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On the classical limit of self-interacting quantum field Hamiltonians with cutoffs

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Abstract. We study, using Hepp's method, the propagation of coherent states for a general class of self interacting bosonic quantum field theories with spatial cutoffs. This includes models with non-polynomial interactions in the field variables. We show indeed that the time evolution of coherent states, in the classical limit, is well approximated by time-dependent affine Bogoliubov unitary transformations. Our analysis relies on a non-polynomial Wick quantization and a specific hypercontractive estimate.

Key words: Classical limit, Coherent states, QFT, Wick quantization, $P(\varphi)_2$ model.

1. Introduction

In the early days of quantum mechanics Niels Bohr formulated the correspondence principle stating that classical physics and quantum physics agree in the limit of large quantum numbers. Later Erwin Schrödinger discovered the so-called coherent states which provide a bridge between the quantum and the classical theory. The quantum dynamics of these states are indeed closely localized around the classical trajectories although the uncertainty principle asserts that it is not possible to find a compactly supported wave function both in the position and the impulsion representation. Nevertheless, coherent states are the best minimizers of the uncertainty inequality with respect to the position and momentum observables and hence they are the most classically localized states in the phase-space.

The physical intuition behind the coherent states and its usefulness for the classical limit was put on a firm mathematical ground by K. Hepp in his remarkable work [16]. Nowadays coherent states are widely used in physics, for instance in quantum optics [23], as well as in the mathematical literature [7]. It is in a certain sense an effective and yet simple tool for microlocalisation (see for instance [8], [15], [20], [31]).

It was noticed in [16] that the classical limit can be derived not only

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for one particle Schrödinger dynamics but also for many-body Hamiltonians and models of quantum field theory (see also [10]). Thus, the coherent states method is also effective for infinite dimensional phase-space analysis. However, the classical limit of quantum field theories attracted a less attention compared to the successful semiclassical analysis in finite dimensions and to the fast growing subject of mean field theory (see [2], [13], [26] and references therein).

The purpose of the present paper is to study, through propagation of coherent states, the classical limit of self interacting Bose field theories. We extend indeed the result of [16] so that it holds true for all coherent states, for all times and for a general class of quantum field Hamiltonians with possibly unbounded non-polynomial interactions. We also clarify the classical field equation obtained in the limit which seems to be set inaccurately in [16]. Our results apply to the models $(\varphi^4)_2$, $(\varphi^{2n})_2$ and more generally $P(\varphi)_2$ boson field Hamiltonians as well as some variant of the Høegh-Krohn model (see [17], [18]) and some recently studied models in [11], [12]. The construction of such Hamiltonians was one of the beautiful results of mathematical physics established by the late sixties (see e.g. [14], [19], [27], [28], [30]).

More precisely we show that the quantum evolution of a coherent state localized around a point φ_0 on the phase-space is well approximated in the classical limit by a sequezeed coherent states centered around φ_t (the classical orbit starting from φ_0 a time $t = 0$) and deformed by a time-dependent unitary Bogoliubov transformation. As a consequence the classical limit of the expectation values of the Weyl operators on time-evolved coherent states are the exponentials of the classical field orbit in phase space.

The classical limit can be addressed either from a dynamical point of view or a variational perspective. Here we focus on dynamical issues while variational questions were studied in [1], [4]. It is also worth mentioning that an alternative method was developed in [3], extending Wigner (or semiclassical) measures to the infinite dimensional phase-space framework. However, it was applied only to many-body Hamiltonians with conserved number of particles. Its adaptation to models of quantum field theory will be considered elsewhere.

Overview of the paper: In Section 2 we fix some notations and state our main results on propagation of coherent states in the classical limit. The proof of the main theorem (Theorem 2.1) is presented in Section 4 where we also establish existence of global solutions for the classical equation and study a related time-dependent quadratic dynamic. In Section 3 we introduce a specific Wick quantization, establish an hypercontractivity type inequality and present some models of quantum field Hamiltonians covered by the present analysis.

2. Preliminaries and main results

The Hamiltonians of quantum field models can be described either in the particle or in the wave representation. In fact, the free Bose fields Hamiltonians are simply expressed in the symmetric Fock space while the interaction is a multiplication by a measurable function on a space $L^2(M,\mu)$ related to the representation of random Gaussian processes indexed by real Hilbert spaces.

The general framework is as follows. Let $\mathscr{Z}$ denote a separable Hilbert space with a scalar product $\langle\cdot,\cdot\rangle$ which is anti-linear in the left argument and with the associated norm $|z|=\sqrt{\langle z,z\rangle}$. We assume that $\mathscr{Z}$ is equipped with a complex conjugation $\mathfrak{c}: z\mapsto \mathfrak{c}(z)$ compatible with the Hilbert structure (*i.e.,* $\mathfrak{c}$ is antilinear, $\mathfrak{c}\circ\mathfrak{c}(z)=z$ and $|\mathfrak{c}(z)|=|z|$). From now on we denote

$$\overline{z}:=\mathfrak{c}(z),\quad \forall z\in\mathscr{Z},$$

and consider $\mathscr{Z}_0$ to be the real subspace of $\mathscr{Z}$, *i.e.*,

$$\mathscr{Z}_0:=\{z\in\mathscr{Z};\bar z=z\}. \tag{2.1}$$

The symmetric Fock space over $\mathscr{Z}$ is the direct Hilbert sum

$$\Gamma_s(\mathscr{Z})=\bigoplus_{n=0}^{\infty}\otimes_s^n\mathscr{Z}. \tag{2.2}$$

A particularly convenient dense subspace of $\Gamma_s(\mathscr{Z})$ is the space of finite particle states given by the algebraic direct sum

$$\mathcal{D}_f=\bigoplus_{n=0}^{alg}\otimes_s^n\mathscr{Z}. \tag{2.3}$$

It is well-known that $\Gamma_s(\mathscr{Z})$ carries a Fock unitary representation of the Weyl commutation relations, namely there exists a mapping $f\mapsto W(f)$ from $\mathscr{Z}$

into unitary operators on $\Gamma_s(\mathcal{Z})$ satisfying

$$W(f_1)W(f_2) = e^{-(i\varepsilon/2)\mathrm{Im}\langle f_1, f_2\rangle} W(f_1 + f_2), \quad \forall f_1, f_2 \in \mathcal{Z}. \tag{2.4}$$

Here ε is a positive sufficiently small (semiclassical) parameter and $\mathrm{Im}\langle\cdot,\cdot\rangle$ is the imaginary part of the scalar product on $\mathcal{Z}$ which is in particular a symplectic form. The so-called Weyl operators $W(f)$ are given by $W(f) = e^{i\Phi_s(f)}$ for all $f \in \mathcal{Z}$ where $\Phi_s(f) = (1/\sqrt{2})(a^*(f) + a(f))$ are the Segal field operators and $a^*(\cdot), a(\cdot)$ are the ε-dependent creation-annihilation operators satisfying

$$[a(f), a^*(g)] = \varepsilon\langle f, g\rangle 1\!\!1, \quad [a(f), a(g)] = 0 = [a^*(f), a^*(g)], \quad \forall f, g \in \mathcal{Z}.$$

In this framework, the coherent states are the total family of vectors in the Fock space $\Gamma_s(\mathcal{Z})$ given by

$$W\Big(-i\frac{\sqrt{2}}{\varepsilon}z\Big)\Omega = e^{-(|z|^2/2\varepsilon)} \sum_{n=0}^{\infty} \varepsilon^{-(n/2)} \frac{z^{\otimes n}}{\sqrt{n!}}, \quad \forall z \in \mathcal{Z}, \tag{2.5}$$

where Ω is the vacuum vector (i.e., $\Omega = (1, 0, \dots) \in \Gamma_s(\mathcal{Z})$).

