simpler Gauss hypergeometric equation

$$\frac{\mathrm{d}^2 y(z)}{\mathrm{d}z^2} + \left(\frac{C}{z} + \frac{A+B-C+1}{z-1} \right) \frac{\mathrm{d}y(z)}{\mathrm{d}z} + \frac{AB}{z\,(z-1)}\, y(z) = 0 \tag{4}$$

and its hypergeometric ${}_2\mathrm{F}_1$ solutions are much better understood. Some Heun equations are pull-back transformations

$$z \longmapsto \varphi(x), \qquad y(z) \longmapsto Y(x) = \theta(x)\, y(\varphi(x)), \tag{5}$$

of a specialized Gauss hypergeometric equation. Here $\varphi(x)$ is a rational function and $\theta(x)$ is a radical function (i.e., a product of powers of rational functions.) Monodromy of those Heun equations is then easily computed from the monodromy of the hypergeometic equations to which a pull-back is applied. Systematic classification of these pull-back transformations to Heun equations from hypergeometric equations with a continuous parameter but *no Liouvillian solutions* has been done in [13]. In total, there are 61 such transformations up to fractional-linear transformations (of both hypergeometric and Heun equations) and algebraic conjugation. For example, the quadratic transformation P1 puts no restrictions on the parameters of the ${}_2\mathrm{F}_1$ function, and gives the formula

$$\mathrm{Hn}\left(\begin{array}{c|c|c} -1 & 2a,\, 2b & \\ 0 & 2c-1;\, a+b-c+1 & x \end{array} \right) = {}_2\mathrm{F}_1\left(\begin{array}{c|c} a,\, b & \\ c & x^2 \end{array} \right). \tag{6}$$

The 61 transformations employ 48 different rational functions $\varphi(x)$ up to

Möbius transformations and algebraic conjugation [12]. Those rational functions are all *Belyi functions* (or *Belyi coverings* $\mathbb{P}^1 \to \mathbb{P}^1$) as they branch (that is, $\varphi'(x) = 0$) only above $\varphi(x) \in \{0, 1, \infty\}$. The branching fibers are exactly the singularities $z = 0$, $z = 1$, $z = \infty$ of the hypergeometric equation.

This paper complements [12], [13] by showing all pull-back transformations to Heun equations from hypergeometric equations with Liouvillian solutions and a continuous parameter. The hypergeometric-to-Heun pull-back transformations with no parameters and no Liouvillian solutions are classified in [1]. There are 366 Galois orbits up to Möbius transformations. Liouvillian Heun functions with no parameters include algebraic Heun functions that are not classified yet. Kleinian pull-back transformations [2] for their Heun equations (with finite monodromy) are of particular interest.

2. Recalling the classification of pull-backs

Before recalling the classification scheme [12] of hypergeometric-to-Heun transformations, let us introduce some notation used there (and in [13]). Let $E(\alpha, \beta, \gamma)$ denote a hypergeometric equation with the exponent differences α, β, γ assigned to the singular points in any order. The exponent differences for (4) are $1-c$, $c-a-b$, $a-b$, since Riemann's P-symbol is

$$P\left\{\begin{array}{ccc|c} 0 & 1 & \infty & z \\ \hline 0 & 0 & a & \\ 1-c & c-a-b & b & \end{array}\right\}. \tag{7}$$

Reflecting the generic set of 24 Kummer's ${}_2\mathrm{F}_1$ solutions of (4), the notation $E(\alpha, \beta, \gamma)$ denotes any of the 24 corresponding hypergeometric equations. Similarly, let $E(\alpha, \beta)$ denote a Fuchsian equation with two singularities and the exponent differences α, β. It exists if and only if $\alpha = \pm\beta$, formally coinciding then with $E(1, \alpha, \alpha)$. Otherwise, the equation $E(1, \alpha, \beta)$ has a logarithmic singularity. Finally, let $HE(\alpha, \beta, \gamma, \delta)$ denote a Heun equation with its exponent differences α, β, γ, δ in some order. The exponent differences for (1) are

$$1-c, \quad 1-d, \quad c+d-a-b, \quad a-b, \tag{8}$$

as evident from (2).The analogue of $24 = 2^{3-1} \cdot 3!$ Kummer's hypergeometric solutions is the set of $192 = 2^{4-1} \cdot 4!$ Maier's Heun solutions [5]. They are related by fractional-linear transformations that permute the 4 singular points and interchange the local Heun solutions at them [13, Appendix B]. The notation $HE(\alpha,\beta,\gamma,\delta)$ does not specify the parameters t,q, hence it does not determine the monodromy.

