

## On topology of some Riemannian manifolds of negative curvature with a compact Lie group of isometries

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**Abstract.** We topologically characterize negatively curved Riemannian manifolds which are of cohomogeneity two under the action of a compact Lie group of isometries.

*Key words:* Manifold, Lie group, Isometry.

### 1. Introduction

A  $G$ -manifold of cohomogeneity  $k$  is a manifold  $M$  with smooth action of a Lie group  $G$  such that the maximum of the dimensions of the  $G$ -orbits is  $\dim M - k$ . When  $k$  is small, Lie group theory can be applied to study geometric and topological properties of  $M$ . This is one of the reasons that actions of low cohomogeneity are of so much interest to mathematicians. If  $M$  has negative curvature and  $k = 0$  or  $k = 1$ , it is proved that either  $M$  is simply connected or  $\pi_1(M) = \mathbb{Z}^p$  for some positive integer  $p$  ([7], [13]). We are interested in the topological properties of cohomogeneity two Riemannian manifolds of negative curvature. This article follows previous papers [9], [10], [11], where we proved various results about topological properties of cohomogeneity two negatively curved  $G$ -manifold  $M$  under some special conditions on  $M$  or  $G$ . Here, we study the topology of  $M$  under the assumption that  $G$  is compact and the action does not have singular orbits of positive dimension. Our main result is Theorem 3.5. The ideas of the proof are similar to those of the main result of [11].

### 2. Preliminary

Later we will need the following classical result by H. Weyl.

**Theorem 2.1** ([15]) *If  $G$  is a connected compact semisimple Lie group, its universal covering group  $\tilde{G}$  is compact.*

In the following we will denote by  $S^1$  and  $B$  the unite circle and the Moebius band, respectively. We also denote by  $M^G \subset M$  the fixed point set of the action of  $G$  on  $M$ .

**Theorem 2.2** ([9]) *Let  $M^{n+2}$  be a nonsimply connected and complete Riemannian manifold of negative curvature which is of cohomogeneity two under the action of a closed and connected Lie group  $G$  of isometries with  $M^G \neq \emptyset$ . Then  $M$  is diffeomorphic either to  $S^1 \times R^{n+1}$  or  $B \times R^n$ .*

If  $M$  is a simply connected Riemannian manifold of nonpositive curvature and  $\gamma$  is a geodesic in  $M$ , then we denote by  $[\gamma]$  the asymptotic class of geodesics containing  $\gamma$  (see [6]). The following set is by definition the infinity of  $M$ :

$$M(\infty) = \{[\gamma] : \gamma \text{ is a geodesic in } M\}.$$

For any  $x \in M$  and  $[\gamma] \in M(\infty)$ , there exists a unique (up to parametrization) geodesic  $\gamma_x \in [\gamma]$  such that  $x \in \gamma_x$  and there is a unique hypersurface  $S_x$ , which contains  $x$  and is perpendicular to all elements of  $[\gamma]$ . The hypersurface  $S_x$  is called horosphere determined by  $x$  and  $[\gamma]$ .

For any isometry  $\delta : M \rightarrow M$  of a Riemannian manifold  $M$ , let us denote by  $d_\delta^2 : M \rightarrow R$  the squared displacement function namely

$$d_\delta^2(x) = (\text{dist}(x, \delta(x)))^2$$

We recall the following (see [1] and [5])

**Theorem 2.3** *Let  $M$  be a complete, simply connected Riemannian manifold of negative curvature and  $\delta \in \text{Iso}(M)$ . Then the squared displacement function  $d_\delta^2$  satisfies one and only one of the following properties:*

- (1)  $d_\delta^2$  has no minimum point;
- (2) the minimum point set of  $d_\delta^2$  coincides with the fixed point set of  $\delta$ ;
- (3) the minimum point set of  $d_\delta^2$  coincides with the image of exactly one geodesic  $\gamma$ , translated by  $\delta$  (i.e., there is a positive real number  $t_0$  such that  $\delta(\gamma(t)) = \gamma(t + t_0)$  for all  $t \in R$ ).

The isometries, for which property (1), (2) or (3) holds, are called parabolic, elliptic or axial, respectively. The geodesic  $\gamma$  in (3) is called the axes of  $\delta$ . We remark that if  $g$  is axial, there exists exactly one geodesic

$\gamma$  such that  $g(\gamma) = \gamma$ , while if  $g$  is parabolic, there exists exactly one  $[\gamma] \in M(\infty)$  such that  $g[\gamma] = [\gamma]$ .