The free Bose field Hamiltonian in this representation is given by the second quantized operator $\mathrm{d}\Gamma(A)$ defined for any self-adjoint operator A on $\mathcal{Z}$ as

$$\mathrm{d}\Gamma(A)_{|\otimes_s^n \mathcal{Z}} := \varepsilon \sum_{i=1}^{n} 1\!\!1 \otimes \cdots \otimes \underbrace{A}_{i^{\text{th}}\text{ position}} \otimes \cdots \otimes 1\!\!1. \tag{2.6}$$

In particular the ε-dependent number operator is defined by $N = \mathrm{d}\Gamma(1\!\!1)$.

We will sometimes use the lifting operation of an operator A on $\mathcal{Z}$ to $\Gamma_s(\mathcal{Z})$ given by $\Gamma(A)_{|\otimes_s^n \mathcal{Z}} := A \otimes \cdots \otimes A$. For instance $\Gamma(\mathfrak{c})$ defines a conjugation on the Fock space $\Gamma_s(\mathcal{Z})$.

It is also well-known that there exist a probability space $(M, \mathfrak{T}, \mu)$ and an ε-independent unitary map $\mathcal{R} : \Gamma_s(\mathcal{Z}) \to L^2(M, \mu)$ such that $\mathcal{R}\Omega = 1$ and $\mathcal{Z}_0 \ni f \mapsto \Phi(f) = \sqrt{2/\varepsilon}\mathcal{R}\Phi_s(f)\mathcal{R}^*$ is an $\mathbb{R}$-linear mapping taking values into centered gaussian random variables on M with variance $|f|^2$ (see Theorem 3.9). This means that any $\Phi(f) = \sqrt{2/\varepsilon}\mathcal{R}\Phi_s(f)\mathcal{R}^*$ is a (self-adjoint) multiplication operator on the space $L^2(M, \mu)$ for every $f \in \mathcal{Z}_0$.

The mapping $\mathcal{R}$ provides an unitary equivalent Fock representation of the Weyl commutation relations on the space $L^2(M,\mu)$ called the wave representation. We observe that for any $V \in \Gamma_s(\mathcal{Z})$ satisfying $\Gamma(\mathfrak{c})V = V$, $\mathcal{R}(V)$ is a real-valued function belonging to $L^2(M,\mu)$. Therefore $\mathcal{R}(V)$ can be considered as a self-adjoint multiplication operator on $L^2(M,\mu)$ which we denote by $\mathcal{M}_{\mathcal{R}(V)}$. It turns that these operators $\mathcal{M}_{\mathcal{R}(V)}$ on $L^2(M,\mu)$ are Wick operators when they are transformed to the Fock space $\Gamma_s(\mathcal{Z})$ via the unitary transform $\mathcal{R}$. Indeed, we have the relation

$$\mathcal{R} F_V^{Wick} \mathcal{R}^* = \mathcal{M}_{\mathcal{R}(\Gamma(\sqrt{\varepsilon})V)},$$

where F_V^{Wick} is an ε-dependent Wick operator (possibly non-polynomial) with an explicit Wick symbol given by

$$F_V(z) = \sum_{n=0}^{\infty} \left\langle \frac{(z+\bar{z})^{\otimes n}}{\sqrt{n!}}, V^{(n)} \right\rangle \quad \text{and}$$

$$V^{(n)} \in \otimes_s^n \mathcal{Z} \quad \text{with} \quad V = \bigoplus_{n=0}^{\infty} V^{(n)} \in \Gamma_s(\mathcal{Z}). \tag{2.7}$$

The relation between symbols and Wick operators is studied in details in Section 3. We warn the reader that the Wick quantization here is ε-dependent.

We shall consider the general class of Hamiltonians given by the "sum"

$$H := \mathrm{d}\Gamma(A) + F_V^{Wick}, \tag{2.8}$$

where A is a self-adjoint operator on $\mathcal{Z}$ satisfying:

(A1) $\mathfrak{c}A = A\mathfrak{c}$ and $A \geq m\mathbb{1}$ for some $m > 0$.

The multiplication operator $\mathcal{M}_{\mathcal{R}(\Gamma(\sqrt{\varepsilon})V)}$ by the function $\mathcal{R}(\Gamma(\sqrt{\varepsilon})V)$ on the wave representation, which transforms unitarily to F_V^{Wick} on the Fock representation, verifies

(A2) $\mathcal{R}(\Gamma(\sqrt{\varepsilon})V)$ is a real-valued function in $L^q(M,\mu)$ for some $q > 2$ and $e^{-t\mathcal{R}(\Gamma(\sqrt{\varepsilon})V)} \in L^1(M,\mu)$ for any $t > 0$ and $\varepsilon \in (0,1)$.

Here $\mathcal{R}$ is the transform given by Theorem 3.9. The operator H depends on the parameter ε and it is self-adjoint under assumptions **(A1)** and **(A2)** (see Theorem 3.19).

Our main result is the following theorem.

Theorem 2.1 *Assume* **(A1)**–**(A2)** *and that* $V \in \mathcal{D}(e^{\alpha\Gamma(\lambda)})$ *for some* $\lambda > 1$ *and* $\alpha > 0$. *Let* $\varphi_0 \in \mathcal{Z}$ *and* $\Psi \in \mathcal{D}_f$ *then there exists for every* $t \in \mathbb{R}$ *a finite* ε*-independent bound* $c(t,\Psi) > 0$ *such that the inequality*

$$\left\| e^{-i(t/\varepsilon)H} W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_0\Big)\Psi - e^{i(\omega(t)/\varepsilon)} W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_t\Big) U_2(t,0)\Psi \right\|_{\Gamma_s(\mathcal{Z})} \leq c(t,\Psi)\sqrt{\varepsilon},$$

holds uniformly in $\varepsilon \in (0,1)$ *as long as* φ_t *is the mild solution of the field equation*

$$i\partial_t \varphi_t = A\varphi_t + \partial_{\bar{z}} F_V(\varphi_t) \tag{2.9}$$

with initial data φ_0, *the function* $\omega(t)$ *is given by*

$$\omega(t) = \int_0^t \sum_{k=0}^{\infty} \frac{(k-2)}{2} \Big\langle \frac{(\varphi_s + \bar{\varphi}_s)^{\otimes k}}{\sqrt{k!}}, V^{(k)} \Big\rangle ds,$$

and $U_2(t,s)$ *is the unitary propagator of a time-dependent quadratic Hamiltonian given by Corollary* 4.6.

Remark 2.2 (i) Explicitly the assumption $V \in \mathcal{D}(e^{\alpha\Gamma(\lambda)})$ means $\sum_{n=0}^{\infty} e^{2\alpha\lambda^n} \|V^{(n)}\|^2 < \infty$. It is mainly due to the weak regularity properties of $U_2(t,s)$, see Proposition 4.5.

(ii) Using hypercontractive estimates (see Lemma 3.13), the condition $V \in \mathcal{D}(e^{\alpha\Gamma(\lambda)})$ implies $\mathcal{R}(V) \in \bigcap_{p\geq 2} L^p(M,\mu)$ and hence $\mathcal{R}(\Gamma(\sqrt{\varepsilon})V) \in \bigcap_{p\geq 2} L^p(M,\mu)$.

(iii) We give an example fulfilling the assumptions. Let $V = \sum_{n=0}^{\infty} b_{2n}\varepsilon^{-n}\Phi_s(\varphi)^{2n}\Omega$ with $\varphi \in \mathcal{Z}$, $0 \leq b_{2n} \leq e^{-\beta\lambda^{2n}}$ for some $\lambda > 1$ and $\beta > \alpha > 0$. Then the Wick ordering implies that $V = \bigoplus_{n=0}^{\infty} a_{2n}\varphi^{\otimes 2n}$ such that $a_m \geq 0$ and $\sum_{m=0}^{\infty} a_m^2 e^{2\alpha\lambda^m}|\varphi|^{2m} < \infty$. Hence $V \in \mathcal{D}(e^{\alpha\Gamma(\lambda)})$ and assumption **(A2)** is satisfied since $\mathcal{R}(V) \geq 0$ and $\Gamma(\sqrt{\varepsilon})$ is a positivity preserving operator. Moreover $\mathcal{R}(V) \notin L^\infty(M,\mu)$ if $V \neq 0$.