Generally, a pull-back transformation of a hypergeometric equation $E(\alpha,\beta,\gamma)$ with respect to a covering $\varphi:\mathbb{P}^1\to\mathbb{P}^1$ has singularities at the branching points of φ and above the singularities $z=0$, $z=1$, $z=\infty$ of the hypergeometric equation. To have just 4 singularities after the pull-back (when $\deg\varphi>2$), we must have regular points with $\varphi(x)\in\{0,1,\infty\}$. As explained in [12, Section 2], an x-point with $\varphi(x)\in\{0,1,\infty\}$ is regular for the pulled-back equation only if the exponent difference of $E(\alpha,\beta,\gamma)$ at $\varphi(x)$ equals $\pm 1/k$, where k is the branching order of φ at the x-point. The method [12] to get hypergeometric-to-Heun transformations is to restrict some of α,β,γ to reciprocals of integers k_z, and look for coverings $\mathbb{P}^1\to\mathbb{P}^1$ with sufficiently many points of branching order k_z in respective fibers. To leave a free parameter, at most two numbers among α,β,γ are restricted. We refer to the fibers of the singularities with restricted exponent differences as *restricted fibers*. A useful consequence of the Hurwitz formula is this.

Lemma 2.1 *If $\varphi(x)$ is a Belyi function of degree d, the number of distinct points with $\varphi(x)\in\{0,1,\infty\}$ equals $d+2$. Otherwise the number of distinct points is greater. If the number of distinct points with $\varphi(x)\in\{0,1,\infty\}$ equals $d+3$, there is exactly one branching point with $\varphi(x)\notin\{0,1,\infty\}$, and its branching order equals 2.*

Proof. See [12, Lemma 2.2]. Though the definition of Belyi function used in [12] allows branching above any set of 3 points, not necessarily $\{0,1,\infty\}$, its proof applies here. □

The following Diophantine inequality is derived in [12, Section 3.1] for pull-back functions $\varphi(x)$ in hypergeometric-to-Heun transformations:

$$\frac{2}{d}+\sum_{z\in S}\frac{1}{k_z}\geq 1. \tag{9}$$

Here $d=\deg\varphi$, all $k_z>0$, and $S\subset\{0,1,\infty\}$ is the set of points with restricted exponent differences. The tables in [12, Section 3] skip the following two cases:

- $|S| = 1$ and the sole $k_z = 1$. The singularity of $E(1, \alpha, \beta)$ with the exponent difference 1 must be non-logarithmic (in order to have regular points in its fiber after the pull-back), hence $\beta = \pm\alpha$. The equation $E(1, \alpha, \alpha)$ actually has just two singularities, up to transformation (5) with $\varphi(x) \in \{x, 1/x\}$. The monodromy representation is completely reducible, generated by one element (thus *cyclic*).
- $|S| = 2$ and both $k_z = 2$. The monodromy of $E(1/2, 1/2, \alpha)$ is a dihedral group; it is infinite when $\alpha \notin \mathbb{Q}$.

The hypergeometric equations $E(1, \alpha, \alpha)$ and $E(1/2, 1/2, \alpha)$ have bases of Liouvillian solutions. The focus of the remaining sections is on pull-back transformation from them to Heun equations. As we will see, there are pull-back transformations of any degree $d > 1$ in both cases. The pull-back coverings $\varphi(x)$ are Belyi functions, except a series of composite coverings for pull-backs from $E(1/2, 1/2, \alpha)$, described in Remark 4.1 here. The obtained pull-back transformations give interesting parametric cases of Heun equations with Liouvillian solutions.

Remark 2.2 The full list of hypergeometric-to-Heun pull-backs with a continuous parameter and Liouvillian solutions is formed by the transformations described in Theorems 3.1, 4.2 here, and Liouvillian specializations of the pull-backs P1, P2, P3, P15, P19, P20, P51 in [13] with more than one parameter.

Of particular interest are transformations between hypergeometric and Heun polynomials. For example, one may assume a is a non-positive integer in (6). Here are instances of polynomial formulas induced by P15, P19 and P20:

$$\mathrm{Hn}\left(\begin{matrix} 1/4 \\ -9na/4 \end{matrix}\,\middle|\, \begin{matrix} -3n,\, 3a \\ \frac{1}{2};\, a-n+1/2 \end{matrix} \,\middle|\, x\right) = {}_2\mathrm{F}_1\left(\begin{matrix} -n,\, a \\ 1/2 \end{matrix}\,\middle|\, x(4x-3)^2\right), \tag{10}$$

$$\mathrm{Hn}\left(\begin{matrix} 9 \\ \widehat{q}_1 \end{matrix}\,\middle|\, \begin{matrix} -3n,\, a-2n \\ a-n+1/3;\, 1-2n-2a \end{matrix} \,\middle|\, x\right)$$
$$= (1-x)^{2n}\, {}_2\mathrm{F}_1\left(\begin{matrix} -n,\, a \\ a-n+1/3 \end{matrix}\,\middle|\, -\frac{x(x-9)^2}{27(x-1)^2}\right), \tag{11}$$

$$\mathrm{Hn}\left(\begin{matrix} 9/8 \\ \widehat{q}_2 \end{matrix}\,\middle|\, \begin{matrix} -4n,\, a-3n \\ 3a-3n-1/2;\, a-n+1/2 \end{matrix} \,\middle|\, x\right)$$

$$= \left(1 - \frac{8x}{9}\right)^{3n} {}_2\mathrm{F}_1\left(\left.\begin{array}{c} -n,\, a \\ a-n+1/2 \end{array}\right| \frac{64x^3(x-1)}{(8x-9)^3}\right), \tag{12}$$

where $\widehat{q}_1 = 9na + 18n^2 - 6n$, $\widehat{q}_2 = -9na + 9n^2 + 3n/2$. Reducible monodromy always leads to a polynomial (up to a power factor) solution. A necessary reducibility condition for Heun's equation (1) is $a, b, c-a, c-b, d-a, d-b, c+d-a$ or $c+d-b \in \mathbb{Z}$; check this in [5].