The argument of [3, Lemma 3], shows that if  $g \in Iso(M)$  and  $g([\gamma]) = [\gamma]$  for some  $[\gamma] \in M(\infty)$ , then either  $g$  maps each horosphere  $S$  determined by  $[\gamma]$  into itself or it is axial with axis in the class  $[\gamma]$ .

From Lemma 3.1 and Remark 3.2 in [14], the next lemma follows.

**Lemma 2.4** *Let  $M$  be a complete simply connected Riemannian manifold of negative curvature and  $G \subset Iso(M)$  a closed connected group of isometries with  $M^G = \emptyset$ . Then there is at most one totally geodesic  $G$ -orbit in  $M$ .*

**Theorem 2.5** ([7]) *A homogeneous Riemannian manifold of negative curvature is simply connected.*

By [13, Theorems 3.5 and 3.7], we have the following

**Proposition 2.6** *Let  $M$  be a complete nonsimply connected Riemannian manifold of negative curvature, with  $\dim M \geq 3$  and of cohomogeneity one w.r.t. the action of a group  $G \subset Iso(M)$ . Then:*

- (a) *If there is a singular orbit, it is unique and diffeomorphic to  $S^1$ , so that all orbits except one, are of dimension  $n - 1$ ,  $n = \dim M$ .*
- (b) *If there is no singular orbit, then there exists an integer  $p$  such that each orbit is diffeomorphic to  $T^p \times R^{n-p-1}$  and  $M$  is diffeomorphic to  $T^p \times R^{n-p}$ .*

**Theorem 2.7** (see [3], [14]) *Let  $G$  be a connected and solvable Lie subgroup of isometries of a simply connected and negatively curved Riemannian manifold  $M$ . Then one of the following claims is true:*

- (1)  $M^G \neq \emptyset$ .
- (2) *There is a unique  $G$ -invariant geodesic.*
- (3) *There is a unique class of asymptotic geodesics  $[\gamma]$  such that  $G[\gamma] = [\gamma]$ .*

**Fact 2.8** (see [5, p. 57, 58]) *Let  $\widetilde{M}$  be a complete and simply connected Riemannian manifold of strictly negative curvature, i.e., with curvature  $K \leq c < 0$  for some negative constant  $c$ , and let  $S$  be a horosphere in  $\widetilde{M}$  determined by asymptotic class of geodesics  $[\gamma]$ . The function  $f : \widetilde{M} \rightarrow R$ ,  $f(p) = \lim_{t \rightarrow \infty} d(p, \gamma(t)) - t$ , is called a Busemann function.*

For each point  $p \in \widetilde{M}$  there is a point  $\eta_S(p)$  in  $S$ , which is the unique

point in  $S$  of minimal distance from  $p$ , and the following map is a homeomorphism:

$$\phi : \widetilde{M} \rightarrow S \times R, \quad \phi(p) = (\eta_S(p), f(p)).$$

**Fact 2.9** (see [2], [10]) Let  $M$  be a Riemannian manifold and  $G$  be a connected subgroup of  $Iso(M)$ , and let  $\widetilde{M}$  be the universal Riemannian covering manifold of  $M$  with covering map  $\kappa : \widetilde{M} \rightarrow M$  and deck transformation group  $\Delta$ . Then there is a connected Lie group  $\widetilde{G}$  and covering map  $\pi : \widetilde{G} \rightarrow G$  such that  $\widetilde{G}$  acts isometrically on  $\widetilde{M}$  and

- (1) Each  $\delta \in \Delta$  maps  $\widetilde{G}$ -orbits onto  $\widetilde{G}$ -orbits.
- (2) If  $x \in M$  and  $\tilde{x} \in \pi^{-1}(x)$  then  $\kappa(\widetilde{G}(\tilde{x})) = G(x)$ .
- (3)  $\widetilde{M}^{\widetilde{G}} = \kappa^{-1}(M^G)$ .
- (4) The deck transformation group centralizes  $\widetilde{G}$  (i.e., for each  $\delta \in \Delta$  and  $\tilde{g} \in \widetilde{G}$ ,  $\delta\tilde{g} = \tilde{g}\delta$ ).
- (6) If  $\widetilde{M}^{\widetilde{G}}$  is a one point set then  $M = \widetilde{M}$ .