(iv) The above theorem holds with the following explicit bound for $t > 0$

(with similar bound if $t < 0$)

$$c(t,\Psi) = C\|e^{\alpha\lambda^{N/\varepsilon}}V\| \int_0^t e^{4\|\varphi_s\|_{\mathcal{Z}}^2} \times \left[\left\|\sqrt{g_s\Big(\frac{N}{\varepsilon}\Big)}\Psi\right\|^2 + g_s'(0)\int_0^s \|V_2(r)\|dr\|\Psi\|^2\right]^{1/2} ds,$$

with $C > 0$ depending only on (α, λ) and $V_2(r) \in \otimes_s^2\mathcal{Z}$ is defined by (4.6). The functions g_t and g_t' are given by

$$g_t(r) = \sum_{k=0}^{\infty} e^{-\alpha_0\lambda^k} e^{2\sqrt{2}\lambda_0^k\int_0^t\|V_2(s)\|ds}(r+1)^k \quad \text{and} \quad g_t'(r) = \frac{d}{dr}g_t(r),$$

for arbitrary λ_0 and α_0 such that $1 < \lambda_0 < \lambda$ and $0 < \alpha_0\lambda^2 < \alpha$.

(v) Furthermore, $V_2(r) \in \otimes_s^2\mathcal{Z}$ given by (4.6) satisfies

$$\|V_2(r)\|_{\Gamma_s(\mathcal{Z})} \leq \left\|\Big(\frac{N}{\varepsilon}\Big)^4 V\right\|_{\Gamma_s(\mathcal{Z})} e^{4\|\varphi_r\|_{\mathcal{Z}}^2}.$$

Corollary 2.3 *Assume* **(A1)**–**(A2)** *and that* $V \in \mathcal{D}(e^{\alpha\Gamma(\lambda)})$ *for some* $\lambda > 1$ *and* $\alpha > 0$*. We have for any* $\xi \in \mathcal{Z}$ *and* $\varphi_0 \in \mathcal{Z}$ *the strong limit*

$$s - \lim_{\varepsilon\to 0} W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_0\Big)^* e^{i(t/\varepsilon)H}W(\xi)e^{-i(t/\varepsilon)H}W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_0\Big) = e^{i\sqrt{2}\mathrm{Re}\langle\xi,\varphi_t\rangle}\mathbb{1},$$

with φ_t *solving the classical field equation* (2.9) *with initial data* φ_0.

Proof. It is enough to prove the limit

$$\lim_{\varepsilon\to 0}\Big\langle e^{-i(t/\varepsilon)H}W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_0\Big)\Psi, W(\xi)e^{-i(t/\varepsilon)H}W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_0\Big)\Phi\Big\rangle = e^{i\sqrt{2}\mathrm{Re}(\xi,\varphi_t)}\langle\Psi,\Phi\rangle,$$

for any $\Psi, \Phi \in \mathcal{D}_f$. Now applying Theorem 2.1 for a fixed time t yields

$$\Big\langle e^{-i(t/\varepsilon)H}W\Big(\frac{\sqrt{2}}{i\varepsilon}\varphi_0\Big)\Psi, W(\xi)e^{-i(t/\varepsilon)H}W\Big(\frac{\sqrt{2}}{i\varepsilon}\varphi_0\Big)\Phi\Big\rangle$$
$$= \Big\langle W\Big(\frac{\sqrt{2}}{i\varepsilon}\varphi_t\Big)U_2(t,0)\Psi, W(\xi)W\Big(\frac{\sqrt{2}}{i\varepsilon}\varphi_t\Big)U_2(t,0)\Phi\Big\rangle + O(\sqrt{\varepsilon}).$$

But using the Weyl commutation relations (2.4) we obtain

$$\Big\langle W\Big(\frac{\sqrt{2}}{i\varepsilon}\varphi_t\Big)U_2(t,0)\Psi, W(\xi)W\Big(\frac{\sqrt{2}}{i\varepsilon}\varphi_t\Big)U_2(t,0)\Phi\Big\rangle$$
$$= \langle U_2(t,0)\Psi, W(\xi)U_2(t,0)\Phi\rangle e^{i\sqrt{2}\mathrm{Re}\langle\xi,\varphi_t\rangle}.$$

Thus, we obtain the claimed limit since $s-\lim_{\varepsilon\to 0} W(\xi) = \mathbb{1}$ and $U_2(t,0)$ is ε-independent unitary operator. □

Outline of the proof of Theorem 2.1. The proof of our main result relies on a Taylor expansion of the Hamiltonian H around the classical orbit φ_t satisfying the field equation (2.9). Formally the Hamiltonian H is a Wick quantization of the function

$$h(z) = \langle z, Az\rangle + F_V(z).$$

The symbol of the translated operator $h(z+\varphi_t)^{\mathrm{Wick}}$ of H in the phase-space can be expanded as a sum of three terms $h(\varphi_t)$, a field operator and a time-dependent quadratic Hamiltonian, plus higher order terms on creation-annihilation operators. The first and the second terms provide an approximation for the evolution of coherent states. More precisely to show Theorem 2.1, we formally differentiate the quantity

$$\mathcal{Y}(t) = e^{i(t/\varepsilon)H}e^{i(\omega(t)/\varepsilon)}W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_t\Big)U_2(t,0).$$

So, we obtain

$$-i\varepsilon\partial_t\mathcal{Y}(t) = e^{i(t/\varepsilon)H}e^{i(\omega(t)/\varepsilon)}W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_t\Big)$$
$$\times\big[h(z+\varphi_t)^{Wick} - A_0(t) - \sqrt{\varepsilon}A_1(t) - \varepsilon A_2(t)\big]U_2(t,0),$$

where we have used $W(-i(\sqrt{2}/\varepsilon)\varphi_t)^* H W(-i(\sqrt{2}/\varepsilon)\varphi_t) = h(z+\varphi_t)^{Wick}$ and $A_0(t)$, $A_1(t)$, $A_2(t)$ are ε-independent Wick monomials. It turns that $F_{R(t)}^{Wick} = h(z+\varphi_t)^{Wick} - A_0(t) - \sqrt{\varepsilon}A_1(t) - \varepsilon A_2(t)$ is a Wick operator of order $\varepsilon^{3/2}$. This leads to the formal estimate for $t > 0$

$$\left\|\mathcal{Y}(t)\Psi - W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_0\Big)\Psi\right\|_{\Gamma_s(\mathcal{Z})} \leq \varepsilon^{-1}\int_0^t \left\|F_{R(s)}^{Wick}U_2(s,0)\Psi\right\|_{\Gamma_s(\mathcal{Z})} ds. \tag{2.10}$$

Hence, we get the expected estimate. However, there are several domain problems that need to be handled carefully. In particular, the regularity with respect to powers of the number operator for the propagator $U_2(t,s)$ is crucial.

3. Wick quantization

We first recall the definition of Wick monomials on the Fock space. For further information we refer the reader to [3], [9]. Later on, we will use the wave representation in order to extend the Wick quantization to non-polynomial symbols.