Non-reducible hypergeometric or Heun equations with a continuous parameter have dihedral monodromy. The hypergeometric equation is then $E(k+1/2, \ell+1/2, \alpha)$ with $k, \ell \in \mathbb{Z}$. The continuous parameter is preserved under these transformations from [13]:

- the quadratic pull-back P1 transforms to $HE(k+1/2, k+1/2, 2\ell+1, 2\alpha)$ or $HE(2k+1, 2\ell+1, \alpha, \alpha)$;
- the quartic pull-back P2 transforms $E(1/2, k+1/2, \alpha)$ to $HE(2k+1, 2k+1, 2\alpha, 2\alpha)$;
- the quartic pull-back P3 transforms $E(1/2, k+1/2, \alpha)$ to $HE(k+1/2, k+1/2, 2k+1, 4\alpha)$ or $HE(4k+2, \alpha, \alpha, 2\alpha)$;
- the cubic pull-back P15 transforms $E(1/2, k+1/2, \alpha)$ to $HE(1/2, k+1/2, 2k+1, 3\alpha)$ or $HE(1/2, 3k+3/2, \alpha, 2\alpha)$;
- the quartic pull-back P19 transforms $E(1/2, k+1/2, \alpha)$ to $HE(k+1/2, 3k+3/2, \alpha, 3\alpha)$.

This gives examples of Heun equations with cyclic or dihedral monodromy. Additional such examples arise in pull-backs from general hypergeometric equations with cyclic monodromy [8], that is, $E(k, \alpha, \alpha+\ell)$ with $k, \ell \in \mathbb{Z}$, $|k| > |\ell|$. The continuous parameter is preserved only under the quadratic transformation P1, giving $HE(2k, \alpha, \alpha, 2\alpha+2\ell)$ or $HE(k, k, 2\alpha, 2\alpha+2\ell)$.

3. Transformations from the cyclic $E(1, \alpha, \alpha)$

As already mentioned, the equations $E(1, \alpha, \alpha)$ are degenerate hypergeometric equations with (essentially) two singularities. The monodromy representation is completely reducible, cyclic. The 24 Kummer's solutions are constant or power functions, except

$${}_2\mathrm{F}_1\left(\left.\begin{array}{c} 1-a,\, 1 \\ 2 \end{array}\right| z\right) = \begin{cases} \frac{1-(1-z)^a}{a\,z}, & \text{if } a \neq 0, \\ -\frac{1}{z}\,\log(1-z), & \text{if } a = 0. \end{cases} \tag{13}$$

The similar formula in [9, (29)] has to be corrected by the factor -1 in the $a \neq 0$ case. As presented in [9, Section 5], there are pull-back transformations $E(1,\alpha,\alpha) \overset{n}{\longleftarrow} E(1,n\alpha,n\alpha)$. Up to Möbius transformations, they are the cyclic coverings $z \mapsto x^n$. A nontrivial transformation formula is

$$ {}_2\mathrm{F}_1\left(\begin{matrix} 1-na,\ 1 \\ 2 \end{matrix} \,\middle|\, x \right) = \psi(x)\, {}_2\mathrm{F}_1\left(\begin{matrix} 1-a,\ 1 \\ 2 \end{matrix} \,\middle|\, nx\psi(x) \right), \qquad \text{with } \psi(x) = \frac{1-(1-x)^n}{n\,x}. \tag{14} $$

The expression $\psi(x)$ is a polynomial of degree $n-1$. The pull-back coverings for transforming $E(1,\alpha,\alpha)$ to Heun's equation have similar expressions. Most of the Heun's solutions are trivial power functions.

Theorem 3.1 (i) *Pull-back transformations from $E(1,\alpha,\alpha)$ to Heun equations exist and are unique up to Möbius transformations for any pair (M,N) of positive integers. The transformed Heun equation is $HE(2,N\alpha,M\alpha,D\alpha)$, where $D=M+N$.*