**Fact 2.10** Let  $M$  be a complete non simply connected Riemannian manifold of strictly negative curvature and  $S$  be a horosphere related to an asymptotic class of geodesics  $[\gamma]$  in  $\widetilde{M}$  such that  $\Delta S = S$ . Then

- (1)  $M$  is homeomorphic to  $S/\Delta \times R$ .
- (2) If for each  $g \in \widetilde{G}$ ,  $g[\gamma] = [\gamma]$  then  $\widetilde{G}S = S$ .

*Proof.* (1) let  $\eta_S$ ,  $f$  and  $\phi$  be the maps mentioned in Fact 2.8. Since  $\Delta S = S$ , the homeomorphism  $\phi : \widetilde{M} \rightarrow S \times R$  induces a homeomorphism  $\phi_1 : \widetilde{M}/\Delta = M \rightarrow S/\Delta \times R$ .

(2) If  $g \in \widetilde{G}$  and  $gS \neq S$  then  $g$  is axial isometry and there is a unique geodesic  $\lambda$  in  $[\gamma]$  such that  $g$  translates it. Since  $g$  commutes with all elements of  $\Delta$ , we get from uniqueness of  $\lambda$  that for each  $\delta \in \Delta$ ,  $\delta\lambda = \lambda$ . But, intersection of  $\lambda$  and  $S$  is a one point set. So, we get from  $\delta S = S$  that  $\delta$  has a fixed point, which is a contradiction for the elements of  $\Delta$  that are different from the identity. Therefore,  $gS = S$ .  $\square$

**Fact 2.11** (See [3, Lemma 3]) If  $\delta$  is a parabolic isometry of  $\widetilde{M}$  such that  $\delta[\gamma] = [\gamma]$  then for each horosphere  $S$  determined by  $[\gamma]$ ,  $\delta S = S$ .

### 3. Results

In the following lemmas and theorem, we keep the notations of Fact 2.9.

**Lemma 3.1** *If  $M$  has negative curvature and all orbits are of dimension bigger than 1, then for each  $\delta \in \Delta$  and each  $x \in \widetilde{M}$ ,  $\delta\widetilde{G}(x) = \widetilde{G}(x)$ .*

*Proof.* Suppose that there is a  $\delta \in \Delta$  and  $x \in \widetilde{M}$  such that  $\delta\widetilde{G}(x) \neq \widetilde{G}(x)$ . It is proved in [1] that the squared displacement function  $f_\delta$  is strictly convex except at the minimum point set, which is at most the image of a geodesic. Let  $\gamma$  be a geodesic in  $\widetilde{M}$ , which is not the minimum point set of  $f_\delta$  and joins a point in  $\widetilde{G}(x)$  to a point in  $\delta\widetilde{G}(x)$ . We get from strict convexity criterion that for each  $t \in \mathbb{R}$ ,  $(f_\delta(\gamma(t)))'' > 0$ . Consider  $g_1, g_2 \in \widetilde{G}$  with the property that  $\gamma(0) = g_1(x) \in \widetilde{G}(x)$  and  $\gamma(1) = \delta g_2(x) \in \delta\widetilde{G}(x)$  and put  $h(t) = f_\delta(\gamma(t))$ . Since  $\delta$  commutes with the elements of  $\widetilde{G}$ , we have

$$\begin{aligned} h(1) &= f_\delta(\gamma(1)) = f_\delta(\delta g_2(x)) = d^2(\delta g_2(x), \delta \delta g_2(x)) = d^2(\delta g_2(x), \delta g_2 \delta(x)) \\ &= d^2(x, \delta(x)) = d^2(g_1(x), g_1 \delta(x)) = d^2(g_1(x), \delta g_1(x)) = f_\delta(\gamma(0)) = h(0) \end{aligned}$$

$h$  is strictly convex, so it has a (unique) minimum point  $t_0$  between 0, 1. Since  $f_\delta$  is constant along orbits,  $\widetilde{G}(\gamma(t_0))$  is the minimum point set of  $f_\delta$  and  $\widetilde{G}(\gamma(t_0))$  must be at most the image of a geodesic. Then  $\dim \widetilde{G}(\gamma(t_0)) \leq 1$ , which is a contradiction.  $\square$

**Lemma 3.2** ([9]) *If  $M$  is a connected and complete cohomogeneity  $k$  Riemannian  $G$ -manifold, then  $k > \dim M^G$ .*