3.1. Polynomial Wick operators

Definition 3.1 We say that a function $b : \mathcal{Z} \to \mathbb{C}$ is a continuous (p,q)-homogeneous polynomial in the class $\mathcal{P}_{p,q}(\mathcal{Z})$ if and only if there exists a hermitian form $\mathfrak{Q} : \otimes_s^q \mathcal{Z} \times \otimes_s^p \mathcal{Z} \to \mathbb{C}$ such that

$$\exists C > 0, \quad \left|\mathfrak{Q}(\zeta,\eta)\right| \leq C\|\zeta\|_{\otimes_s^q \mathcal{Z}} \cdot \|\eta\|_{\otimes_s^p \mathcal{Z}}, \quad \forall(\zeta,\eta) \in \otimes_s^q \mathcal{Z} \times \otimes_s^p \mathcal{Z} \tag{3.1}$$

$$\mathfrak{Q}(\lambda\zeta,\mu\eta) = \overline{\lambda}^q \mu^p \mathfrak{Q}(\zeta,\eta), \quad \forall(\zeta,\eta) \in \otimes_s^q \mathcal{Z} \times \otimes_s^p \mathcal{Z}, \quad \forall\lambda,\mu \in \mathbb{C} \tag{3.2}$$

and

$$b(z) = \mathfrak{Q}(z^{\otimes q}, z^{\otimes p}), \quad \forall z \in \mathcal{Z}. \tag{3.3}$$

The vector space spanned by all these polynomials will be denoted by $\mathcal{P}$.

We notice that the hermitian form $\mathfrak{Q}$ associated to b in the above definition is unique by a polarization identity. Consequently, for any $b \in \mathcal{P}_{p,q}(\mathcal{Z})$ there exists a unique bounded operator $\tilde{b} \in \mathcal{L}(\otimes_s^p \mathcal{Z}, \otimes_s^q \mathcal{Z})$ such that

$$b(z) = \langle z^{\otimes q}, \tilde{b} z^{\otimes p} \rangle, \quad \forall z \in \mathcal{Z}. \tag{3.4}$$

Next we recall the definition of Wick quantization for symbols in $\mathcal{P}_{p,q}(\mathcal{Z})$, $p, q \in \mathbb{N}$. The whole analysis depends on a small parameter ε which we can choose sufficiently small or at least in $(0, 1]$. Let $\mathcal{S}_n$ denote the orthogonal projection on the symmetric tensor product $\otimes_s^n \mathcal{Z}$ given by

$$\mathcal{S}_n(\zeta_1 \otimes \zeta_2 \cdots \otimes \zeta_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \zeta_{\sigma(1)} \otimes \zeta_{\sigma(2)} \otimes \cdots \otimes \zeta_{\sigma(n)}, \tag{3.5}$$

where $\mathfrak{S}_n$ is the symmetric group of n elements.

Definition 3.2 The *Wick monomial* of a *symbol* $b \in \mathcal{P}_{p,q}(\mathcal{Z})$ is the closure of the ε-dependent linear operator $b^{Wick} : \mathcal{D}_f \to \mathcal{D}_f \subset \Gamma_s(\mathcal{Z})$ defined by

$$b^{Wick}_{|\otimes_s^n \mathcal{Z}} = 1_{[p,+\infty)}(n) \frac{\sqrt{n!(n+q-p)!}}{(n-p)!} \varepsilon^{(p+q)/2} \mathcal{S}_{n-p+q}\big(\tilde{b} \otimes 1\!\!1^{\otimes(n-p)}\big), \tag{3.6}$$

where $\tilde{b} \in \mathcal{L}(\otimes_s^p \mathcal{Z}, \otimes_s^q \mathcal{Z})$ verifying (3.4).

Remark 3.3 1) For any $b \in \mathcal{P}_{p,q}(\mathcal{Z})$ the monomial $\bar{b}(z) := \overline{b(z)}$ belongs to $\mathcal{P}_{q,p}(\mathcal{Z})$ and the relation $\bar{b}^{Wick} \subset (b^{Wick})^*$ holds. Therefore (3.6) defines a closable operator on $\Gamma_s(\mathcal{Z})$ and in all the sequel b^{Wick} denotes a closed operator.

2) The ε-dependent annihilation-creation operators can be written as

$$a^*(f) = \langle z, f \rangle^{Wick}, \quad a(f) = \langle f, z \rangle^{Wick}.$$

3) The Wick operator $\langle z, z \rangle^{Wick}$ is the number operator and more generally $\mathrm{d}\Gamma(A) = \langle z, Az \rangle^{Wick}$.

The composition of two Wick polynomials with symbols in $\mathcal{P}_{p,q}(\mathcal{Z})$ is meaningful in the subspace $\mathcal{D}_f$. In fact, one can show that for any $b_i \in \mathcal{P}$, $(i = 1, 2)$, there exists a unique $c \in \mathcal{P}$ such that

$$b_1^{Wick} \, b_2^{Wick}{}_{|\mathcal{D}_f} = c^{Wick}{}_{|\mathcal{D}_f}. \tag{3.7}$$

The explicit formula of composition is presented in [3, Proposition 2.7].

The Wick quantization of the real canonical variables are the so-called

Segal field operators $\Phi_s(f) = \sqrt{2}(\mathrm{Re}\langle f, z\rangle)^{Wick}$, which are self-adjoint. Furthermore, we observe that for any $b \in \mathcal{P}$, the polynomial $z \mapsto b(e^{-itA}z)$ belongs to $\mathcal{P}$ with the following formula holds true

$$e^{i(t/\varepsilon)\mathrm{d}\Gamma(A)} b(\cdot)^{Wick} e^{-i(t/\varepsilon)\mathrm{d}\Gamma(A)} = \big(b(e^{-itA}\cdot)\big)^{Wick}.$$

We recall the standard number estimate (see *e.g.*, [3, Lemma 2.5]). Uniformly in $\varepsilon > 0$, the inequality

$$\big|\langle \Psi, b^{Wick}\Phi\rangle\big| \leq \|\tilde{b}\|_{\mathcal{L}(\otimes_s^p \mathcal{Z}, \otimes_s^q \mathcal{Z})} \|\langle N\rangle^{q/2}\Psi\| \times \|\langle N\rangle^{p/2}\Phi\|, \tag{3.8}$$

holds for any $b \in \mathcal{P}_{p,q}(\mathcal{Z})$.

We set

$$\mathcal{D}_c := \mathrm{vect}\{W(\varphi)\Omega; \varphi \in \mathcal{Z}_0\}. \tag{3.9}$$

Lemma 3.4 *The subspace $\mathcal{D}_c$ is dense in the symmetric Fock space $\Gamma_s(\mathcal{Z})$.*

Proof. Let $\Psi = \{\Psi^{(n)}\}_{n\geq 0}$ be a vector in $\Gamma_s(\mathcal{Z})$ orthogonal to the set $\mathcal{D}_c$. In particular, we have $\langle \Psi, W(\lambda\varphi)\Omega\rangle_{\Gamma_s(\mathcal{Z})} = 0$ for any $\lambda \in \mathbb{R}$ and $\varphi \in \mathcal{Z}_0$. An explicit computation yields

$$\langle \Psi, W(\lambda\varphi)\Omega\rangle_{\Gamma_s(\mathcal{Z})} = e^{-(\varepsilon/4)\lambda^2|\varphi|^2} \sum_{n=0}^{\infty} i^n \varepsilon^{n/2} \langle \Psi^{(n)}, \varphi^{\otimes n}\rangle \frac{\lambda^n}{\sqrt{2^n n!}},$$

and hence the function $\lambda \mapsto \langle \Psi, W(\lambda\varphi)\Omega\rangle_{\Gamma_s(\mathcal{Z})}$ is real-analytic. It follows that $\langle \Psi^{(n)}, \varphi^{\otimes n}\rangle = 0$ for all $n \in \mathbb{N}$ and $\varphi \in \mathcal{Z}_0$. Since the set $\{\varphi^{\otimes n}, \varphi \in \mathcal{Z}_0\}$ is total in $\otimes_s^n \mathcal{Z}$ for all $n \in \mathbb{N}$, we conclude that $\Psi = 0$. □

Lemma 3.5 *For any $b \in \mathcal{P}$ the subspace $\mathcal{D}_c$ is a core for b^{Wick}.*