(ii) *The pull-back covering is a Belyi map of degree D. Up to Möbius transformations, it is*

$$ \varphi(x) = 1-(1-x)^N \left(1+\frac{Nx}{M}\right)^M. \tag{15} $$

(iii) *The following identities with Heun's functions hold, for non-zero $a, b, a+b$:*

$$ \mathrm{Hn}\left(\begin{matrix} 1+b/a \\ -b\,(a+1) \end{matrix} \,\middle|\, \begin{matrix} a,\ -b \\ 1+a; -1 \end{matrix} \,\middle|\, 1-x \right) = \left(\frac{a\,x+b}{a+b} \right)^b, \tag{16} $$

$$ \mathrm{Hn}\left(\begin{matrix} 1+a/b \\ -a\,(b+1) \end{matrix} \,\middle|\, \begin{matrix} -a,\ b \\ 1+b; -1 \end{matrix} \,\middle|\, 1+\frac{ax}{b} \right) = \left(\frac{a\,(1-x)}{a+b} \right)^a, \tag{17} $$

$$ \mathrm{Hn}\left(\begin{matrix} -a/b \\ (b^2-a^2)((a-1)/b+1) \end{matrix} \,\middle|\, \begin{matrix} -a-b,\ 2-a-b \\ 1-a-b; 1-a \end{matrix} \,\middle|\, \frac{1}{x} \right) = \left(1-\frac{1}{x}\right)^a \left(1+\frac{b}{ax}\right)^b, \tag{18} $$

$$\frac{a\,(a+b)}{2b}x^2\,\mathrm{Hn}\left(\begin{matrix}-b/a\\2(1-b/a)\end{matrix}\,\middle|\,\begin{matrix}2,\,2-a-b\\3;\,1-a\end{matrix}\,\middle|\,x\right)$$
$$= 1-(1-x)^a\left(1+\frac{ax}{b}\right)^b, \tag{19}$$

in neighborhoods of, respectively, $x=1$, $x=-b/a$, $x=\infty$, $x=0$. These functions are solutions of HE$(2,a,b,a+b)$. This Heun equation is a pull-back transformation of $E(1,\alpha,\alpha)$ if and only if the ratio a/b is a rational number. The minimal degree of the pull-back covering is equal to the sum of the numerator and the denominator of a/b.

Proof. Let $D=\deg\varphi$. Let k denote the number of distinct points above the two α-points of $E(1,\alpha,\alpha)$. We have $k\in\{2,3,4\}$. If $k=2$, the covering is cyclic and the pulled-back equation is $E(1,D\alpha,D\alpha)$. If $k=4$, there are other branching points by Lemma 2.1, and those points would be excess singularities. If $k=3$, we are led to the branching pattern $M+N=D=2+[1]_{D-2}$ in the notation of [12]. By Möbius transformations, we assign the points $x=\infty$, $x=0$, $x=1$ to the branching orders D, 2, N (respectively) above $\varphi=\infty$, $\varphi=0$, $\varphi=1$ (respectively). The Belyi function has then the form $1-\varphi(x)=(1-x)^N(1-sx)^M$. The branching at $x=0$ gives $s=-N/M$. Parts (i) and (ii) are shown.

Let $a=N\alpha$, $b=M\alpha$. The formulas in (iii) are derived by writing down Heun-to-hypergeometric identities similar to (6) and (14), then evaluating the hypergeometric function using (13) and dropping the condition $a/b\in\mathbb{Q}$ by the analytic continuation argument. For example, the least trivial identity is

$$\mathrm{Hn}\left(\begin{matrix}-M/N\\2(1-M/N)\end{matrix}\,\middle|\,\begin{matrix}2,\,2-D\alpha\\3;\,1-N\alpha\end{matrix}\,\middle|\,x\right)=\frac{2M\,\varphi(x)}{ND\,x^2}\,{}_2\mathrm{F}_1\left(\begin{matrix}1-\alpha,1\\2\end{matrix}\,\middle|\,\varphi(x)\right). \tag{20}$$

The formulas in (iii) can be checked as follows. The Heun equation (1) with $(a,b,c,d,t,q)=(0,-a-b,-1,1-a,-m/n,0)$ has this general solution:

$$y(x)=C_1+C_2(x-1)^a(ax+b)^b.$$

Formulas (16)–(19) identify the most interesting Heun solutions in the orbit of Maier's 192 Heun solutions [5]. The degree of a pull-back covering to

$HE(2,a,b,a+b)$ with fixed $a/b \in \mathbb{Q}$ must be a multiple of the sum of the numerator and the denominator. $\square$

If the numbers N, M are not co-prime, the covering of Theorem 3.1 is the composition

$$E(1,\alpha,\alpha) \xleftarrow{n} E(1,n\alpha,n\alpha) \xleftarrow{D/n} HE(2,N\alpha,M\alpha,D\alpha), \tag{21}$$

where n is a divisor of $\gcd(N,M)$. Putting $\alpha = 1/D$ changes the pulled-back Heun equation to the hypergeometric $E(2,N/D,M/D)$. The suitable Belyi covering can be found in [9, Section 5] as formula [9, (33)] with $(k,\ell,n,m) = (D,N,0,1)$:

$$(1-z) \mapsto \frac{(1-x)^N}{(1-Nx/D)^D}. \tag{22}$$

The covering $\varphi(x)$ in (15) differs by the Möbius transformation $x \mapsto Dx/(Nx+M)$. The pull-back covering can be written as

$$\varphi(x) = \frac{ND}{2M}\, x^2\, \mathrm{Hn}\left(\begin{matrix} -M/N \\ 2(1-M/N) \end{matrix} \left| \begin{matrix} 2, 2-D \\ 3; 1-N \end{matrix} \right| x \right). \tag{23}$$

This is explained as follows. The specialization $\alpha = 1$ gives a pull-back of the the trivial "hypergeometric" equation $y'' = 0$ to $HE(2,N,M,D)$. We have here the Kleinian pull-back [2] for $HE(2,N,M,D)$; its expression in terms of Schwarz maps (as in [1, Section 5]) gives (23).