**Lemma 3.3** *Let  $M$  be a complete and nonsimply connected Riemannian  $G$ -manifold of negative curvature such that  $\dim M > 3$  and all  $G$ -orbits are of dimension equal to  $\dim M - 2$ . If there is a nontrivial totally geodesic submanifold  $W$  of  $\widetilde{M}$  such that  $\widetilde{G}W = W$  then  $\pi_1(M) = \mathbb{Z}^p$ , for some positive integer  $p$ .*

*Proof.* Let  $\dim M = n + 2$ ,  $n > 1$ . If  $\dim W = 1$  then we get from  $\widetilde{G}W = W$  that there is an orbit of dimension  $\leq 1$ , which is in contradiction with hypotheses. Then,  $\dim W > 1$ . Since  $W$  is a union of  $\widetilde{G}$ -orbits and all orbits are of dimension  $n$ , then  $n \leq \dim W < n + 2$ , so  $\dim W = n$  or  $\dim W = n + 1$ . In the first case,  $W$  is a  $\widetilde{G}$ -orbit and by uniqueness criterion in Lemma 2.4, we get that  $\Delta(W) = W$ , so  $\kappa(W) = W/\Delta$ . But

$\kappa(W)$  is a totally geodesic  $G$ -orbit in  $M$  and by Theorem 2.5, it must be simply connected. Then  $\Delta$  is trivial and  $M$  is simply connected, which is a contradiction. In the second case ( $\dim W = n + 1$ ),  $W$  is a cohomogeneity one  $\tilde{G}$ -manifold. By Lemma 3.1, for all  $\delta \in \Delta$  and all  $x \in \tilde{M}$ ,  $\delta\tilde{G}(x) = \tilde{G}(x)$ . Then  $\Delta(W) = W$  and  $\pi_1(M) = \pi_1(\kappa(W))$ . But,  $\kappa(W)$  is a totally geodesic cohomogeneity one  $G$ -submanifold of  $M$  without singular orbits (because, if there is a singular orbit then by Proposition 2.6, it must be of dimension one). So, there is a positive integer  $p$  such that each  $G$ -orbit in  $\kappa(W)$  is diffeomorphic to  $T^p \times R^s$ ,  $p + s = \dim W - 1 = n$ , and  $\kappa(W)$  is diffeomorphic to  $T^p \times R^{s+1}$ . Then,  $\pi_1(M) = \pi_1(\kappa(W)) = Z^p$ .  $\square$

**Lemma 3.4** *Let  $M$  be a nonsimply connected Riemannian manifold of strictly negative curvature such that  $\dim M > 3$  and all  $G$ -orbits are of dimension  $\dim M - 2$ . If there is a unique class  $[\gamma]$  of asymptotic geodesics in  $\tilde{M}$  such that  $\tilde{G}[\gamma] = [\gamma]$  then  $M$  is a parabolic manifold homeomorphic to  $S/(\pi_1(M)) \times R$ , where  $S$  is a horosphere in  $\tilde{M}$  and  $S/(\pi_1(M))$  is a cohomogeneity one  $G$ -manifold.*

*Proof.* If  $\delta \in \Delta$  then  $\delta$  has no fixed point, so it is not elliptic. If there is a unique geodesic  $\lambda$  such that  $\delta\lambda = \lambda$  then for each  $g \in \tilde{G}$ ,  $\delta(g\lambda) = g\delta\lambda = g\lambda$ . Thus, we get from uniqueness of  $\lambda$  that  $g\lambda = \lambda$ . So,  $\lambda$  must be a  $\tilde{G}$ -orbit, which is a contradiction (because, all orbits are of dimension  $\dim M - 2 > 1$ ). Thus,  $\delta$  is not axial. This means that the elements of  $\Delta$  are parabolic. Since the elements of  $\Delta$  commute with the elements of  $\tilde{G}$ , we get from uniqueness of  $[\gamma]$  that  $\Delta[\gamma] = [\gamma]$ . By Fact 2.11, for each  $\delta \in \Delta$  and each horosphere  $S$  determined by the asymptotic class  $[\gamma]$ ,  $\delta S = S$ . Now, by Fact 2.10,  $M$  is diffeomorphic to  $S/(\pi_1(M)) \times R$  and all  $\tilde{G}$ -orbits of  $\tilde{M}$  are included in horospheres. Thus,  $S$  is a cohomogeneity one  $\tilde{G}$ -manifold and  $S/(\pi_1(M))$  is a cohomogeneity one  $G$ -manifold.  $\square$