Proof. It is enough to show this property only for Wick monomials and with $\varepsilon = 1$. Recall that the subspace $\mathcal{G}_0 := \mathrm{Vect}\{W(f)\Psi, \Psi \in \mathcal{D}_f, f \in \mathcal{Z}\}$ is a core for b^{Wick} (see [3, Proposition 2.10]) and it contains $\mathcal{D}_c$. The Wick identity

$$[\langle z+\bar{z}, \varphi\rangle^n]^{Wick} = \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{r!(n-2r)!} \left(\frac{|\varphi|}{\sqrt{2}}\right)^{2r} \left[\langle z+\bar{z}, \varphi\rangle^{Wick}\right]^{n-2r},$$

holds true for any $\varphi \in \mathscr{Z}_0$. In particular, we have

$$\begin{aligned}\varphi^{\otimes n} &= \left[\left\langle \frac{(z+\bar z)^{\otimes n}}{\sqrt{n!}}, \varphi\right\rangle\right]^{Wick}\Omega \\ &= \sum_{r=0}^{[n/2]} (-1)^r \frac{\sqrt{n!}}{r!(n-2r)!}\left(\frac{|\varphi|}{\sqrt{2}}\right)^{2r} \left[\langle z+\bar z, \varphi\rangle^{Wick}\right]^{n-2r}\Omega \\ &= \lim_{t\to 0}\sum_{r=0}^{[n/2]} (-1)^r \frac{\sqrt{n!}}{r!(n-2r)!}\left(\frac{|\varphi|}{\sqrt{2}}\right)^{2r} \left[\frac{W(\sqrt{2}t\varphi)-1}{it}\right]^{n-2r}\Omega, \end{aligned} \tag{3.10}$$

where in the last equality we have used the fact that Ω is C^∞-vector for the field operator $\Phi_s(\varphi)$. Hence we have at hand an explicit sequence $\varphi(t) \in \mathcal{D}_c$, for $t \in \mathbb{R} \setminus \{0\}$, given by (3.10) approximating each element of the total family $\{\varphi^{\otimes n},\ \varphi \in \mathscr{Z}_0, n \in \mathbb{N}\}$. Moreover, $\lim_{t\to 0} b^{Wick}\varphi(t) = b^{Wick}\varphi^{\otimes n}$ since $\varphi(t), \varphi^{\otimes n} \in \mathcal{D}(b^{Wick})$ and b^{Wick} is closed. Therefore it follows that the closure of the graph of $(b^{Wick})_{|\mathcal{D}_c}$ contains the graph of b^{Wick}. $\square$

3.2. Non-polynomial Wick operators

We set

$$\mathcal{K} := \left\{F : \mathscr{Z} \to \mathbb{C}; \exists\, V = \bigoplus_{n=0}^{\infty} V^{(n)} \in \Gamma_s(\mathscr{Z}); F(z) = \sum_{n=0}^{\infty}\left\langle \frac{(z+\bar z)^{\otimes n}}{\sqrt{n!}}, V^{(n)}\right\rangle\right\}. \tag{3.11}$$

The mapping

$$\begin{aligned}\Xi : \Gamma_s(\mathscr{Z}) &\longrightarrow \mathcal{K}\\ V = \bigoplus_{n=0}^{\infty} V^{(n)} &\longmapsto F_V(z) := \sum_{n=0}^{\infty}\left\langle \frac{(z+\bar z)^{\otimes n}}{\sqrt{n!}}, V^{(n)}\right\rangle,\end{aligned}$$

defines a Hilbert spaces isomorphism between $\Gamma_s(\mathscr{Z})$ and $\mathcal{K}$ when the latter is endowed with the scalar product

$$\langle F_{V_1}, F_{V_2}\rangle_{\mathcal{K}} := \langle V_1, V_2\rangle_{\Gamma_s(\mathscr{Z})}.$$

Moreover, we notice that $(\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$ is a reproducing kernel Hilbert space with the explicit kernel $K(z,w) := e^{\langle z+\bar z, w+\bar w\rangle}$ satisfying the pointwise relation

$$\langle K(w,\cdot), F_V(\cdot)\rangle_{\mathcal{K}} = F_V(w)$$

for all $F_V \in \mathcal{K}$ and $w \in \mathcal{Z}$.

Below we give the definition of Wick operators with symbols in the class $\mathcal{K}$.

Definition 3.6 The Wick operator with symbol F_V in $\mathcal{K}$ is the closure of the ε-dependent linear operator defined by

$$F_V^{Wick}\,(W(\varphi)\Omega) = W(\varphi)\Gamma(\sqrt{\varepsilon})V, \quad \forall \varphi \in \mathcal{Z}_0. \tag{3.12}$$

We observe that for any collection φ_i, $i = 1, \dots, n$ of distinct elements of $\mathcal{Z}_0$ if $\sum_{i=1}^n \lambda_i W(\varphi_i)\Omega = 0$ then $\lambda_i = 0$ for $i = 1, \dots, n$. This implies that F_V^{Wick} is a well-defined linear operator on $\mathcal{D}_c$.

Remark 3.7 1) Since $\mathcal{D}_c$ is dense in $\Gamma_s(\mathcal{Z})$ the operator F_V^{Wick}, $F_V \in \mathcal{K}$, is densely defined.

2) Notice that $(\overline{F_V})^{Wick}_{|\mathcal{D}_c} \subset (F_V^{Wick})^*$ then the operator given by (3.12) is closable.
3) The Wick quantization procedure of Definition (3.2) and (3.6) coincide for symbols in $\mathcal{P} \cap \mathcal{K}$.
4) The classes $\mathcal{K}$ and $\mathcal{P}$ are different. In fact $|z|^2$ belongs to $\mathcal{P}$ but not to $\mathcal{K}$ and $F_{W(\psi)\Omega} \in \mathcal{K}$ for $\psi \neq 0$ and not in $\mathcal{P}$.

The Wick quantization procedure given above have the following further property.

Lemma 3.8 *If $F_V \in \mathcal{K}$ then for all $\varphi \in \mathcal{Z}_0$*

$$W(\varphi)\mathcal{D}(F_V^{Wick}) = \mathcal{D}(F_V^{Wick}) \quad \textit{and} \quad W(\varphi)^* F_V^{Wick} W(\varphi) = F_V^{Wick}.$$

Proof. By Definition 3.6 and using the fact that $W(\varphi)\mathcal{D}_c \subset \mathcal{D}_c$, we verify that

$$F_V^{Wick} W(\varphi)_{|\mathcal{D}_c} = W(\varphi) F_V^{Wick}{}_{|\mathcal{D}_c}, \quad \text{for all } \varphi \in \mathcal{Z}_0. \tag{3.13}$$

Since $\mathcal{D}_c$ is a core for F_V^{Wick}, we see that $W(\varphi)\mathcal{D}(F_V^{Wick}) \subset \mathcal{D}(F_V^{Wick})$. Now the fact that $W(\varphi)$ is unitary with $W(\varphi)^* = W(-\varphi)$, $\varphi \in \mathcal{Z}_0$, yields the equality. Hence (3.13) extends to the domain of F_V^{Wick}. □

The Wick quantization of symbols in $\mathcal{K}$ gives multiplication operators in the wave representation. Therefore it is convenient to switch to such representation when it is advantageous. For reader's convenience, we briefly recall some facts about the wave representation (see [9], [30]).

Let $(M, \mathfrak{T}, \mu)$ be a probability space. A random variable $X : M \to \mathbb{R}$ with a finite variance $\sigma^2 \geq 0$ is called centered gaussian if and only if its characteristic function is

$$\int_M e^{-itX}\mu = e^{-(1/2)\sigma^2 t^2}, \quad t \in \mathbb{R}.$$

Let $\mathfrak{H}$ be a real Hilbert space. A *gaussian random process* indexed by $\mathfrak{H}$ is a map $\mathfrak{H} \ni f \mapsto \Phi(f)$ into centered gaussian random variables on M with variance $|f|^2$ satisfying for any $f_1, f_2, f \in \mathfrak{H}$ and $\lambda \in \mathbb{R}$,

$$\Phi(f_1) + \Phi(f_2) = \Phi(f_1 + f_2) \quad \text{and} \quad \lambda\Phi(f) = \Phi(\lambda f) \quad a.e.$$

The process is called *full* if $\mathfrak{T}$ is the smallest σ-algebra such that $\Phi(f), f \in \mathfrak{H}$, are measurable.