Identity (20) is valid for $\alpha = 0$ as well, but then we must identify the Heun function as

$$-\frac{2M}{NDx^2}\left(N\ln(1-x) + M\ln\left(1+\frac{Nx}{M}\right)\right). \tag{24}$$

Remark 3.2 As is known [7], Belyi functions (up to Möbius transformations) are in a bijective correspondence with *dessins d'enfant* (up to homotopy). The latter are certain bi-coloured graphs on the Riemann sphere (in our case) such that the incidence degrees of cells and vertices of both colors follow the branching pattern. Up to homotopy, a dessin d'enfant of $\varphi(x)$ is the pre-image of the real interval $[0,1]$ with the points above $z=0$, $z=1$

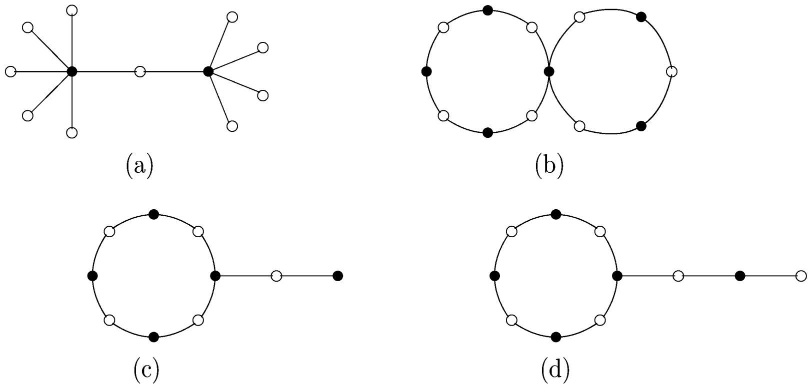

Figure 1. Dessins d'enfant for Belyi pull-backs to $HE(2, N\alpha, M\alpha, D\alpha)$ and $HE(1/2, 3/2, N\alpha, M\alpha)$.

coloured (say) black and white, respectively. The points above $z = \infty$ are represented by the cells. The dessin is a tree when there is a unique cell (of degree D). In particular, the dessin for (15) is depicted in Figure 1(a). The number of "sticks" coming out of the black vertices is $N-1$ and $M-1$. It is straightforward to see that no other dessins are possible for its branching pattern, confirming the uniqueness of $\varphi(x)$.

Remark 3.3 Formulas (16)–(17) are fancy expressions for the power function X^b, yet with a free parameter a. A Möbius transformation gives the straightforward expression

$$\mathrm{Hn}\left(\begin{array}{c} 1+b/a \\ -b\,(a+1) \end{array} \middle| \begin{array}{c} a,\, -b \\ 1+a; -1 \end{array} \middle| \, \frac{a+b}{a}\, x \right) = (1-x)^b. \tag{25}$$

A fractional-linear transformation from [13, (B.3)] and the change $b \mapsto -b$ give the symmetric expression

$$\mathrm{Hn}\left(\begin{array}{c} a/(a-b) \\ \frac{a\,b\,(a+1)}{a-b} \end{array} \middle| \begin{array}{c} a,\, b \\ 1+a; 1+b \end{array} \middle| \, x \right) = (1-x)^{-b}. \tag{26}$$

Since one of the singularities of the considered Heun equation is apparent with the local exponent difference 2, expressions in terms of ${}_3\mathrm{F}_2$ functions in [4, Section 5] apply. The result (with $e = a$ in [4, Theorem 5.3]) is trivial:

$$(1-x)^{-b} = {}_3\mathrm{F}_2\left(\left. \begin{matrix} a, b, a+1 \\ a+1, a \end{matrix} \right| x \right).$$

However, the expression in terms of contiguous ${}_2\mathrm{F}_1$ functions in [4, Corollary 5.3.1] is

$$(1-x)^{-b} = {}_2\mathrm{F}_1\left(\left. \begin{matrix} a, b \\ a+1 \end{matrix} \right| x \right) + \frac{b\,x}{a+1}\,{}_2\mathrm{F}_1\left(\left. \begin{matrix} a+1, b+1 \\ a+2 \end{matrix} \right| x \right). \qquad (27)$$

This relates $HE(2, \alpha, \beta, \alpha+\beta)$ to the contiguous orbit of $E(\alpha, \beta+1, \alpha+\beta)$. Interestingly, the monodromy representation for the Heun equation is completely reducible, while the monodromy representation for the contiguous hypergeometric functions has (generally) only one invariant subspace of dimension 1.