If  $M$  is a  $G$ -manifold of cohomogeneity  $k$  then there are two types of  $G$ -orbits in  $M$ , which are called singular and principal orbits (see [2] or [8] for definitions and details about singular and principal orbits). The dimension of a principal orbit is equal to  $\dim M - k$  and the dimension of a singular orbit is  $\leq \dim M - k$ . The union of all principal orbits is an open and dense subset of  $M$ . If there is no singular orbit or dimension of each singular orbit is zero (i.e, singular orbits are fixed points of  $G$ ), then we say that  $M$  is a  $G$ -manifold of fixed point singular type.

**Theorem 3.5** *Let  $M$  be a complete Riemannian manifold of strictly negative curvature and  $\dim M = n + 2$ ,  $n > 1$ , and let  $G$  be a compact and connected subgroup of  $\text{Iso}(M)$  such that  $M$  is a cohomogeneity two  $G$ -manifold of fixed point singular type. Then one of the following claims is true:*

- (1)  $M$  is simply connected (i.e., it is diffeomorphic to  $R^{n+2}$ ).
- (2)  $M$  is diffeomorphic to  $S^1 \times R^{n+1}$  or  $B \times R^n$  ( $B$  is the moebius band).
- (3) There is a positive integer number  $p$  such that  $\pi_1(M) = Z^p$ .
- (4)  $M$  is a parabolic manifold homeomorphic to  $S/(\pi_1(M)) \times R$ , where  $S$  is a horosphere in the universal Riemannian covering of  $M$  and  $S/(\pi_1(M))$  is a cohomogeneity one  $G$ -manifold.

*Proof.* Suppose that  $M$  is not simply connected. The proof is obtained by considering the following three cases.

(a) If  $\widetilde{M}^{\widetilde{G}} \neq \emptyset$  then  $M^G \neq \emptyset$ , so by Theorem 2.2 we get that (2) is true.

(b) If  $G$  is semi-simple then by Theorem 2.1,  $\widetilde{G}$  is compact. It is well known that a compact connected subgroup of the isometries of a simply connected Riemannian manifold of nonpositive curvature has nonempty fixed point set. So,  $\widetilde{M}^{\widetilde{G}} \neq \emptyset$  and, by (a), we get that (2) is true.

(c) If  $G$  is non-semisimple and  $\widetilde{M}^{\widetilde{G}} = \emptyset$  then by Fact 2.9,  $M^G = \emptyset$ . So, by assumptions of the theorem, there is no singular  $G$ -orbit in  $M$  and all  $G$ -orbits are of dimension  $n$ . Therefore, all  $\widetilde{G}$ -orbits of  $\widetilde{M}$  are  $n$ -dimensional. Since  $G$  is non-semisimple,  $\widetilde{G}$  is non-semisimple. Let  $H$  be a connected solvable normal subgroup of  $\widetilde{G}$  and put  $W = \widetilde{M}^H$ .

If  $W = \emptyset$  then by Theorem 2.7, either there is a unique geodesic  $\gamma$  such that  $H(\gamma) = \gamma$  or there is a unique class of asymptotic geodesics  $[\gamma]$  such that  $H[\gamma] = [\gamma]$ . From normality of  $H$  in  $\widetilde{G}$ , uniqueness of  $\gamma$  in the first case, and uniqueness of  $[\gamma]$  in the second case, we get that  $\widetilde{G}(\gamma) = \gamma$  or  $\widetilde{G}[\gamma] = [\gamma]$ .  $\widetilde{G}(\gamma) = \gamma$  can not occur (because all orbits are of dimension  $n > 1$ ). So,  $\widetilde{G}[\gamma] = [\gamma]$  and, by Lemma 3.4, we get that (4) is true. If  $W \neq \emptyset$  then it is a nontrivial totally geodesic submanifold of  $\widetilde{M}$ . Let  $g \in \widetilde{G}$ ,  $h \in H$  and  $x \in W$ . Since  $H$  is normal in  $\widetilde{G}$  then  $g^{-1}hg \in H$ , so  $g^{-1}hg(x) = x$  and  $hg(x) = g(x)$ . Therefore  $G(W) = W$  and, by Lemma 3.3, we get that (3) is true.  $\square$

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