Let $\mathfrak{M}$ be the abelian Von Neumann algebra generated by the Weyl operators $W(f)$, $f \in \mathcal{Z}_0$. The following theorem gives the wave representation of the canonical commutation relations (see e.g. [30, Theorem I.1]).

Theorem 3.9 *There exist a probability measure space $(M, \mathfrak{T}, \mu)$ and a unitary map $\mathcal{R} : \Gamma_s(\mathcal{Z}) \to L^2(M, \mu)$ such that*

$$\text{(i) } \mathcal{R}\Omega = 1, \qquad \text{(ii) } \mathcal{R}\mathfrak{M}\mathcal{R}^* = L^\infty(M, \mu), \qquad \text{(iii) } \mathcal{R}\Gamma(\mathfrak{c})\psi = \overline{\mathcal{R}\psi}.$$

Moreover, the map

$$\mathcal{Z}_0 \ni f \mapsto \Phi(f) = \mathcal{R}\sqrt{\frac{2}{\varepsilon}}\Phi_s(f)\mathcal{R}^*,$$

is a gaussian full random process indexed by $\mathcal{Z}_0$.

This theorem allows to see the Wick operators with symbols in $\mathcal{K}$ as multiplication operators by unbounded measurable functions when represented in the space $L^2(M, \mu)$, see the following lemma.

Lemma 3.10 *For any $F_V \in \mathcal{K}$ there exists a measurable function $\mathcal{V} \in L^2(M,\mu)$ such that*

$$\mathcal{R}F_V^{Wick}\mathcal{R}^*\psi = \mathcal{V}\psi, \quad \forall\psi \in \mathcal{R}(\mathcal{D}_c) \subset L^2(M,\mu),$$

with $\mathcal{V}$ acting as a multiplication operator on $L^2(M,\mu)$.

Thanks to such identification we obtain the following results.

Lemma 3.11 *For any real-valued $F_V \in \mathcal{K}$, the corresponding Wick operator F_V^{Wick} is essentially self-adjoint on $\mathcal{D}_c$.*

Proof. This follows from the fact that $\mathcal{R}F_V^{Wick}\mathcal{R}^*$ is a densely defined multiplication operator by a μ-a.e. finite real-valued function on $L^2(M,\mu)$ (see [25, Section VIII.3]). □

In the following lemma, we prove that the set of Wick operators with symbols in $\mathcal{K}\cap\mathcal{P}$ is dense, with respect to the strong resolvent topology, in the set of Wick operators with $\mathcal{K}$ symbols.

Lemma 3.12 *Let F_V be a real-valued function in $\mathcal{K}$, $V = \bigoplus_{n=0}^{\infty} V^{(n)} \in \Gamma_s(\mathscr{Z})$. For κ integer we set $V_\kappa = \bigoplus_{n=0}^{\kappa} V^{(n)}$ and $F_{V_\kappa}(z) = \sum_{n=0}^{\kappa} \cdot\langle (z+\bar{z})^{\otimes n}/\sqrt{n!}, V^{(n)}\rangle$. Then the sequence of self-adjoint Wick polynomials $F_{V_\kappa}^{Wick}$ converges to F_V^{Wick} in the strong resolvent sense.*

Proof. By the above lemma we know that $F_{V_\kappa}^{Wick}$ and F_V^{Wick} are self-adjoint operators with a common core $\mathcal{D}_c$. Therefore, it is enough to prove that

$$\lim_{\kappa\to\infty} F_{V_\kappa}^{Wick}\Psi = F_V^{Wick}\Psi, \tag{3.14}$$

for any $\Psi \in \mathcal{D}_c$ in order to get the strong resolvent convergence (see [25, Theorem VIII.25]). Since $F_{V_\kappa} \in \mathcal{K}$ we can apply Lemma 3.8 and hence obtain

$$F_{V_\kappa}^{Wick} W(\varphi)\Omega = W(\varphi)F_{V_\kappa}^{Wick}\Omega = W(\varphi)\sum_{n=0}^{\kappa} \varepsilon^{n/2} V^{(n)}.$$

Taking $\kappa \to \infty$, we get $\lim_{\kappa\to\infty} F_{V_\kappa}^{Wick} W(\varphi)\Omega = W(\varphi)\Gamma(\sqrt{\varepsilon})V = F_V^{Wick} W(\varphi)\Omega$. □

3.3. Hypercontractive estimates

We recall the well-known hypercontractive inequality (see [30, Theorem I.17]).

Lemma 3.13 *Let $1 < p \leq q < \infty$ and $0 < \alpha \leq \sqrt{(p-1)/(q-1)}$. Then for any $\Psi \in \Gamma_s(\mathcal{Z})$,*

$$\|\mathcal{R}\Gamma(\alpha)\Psi\|_{L^q(M,\mu)} \leq \|\mathcal{R}\Psi\|_{L^p(M,\mu)}. \tag{3.15}$$

The following lemma provides an information on the domain of Wick operators with symbols in $\mathcal{K}$.

Lemma 3.14 *Let $V \in \Gamma_s(\mathcal{Z})$ and $\lambda \geq \sqrt{3}$. Then for all $\varepsilon \in (0, 1/3]$ and $\Psi \in \mathcal{D}(\Gamma(\lambda))$:*

$$\left\|F_V^{Wick}\Psi\right\|_{\Gamma_s(\mathcal{Z})} \leq \|V\|_{\Gamma_s(\mathcal{Z})}\|\Gamma(\lambda)\Psi\|_{\Gamma_s(\mathcal{Z})}.$$

Proof. Let $F_V \in \mathcal{K}$, $V \in \Gamma_s(\mathcal{Z})$. Using Hölder inequality, we get for any $\Psi \in \mathcal{D}(\Gamma(\lambda))$

$$\begin{aligned}\left\|F_V^{Wick}\Psi\right\|_{\Gamma_s(\mathcal{Z})} &= \left\|\mathcal{R}F_V^{Wick}\mathcal{R}^*\mathcal{R}\Psi\right\|_{L^2(M,\mu)}\\ &= \left\|(\mathcal{R}\Gamma(\sqrt{\varepsilon})V).(\mathcal{R}\Psi)\right\|_{L^2(M,\mu)}\\ &\leq \left\|\mathcal{R}\Gamma(\sqrt{\varepsilon})V\right\|_{L^4(M,\mu)}\|\mathcal{R}\Psi\|_{L^4(M,\mu)}.\end{aligned}$$

The hypercontractive bound of Lemma 3.13 with $p = 2$ and $q = 4$ yields

$$\begin{aligned}\left\|F_V^{Wick}\Psi\right\|_{\Gamma_s(\mathcal{Z})} &\leq \|\mathcal{R}V\|_{L^2(M,\mu)}\|\mathcal{R}\Gamma(\lambda)\Psi\|_{L^2(M,\mu)}\\ &\leq \|V\|_{\Gamma_s(\mathcal{Z})}\|\Gamma(\lambda)\Psi\|_{\Gamma_s(\mathcal{Z})}.\end{aligned}$$ □

Remark 3.15 (i) In the case $\varepsilon \in [1/3, 1]$, we can show the inequality

$$\left\|F_V^{Wick}\Psi\right\|_{\Gamma_s(\mathcal{Z})} \leq \left\|\Gamma(\sqrt{3})V\right\|_{\Gamma_s(\mathcal{Z})}\left\|\Gamma(\sqrt{3})\Psi\right\|_{\Gamma_s(\mathcal{Z})}.$$