4. Transformations from dihedral $E(1/2, 1/2, \alpha)$

The monodromy representation of $E(1/2, 1/2, \alpha)$ with general $\alpha \in \mathbb{C}$ is an infinite dihedral group, hence these hypergeometric equations and their solutions are said to be *dihedral*. Their hypergeometric solutions are very explicit:

$${}_2\mathrm{F}_1\left(\left. \begin{matrix} \frac{a}{2}, \frac{a+1}{2} \\ a+1 \end{matrix} \right| z \right) = \left(\frac{1+\sqrt{1-z}}{2} \right)^{-a}, \qquad (28)$$

$${}_2\mathrm{F}_1\left(\left. \begin{matrix} \frac{a}{2}, \frac{a+1}{2} \\ 1/2 \end{matrix} \right| z \right) = \frac{(1-\sqrt{z})^{-a} + (1+\sqrt{z})^{-a}}{2}. \qquad (29)$$

General expressions for contiguous ${}_2\mathrm{F}_1$ functions are presented in [10]. As shown in [11, Section 4], there are pull-back transformations $E(1/2, 1/2, \alpha) \xleftarrow{d} E(1/2, 1/2, n\alpha)$ of any degree d. The pull-back coverings have the form $x \mapsto x\theta_2(x)^2/\theta_1(x)^2$, where

$$\theta_1(x) = {}_2\mathrm{F}_1\left(\left. \begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ 1/2 \end{matrix} \right| x \right), \qquad \theta_2(x) = n\,{}_2\mathrm{F}_1\left(\left. \begin{matrix} -\frac{n-1}{2}, -\frac{n-2}{2} \\ 3/2 \end{matrix} \right| x \right). \qquad (30)$$

The components $\theta_1(x)$, $\theta_2(x)$ can be expressed in terms of Chebyshev polynomials $T_n(z)$, $U_{n-1}(z)$, respectively. The identity $(1-\sqrt{x})^n = \theta_1(x) - \theta_2(x)\sqrt{x}$ holds. These transformations can be composed with the simplification $E(1/2,1/2,\alpha) \xleftarrow{2} E(1,\alpha,\alpha)$ to a cyclic monodromy group.

Remark 4.1 Pull-back transformations of $E(1/2,1/2,\alpha)$ to Heun equations may involve non-Belyi coverings. An example is the composition

$$E(1/2,1/2,\alpha) \xleftarrow{2} E(1,\alpha,\alpha) \xleftarrow{N+M} HE(2,N\alpha,M\alpha,(N+M)\alpha), \quad (31)$$

Here the second transformation is as in Theorem 3.1, but the fiber of the "singularity" with the exponent difference 1 may be an arbitrary point of $\mathbb{P}^1_z \setminus \{0,1,\infty\}$. For instance, taking $N=M=1$ gives the covering

$$\varphi_s(x) = \frac{4s\,x\,(2-x)}{(x^2-2x-s)^2} \quad (32)$$

with a parameter s. The branching points lie above 4 points, namely $\varphi(x) \in \{0,1,\infty,4s/(s+1)^2\}$. The transformed Fuchsian equation for

$$\left(1+\frac{2x-x^2}{s}\right)^{-e} {}_2\mathrm{F}_1\left(\left.\begin{array}{c}\frac{e}{2},\frac{e+1}{2}\\ 1+e\end{array}\right|\,\frac{4s\,x\,(2-x)}{(x^2-2x-s)^2}\right) \quad (33)$$

is Heun's equation (1) with $(t,q,a,b,c,d) = (2,0,0,2e,1+e,-1)$. The specialized coverings $\varphi_s(x)$ with $s=\pm 1$ are Belyi coverings. According to [12, Proposition 3.3], the only other case when non-Belyi maps occur in Heun-to-hypergeometric transformations is when the Fuchsian equations have a basis of algebraic solutions.

Theorem 4.2 (i) *The pull-back transformations from $E(1/2,1/2,\alpha)$ to Heun equations with respect to a non-Belyi covering $\varphi(x)$ are compositions* (31)*. The non-Belyi coverings are parametric, and may specialize to Belyi coverings.*

(ii) *Pull-back transformations from $E(1/2,1/2,\alpha)$ to Heun equations with respect to other Belyi coverings $\varphi(x)$ exist and are unique up to Möbius transformations for any pair (M,N) of non-equal positive integers, $M \neq N$. The transformed Heun equation is $HE(1/2,3/2,N\alpha,M\alpha)$.*

(iii) *The degree of the mentioned Belyi covering $\varphi(x)$ is $D = M+N$. Up to Möbius transformations, the Belyi covering is $\varphi(x) = x^3\,\Theta_2(x)^2/\Theta_1(x)^2$, where $\Theta_1(x),\Theta_2(x)$ are the polynomials in x determined by*

$$\left(1+\sqrt{x}\right)^N \left(1-\frac{N\sqrt{x}}{M}\right)^M = \Theta_1(x) + x^{3/2}\,\Theta_2(x). \tag{34}$$