(ii) A crude inequality can be easily proved using the bound $C_n^k \leq 2^n$ and without resorting to hypercontractivity. Indeed for $\alpha > 2$, we can show that

$$\left\|F_V^{Wick}\Psi\right\| \leq \frac{2}{\sqrt{1-4/\alpha^2}}\left\|\Gamma(\sqrt{2})V\right\|_{\Gamma_s(\mathcal{Z})}\|\Gamma(\alpha)\Psi\|_{\Gamma_s(\mathcal{Z})}.$$

Proposition 3.16 *Let $V=\bigoplus_{n=0}^{\infty}V^{(n)}\in\Gamma_s(\mathcal{Z})$ and set $V_\kappa=\bigoplus_{n=0}^{\kappa}V^{(n)}$. Then for $\varepsilon\in(0,1/3]$:*

- (i) *$\mathcal{D}_f$ is a core for F_V^{Wick}.*
- (ii) *For any $\Psi\in\mathcal{D}_f$ the sequence $(F_{V_\kappa}^{Wick}\Psi)_{\kappa\in\mathbb{N}}$ converges to $F_V^{Wick}\Psi$.*

Proof. (i) Since $\mathcal{D}_f\subset\mathcal{D}(\Gamma(\lambda))$ for any $\lambda>0$ we see that $\mathcal{D}_f\subset\mathcal{D}(F_V^{Wick})$ by Lemma 3.14. The explicit formula (2.5) shows that any coherent vector $W(\varphi)\Omega$, $\varphi\in\mathcal{Z}_0$, belongs to $\mathcal{D}(\Gamma(\lambda))$. Moreover the sequence

$$\Psi_\kappa=e^{-|\varphi|^2/4}\sum_{n=0}^{\kappa}\frac{i^n\varepsilon^{n/2}}{\sqrt{2^n n!}}\varphi^{\otimes^n}\in\mathcal{D}_f$$

converges to $W(\varphi)\Omega$, when $\kappa\to\infty$, with respect to the graph norm of $\Gamma(\lambda)$. Therefore, Lemma 3.14 proves that $\lim_\kappa F_V^{Wick}\Psi_\kappa=F_V^{Wick}W(\varphi)\Omega$. So that

$$F_{V}{}_{|\mathcal{D}_c}^{Wick}\subset\overline{F_{V}{}_{|\mathcal{D}_f}^{Wick}}\subset F_V^{Wick}\quad\text{and}\quad F_V^{Wick}=\overline{F_{V}{}_{|\mathcal{D}_c}^{Wick}}.$$

(ii) The inequality in Lemma 3.14 yields

$$\left\|(F_V^{Wick}-F_{V_\kappa}^{Wick})\Psi\right\|_{\Gamma_s(\mathcal{Z})}\leq\|V-V_\kappa\|_{\Gamma_s(\mathcal{Z})}\left\|\Gamma(\sqrt{3})\Psi\right\|_{\Gamma_s(\mathcal{Z})}. \tag{3.16}$$

□

A more specific inequality is needed.

Proposition 3.17 *Let $S(\lambda)=\sum_{k=0}^{\infty}a_k\lambda^k$ be an entire function on $\mathbb{C}$ such that $a_k>0$ for all $k\in\mathbb{N}$. For $\lambda_1>8e$ there exists $C>0$ such that the inequality*

$$\begin{aligned}\left\|F_V^{Wick}\Psi\right\|&\leq 2\left\|\Gamma(\sqrt{\varepsilon})V\right\|\|\Psi\|\\&\quad+C\left(\sum_{n=0}^{\infty}\frac{(\lambda_1\varepsilon)^n}{a_{n+2}}\|V^{(n)}\|^2_{\otimes_s^n\mathcal{Z}}\right)^{1/2}\left\|\sqrt{S\left(\frac{N}{\varepsilon}\right)}\Psi\right\|\end{aligned} \tag{3.17}$$

holds whenever the right hand side is finite.

Proof. For $\Psi,V\in\mathcal{D}_f$ we write the decomposition

$$F_V^{Wick}\Psi = F_{V^{(0)}}^{Wick}\Psi + F_{\bigoplus_{n\geq 1} V^{(n)}}^{Wick}\Psi^{(0)} + F_{\bigoplus_{n\geq 1} V^{(0)}}^{Wick}\left(\bigoplus_{n\geq 1}\Psi^{(n)}\right).$$

The first and second term are bounded by

$$\|V^{(0)}\|\|\Psi\| + \left\|\bigoplus_{n\geq 1}\varepsilon^{n/2}V^{(n)}\right\|\|\Psi^{(0)}\| \leq 2\big\|\Gamma(\sqrt{\varepsilon})V\big\|_{\Gamma_s(\mathcal{Z})}\|\Psi\|_{\Gamma_s(\mathcal{Z})}.$$

Now we can suppose that $V^{(0)} = 0$ and $\Psi^{(0)} = 0$ and write a Taylor expansion

$$F_V^{Wick}\Psi = \sum_{n\geq 1}\frac{\varepsilon^{n/2}}{\sqrt{n!}}\sum_{k=0}^{n} C_n^k \sum_{m\geq n-k}\frac{\sqrt{m!(m+2k-n)!}}{(m+k-n)!}\mathcal{S}_{m+2k-n}V_{n-k,k}^{(n)} \otimes 1\!\!1^{(m+k-n)}\Psi^{(m)},$$

with $V_{n-k,k}^{(n)} \in \mathcal{L}(\otimes_s^{n-k}\mathcal{Z}, \otimes_s^{k}\mathcal{Z})$. Using the bound $\sqrt{m!(m+2k-n)!}/(m+k-n)! \leq \sqrt{m}^{n-k}\sqrt{m+n}^{k}$, we get

$$\begin{aligned}\big\|F_V^{Wick}\Psi\big\|_{\Gamma_s(\mathcal{Z})} &\leq \sum_{n\geq 1}\frac{\varepsilon^{n/2}}{\sqrt{n!}}\sum_{k=0}^{n} C_n^k \sum_{m\geq n-k}\sqrt{m}^{n-k}\sqrt{m+n}^{k}\|V^{(n)}\|\|\Psi^{(m)}\| \\ &\leq \sum_{n\geq 1}\frac{\varepsilon^{n/2}}{\sqrt{n!}}\|V^{(n)}\|\sum_{m\geq 1}\big(\sqrt{m}+\sqrt{m+n}\big)^n\|\Psi^{(m)}\|.\end{aligned}$$

Cauchy-Schwarz inequality gives

$$\begin{aligned}&\sum_{m\geq 1}\big(\sqrt{m}+\sqrt{m+n}\big)^n\|\Psi^{(m)}\| \\ &\quad\leq \left(\sum_{m\geq 1}\frac{(\sqrt{m}+\sqrt{m+n})^{2n}}{S(m)}\right)^{1/2}\left(\sum_{m\geq 1}S(m)\|\Psi^{(m)}\|^2\right)^{1/2}.\end{aligned}$$

Since $S(m) \geq a_{n+2}m^{n+2}$ and $m+n \leq 2nm$ for n, m positive integers, we get the estimate

$$\frac{(\sqrt{m}+\sqrt{m+n})^{2n}}{S(m)} \leq \frac{2^{3n}n^n}{a_{n+2}\,m^2}.$$

Hence by Cauchy-Schwarz

$$\begin{aligned}
&\left\|F_V^{Wick}\Psi\right\|_{\Gamma_s(\mathcal{Z})} \\
&\quad\le \frac{\pi}{\sqrt{6}}\sum_{n\ge 1}\frac{\varepsilon^{n/2}}{\sqrt{n!}}\frac{\sqrt{2}^{3n}n^{n/2}}{\sqrt{a_{n+2}}}\|V^{(n)}\|\left\|S\Big(\frac{N}{\varepsilon}\Big)^{1/2}\Psi\right\| \\
&\quad\le \frac{\pi}{\sqrt{6}}\Big(\sum_{n=1}^{\infty}a_{n+2}^{-1}\lambda_1^n\varepsilon^n\|V^{(n)}\|^2_{\otimes_s^n\mathcal{Z}}\Big)^{1/2}\Big(\sum_{n\ge1}\frac{2^{3n}n^n}{\lambda_1^n n!}\Big)^{1/2}\left\|S\Big(\frac{N}{\varepsilon}\Big)^{1/2}\Psi\right\|.
\end{aligned}$$