(iv) *The polynomials* $\Theta_1(x)$, $\Theta_2(x)$ *are Heun polynomials*:

$$\Theta_1(x) = \mathrm{Hn}\left(\begin{matrix} M^2/N^2 \\ MD/4N \end{matrix}\;\middle|\; \begin{matrix} -\frac{D}{2}, -\frac{D-1}{2} \\ -\frac{1}{2};\ 1-N \end{matrix} \;\middle|\; x\right), \tag{35}$$

$$\Theta_2(x) = \frac{ND(M-N)}{3M^2}\,\mathrm{Hn}\left(\begin{matrix} M^2/N^2 \\ \frac{3}{2}\left(1+\frac{M^2}{N^2}\right) - \frac{5MD}{4N} \end{matrix}\;\middle|\; \begin{matrix} -\frac{D-3}{2}, -\frac{D-4}{2} \\ \frac{5}{2};\ 1-N \end{matrix} \;\middle|\; x\right). \tag{36}$$

Proof. The non-singular points above the singularities of $E(1/2, 1/2, \alpha)$ are the simple branching points in the two fibers with the exponent difference $1/2$. Let k denote the number of simple branching points in those two fibers, and let ℓ denote the number of points in the third fiber. We have $k \le d$, $\ell \le 4$. Besides, $k \ge d-2$ by Lemma 2.1. The case $(k, \ell) = (d-1, 2)$ leads to too many singularities. There are too few points ($< d+2$) in the three fibers when $(k, \ell) \in \{(d, 1), (d-2, 1)\}$. The cases $(k, \ell) \in \{(d, 2), (d-1, 1)\}$ lead to hypergeometric transformations to $E(1, \beta, \beta)$ and $E(1/2, 1/2, \beta)$. The case $(k, \ell) = (d-2, 2)$ leads to the branching patterns $[2]_{d/2-1} = [2]_{(d-4)/2} + 3 + 1 = N + M$ and $[2]_{(d-1)/2} + 1 = [2]_{(d-3)/2} + 3 = N + M$. We can have $\ell = 4$ only with $k = d$, but then we have more singularities outside the 3 fibers. With $\ell = 3$, we can obtain non-Belyi coverings with the branching $[2]_{d/2} = [2]_{d/2} = K + N + M$ in the three fibers, and Belyi coverings with branching patterns $[2]_{d/2} = [2]_{(d-4)/2} + 4 = K + N + M$.

Transformations to Heun equations are obtained when $(k, \ell) = (d-2, 2)$ or $\ell = 3$.

Consider first the non-Belyi coverings in the $\ell = 3$ case. Alter the specialization $\alpha = 1/K$, they would pull-back $E(1/2, 1/2, \alpha)$ to $E(2, N/K, M/K)$. By [8, Section 7], the latter equation has a completely reducibly monodromy (rather than logarithmic singularities) only when $N/K \pm M/K \in \{0, \pm 1\}$. The case $N = M$ is ruled out by the specialization $\alpha = 1/N$, as it leads to a pull-back to $E(2, K/N)$ that exists only when $K = 2N$; but that is an instance of the $N/K + M/K = 1$ case. We conclude that one of the numbers K, N, M is the sum of the other two. The specialization $\alpha = 1$ gives a pull-back to $HE(2, N, M, K)$ with the monodromy smaller than $\mathbb{Z}/2\mathbb{Z}$. The functions fields must factor through the maximal $\mathbb{Z}/2\mathbb{Z}$-invariant field, hence the factorization $E(1/2,1/2,1) \xleftarrow{2} E(1,1,1) \xleftarrow{d/2} HE(2,N,M,K)$, giving (31). As already mentioned in Remark 4.1, the non-Belyi coverings have a parameter.

The same specialization $\alpha \in \{1/K, 1/N, 1\}$ arguments apply to the Belyi coverings with $\ell = 3$. Their dessins d'enfant look like in Figure 1(b). They are a special case of the one-parameter non-Belyi covering.

Finally, the Belyi coverings with $(k,\ell) = (d-2,2)$ give the transformations $E(1/2,1/2,\alpha) \xleftarrow{d} HE(1/2,3/2,N\alpha,M\alpha)$, with $d = N+M$. The specialization $\alpha = 1/M$ gives a transformation to $HE(1/2,3/2,N/M)$, so we can apply [11, Theorem 5.1] with $(k,\ell,m,n) = (1,0,M,N)$. The obtained pull-back covering is described in (iii). In particular, multiplying (34) with its $\sqrt{x}$-conjugate gives

$$(1-x)^N \left(1 - \frac{N^2\,x}{M^2}\right)^M = \Theta_1(x)^2 - x^3\,\Theta_2(x)^2, \tag{37}$$

convincing us that $\varphi(x)$ is a Belyi covering. We see that the fourth singular point of the pull-backed Heun equation is $t = M^2/N^2$. Besides, $\Psi(x)$ of [11, Section 5] is equal, up to a constant multiple, to $N^2x - M^2$. Theorem 5.6 in [11] gives differential equations for $\Theta_1(x), \Theta_2(x)$, and there we recognize Heun's equations $HE(1/2,3/2,N,M)$ in our particular setting. This allows us to identify the expressions in (iv). The covering $\varphi(x)$ degenerates when $M = N$. There is no pull-back transformation $E(1/2,1/2,\alpha) \xleftarrow{2N} HE(1/2,3/2,N\alpha,N\alpha)$ at all, because the specialization $\alpha = 1/N$ gives a pull-back to the non-existent $E(1/2,3/2)$. $\square$

As noticed in [11, Section 5.2], for $N=1$ we have

$$\Theta_1(x) = {}_2\mathrm{F}_1\left(\left. \begin{array}{c} -\frac{M}{2}, -\frac{M+1}{2} \\ -1/2 \end{array} \right| \frac{x}{M^2} \right),$$

$$\Theta_2(x) = \frac{M^2-1}{3M^2}\, {}_2\mathrm{F}_1\left(\left. \begin{array}{c} -\frac{M-2}{2}, -\frac{M-3}{2} \\ 5/2 \end{array} \right| \frac{x}{M^2} \right),$$

and for $N = 2$ we can write $\Theta_1(x), \Theta_2(x)$ as ${}_3\mathrm{F}_2$ polynomials. The dessin d'enfant for the Belyi covering of (iii) is as depicted in Figure 1(c) or (d). The degree of the bounded cell equals $\min(M,N)$.

Like in Theorem (3.1)(iii), Heun solutions of the pulled-back dihedral equation can be expressed by writing down Heun-to-hypergeometric identities and evaluating the hypergeometric function using formulas like (28)–(29). After setting $a = N\alpha$, $b = M\alpha$ and dropping the condition $a/b \in \mathbb{Q}$ by the analytic continuation argument, we get the following formulas:

$$\mathrm{Hn}\left(\begin{array}{c} b^2/a^2 \\ -\frac{b(a+b)}{4a} \end{array}\middle|\begin{array}{c} \frac{a+b}{2},\ \frac{a+b+1}{2} \\ -\frac{1}{2};\, 1+a \end{array}\middle|\, x\right) = \frac{\left(1+\sqrt{x}\right)^{-a}\left(1-\frac{a}{b}\sqrt{x}\right)^{-b} + \left(1-\sqrt{x}\right)^{-a}\left(1+\frac{a}{b}\sqrt{x}\right)^{b}}{2}, \tag{38}$$

$$\mathrm{Hn}\left(\begin{array}{c} \frac{a^2-b^2}{a^2} \\ \frac{(a+b)^2(a+1)}{4a} \end{array}\middle|\begin{array}{c} \frac{a+b}{2},\ \frac{a+b+1}{2} \\ 1+a;\, -\frac{1}{2} \end{array}\middle|\, 1-x\right) = \left(\frac{1+\sqrt{x}}{2}\right)^{-a}\left(\frac{b-a\sqrt{x}}{b-a}\right)^{-b}. \tag{39}$$

Another formula is

$$\mathrm{Hn}\left(\begin{array}{c} b^2/a^2 \\ \frac{5ab(a+b)+6(a^2+b^2)}{4a^2} \end{array}\middle|\begin{array}{c} \frac{a+b+3}{2},\ \frac{a+b+4}{2} \\ \frac{5}{2};\, 1+a \end{array}\middle|\, x\right) = \frac{3b^2\, x^{-3/2}}{2a(a^2-b^2)}\left(\left(1+\sqrt{x}\right)^{-a}\left(1-\frac{a}{b}\sqrt{x}\right)^{-b} - \left(1-\sqrt{x}\right)^{-a}\left(1+\frac{a}{b}\sqrt{x}\right)^{-b}\right). \tag{40}$$

The Heun solutions of the same equation with the argument $1/x$ evaluate as follows, after the change $x \mapsto 1/x$:

$$\mathrm{Hn}\left(\begin{array}{c} a^2/b^2 \\ \frac{(a^2-b^2)^2+a^3+b^3}{4b^2} \end{array}\middle|\begin{array}{c} \frac{a+b}{2},\ \frac{a+b+3}{2} \\ \frac{1}{2};\, 1+a \end{array}\middle|\, x\right) = \frac{\left(1+\sqrt{x}\right)^{-a}\left(1-\frac{b}{a}\sqrt{x}\right)^{-b} + \left(1-\sqrt{x}\right)^{-a}\left(1+\frac{b}{a}\sqrt{x}\right)^{b}}{2}, \tag{41}$$

$$\mathrm{Hn}\left(\begin{array}{c} a^2/b^2 \\ \frac{(a^2-b^2)^2+3(a^3+b^3)+2(a^2+b^2)}{4b^2} \end{array}\middle|\begin{array}{c} \frac{a+b+1}{2},\ \frac{a+b+4}{2} \\ \frac{3}{2};\, 1+a \end{array}\middle|\, x\right) = \frac{a\, x^{-1/2}}{2(b^2-a^2)}\left(\left(1+\sqrt{x}\right)^{-a}\left(1-\frac{b}{a}\sqrt{x}\right)^{-b} - \left(1-\sqrt{x}\right)^{-a}\left(1+\frac{b}{a}\sqrt{x}\right)^{-b}\right). \tag{42}$$

These formulas can be applied to evaluate all solutions of $HE(1/2, 3/2, a, b)$.

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