Since $\lambda_1 > 8e$ the sum $\sum_{n\ge1}(2^{3n}n^n/\lambda_1^n n!)$ is convergent. By Proposition 3.16 the inequality extends to any $V \in \Gamma_s(\mathcal{Z})$ such that $\sum_{n=0}^{\infty} a_{n+2}^{-1}(\lambda_1\varepsilon)^n \cdot\|V^{(n)}\|^2_{\otimes_s^n\mathcal{Z}}$ is finite and $\Psi \in \mathcal{D}_f$ and then to any $\Psi \in \mathcal{D}(\sqrt{S(N/\varepsilon)})$. □

Later we will need to shift by translation a symbol $F_V \in \mathcal{K}$ (*i.e.*, $z \mapsto F_V(z+\varphi)$, $\varphi \in \mathcal{Z}$). Therefore it would be convenient if the translated symbol still in $\mathcal{K}$. Below, we provide a simple sufficient condition ensuring such stability.

Lemma 3.18 *Let F_V be in $\mathcal{K}$, $V = \bigoplus_{n=0}^{\infty} V^{(n)} \in \Gamma_s(\mathcal{Z})$ and assume that*

$$\left\|\Gamma(\sqrt{2})V\right\|_{\Gamma_s(\mathcal{Z})} = \sqrt{\sum_{n=0}^{\infty} 2^n\|V^{(n)}\|^2_{\otimes_s^n\mathcal{Z}}} < \infty. \tag{3.18}$$

Then for any $\varphi \in \mathcal{Z}$ the function $z \mapsto F_V(z+\varphi)$ belongs to $\mathcal{K}$. Moreover, for $\varepsilon \in (0,1/3]$ the following relation holds true

$$W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi\Big)^* F_V(\cdot)^{Wick}W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi\Big) = (F_V(\cdot+\varphi))^{Wick}. \tag{3.19}$$

Proof. Let F be in $\mathcal{K}$, $F = \Xi(V)$ and $V = \bigoplus_{n=0}^{\infty} V^{(n)} \in \Gamma_s(\mathcal{Z})$, such that the sequence $(V^{(n)})_{n\ge0}$ satisfies (3.18). For $\varphi \in \mathcal{Z}$, we have

$$\begin{aligned}
F(z+\varphi) &= \sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\big\langle(z+\bar z+\varphi+\bar\varphi)^{\otimes n}, V^{(n)}\big\rangle \\
&= \sum_{p=0}^{\infty}\Big\langle\frac{(z+\bar z)^{\otimes p}}{\sqrt{p!}}, \sum_{n=p}^{\infty}\sqrt{\frac{n!}{p!}}\frac{1}{(n-p)!}V_p^{(n)}\Big\rangle,
\end{aligned}$$

where $V_p^{(n)}$ are the vectors in $\otimes_s^p \mathcal{Z}$ given by

$$V_p^{(n)} := \mathcal{S}_p \big\langle (\varphi + \bar{\varphi})^{\otimes n-p} \big| \otimes 1\!\!1^{(p)} V^{(n)}.$$

In order to have $F(z + \varphi) \in \mathcal{K}$ it is enough to show that $V_\varphi = \bigoplus_{p=0}^{\infty} \cdot \sum_{n=p}^{\infty} \sqrt{n!/p!}(1/(n-p)!) V_p^{(n)}$ belongs to $\Gamma_s(\mathcal{Z})$, (i.e., $\sum_{p=0}^{\infty} \big\| \sum_{n=p}^{\infty} \sqrt{n!/p!} \cdot (1/(n-p)!) V_p^{(n)} \big\|_{\otimes_s^p \mathcal{Z}} < \infty$). Indeed, we have by Cauchy-Schwarz inequality

$$\begin{aligned}
&\sum_{p=0}^{\infty} \Big\| \sum_{n=p}^{\infty} \sqrt{\frac{n!}{p!}} \frac{1}{(n-p)!} V_p^{(n)} \Big\|^2 \\
&\quad \leq \sum_{p=0}^{\infty} \Big[\sum_{n=p}^{\infty} \sqrt{\frac{2^n}{(n-p)!}} (2\|\varphi\|)^{n-p} \|V^{(n)}\| \Big]^2 \\
&\quad \leq \sum_{p=0}^{\infty} 2^p \Big[\sum_{n=p}^{\infty} \frac{2^{n-p}}{(n-p)!} (2\|\varphi\|)^{2(n-p)} \Big] \sum_{n=p}^{\infty} \|V^{(n)}\|^2 \\
&\quad \leq e^{8\|\varphi\|^2} \sum_{n=0}^{\infty} \|V^{(n)}\|^2 \Big(\sum_{p=0}^{n} 2^p \Big) \\
&\quad \leq 2e^{8\|\varphi\|^2} \sum_{n=0}^{\infty} 2^n \|V^{(n)}\|^2.
\end{aligned}$$

Our next task is to show (3.19). Let $\varphi \in \mathcal{Z}$ and $V_\kappa = \bigoplus_{n=0}^{\kappa} V^{(n)}$ with $\kappa \in \mathbb{N}$. By [3, Proposition 2.10], we know that for any $\Psi \in \mathcal{D}_c$

$$W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi\Big)^* F_{V_\kappa}^{Wick} W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi\Big)\Psi = F_{V_\kappa}(\cdot + \varphi)^{Wick}\Psi.$$

Using the inequality (3.16), we see that the above identity extends to V instead of V_κ. Indeed, we can take the limit $\kappa \to \infty$ in the left and right hand side since $W(-i(\sqrt{2}/\varepsilon)\varphi)\Psi$ belongs to the domain of $\Gamma(\sqrt{3})$ without any assumption on φ. □

3.4. Models of quantum field theory

Our analysis is particulary motivated by two models of quantum field theory, namely the $P(\varphi)_2$ model and the Høegh-Krohn model. The first is an example of scalar boson quantum field theory in two-dimensional space-time with a self interaction given by an even positive polynomial on the neutral field with a spatial cutoff. While the second model is less physical but it has some interest. In particular it provides an example of a non polynomial interaction.

Before presenting these two models we recall a cornerstone result in this subject which provides the essential self-adjointness of the sum (2.8) under some assumptions. The final statement is the theorem below due to I. Segal (see [28, Theorem 2]). It sums up several remarkable contributions by E. Nelson, A. Jaffe, J. Glimm, L. Rosen and many others (see e.g. [14], [19], [27], [28], [30]). It is also one of the beautiful results of mathematical physics which have had an impact on other fields (see [5]).

Theorem 3.19 *Let A be a self-adjoint operator on $\mathcal{Z}$ satisfying* **(A1)** *and $F_V \in \mathcal{K}$ verifying the assumption* **(A2)**. *Then the operator*

$$H = \mathrm{d}\Gamma(A) + F_V^{Wick}, \tag{3.20}$$

defined on $\mathcal{D}(\mathrm{d}\Gamma(A)) \cap \mathcal{D}(F_V^{Wick})$ is essentially self-adjoint.

$\mathbf{P(\varphi)_2}$ model: Consider the following one variable real polynomial

$$P(x) = \sum_{j=0}^{2n} \alpha_j x^j, \quad (\alpha_{2n} > 0).$$

Let $\varphi(x)$ be the neutral scalar-field of mass $m_0 > 0$, *i.e.*:

$$\varphi(x) := \int_{\mathbb{R}} e^{-ikx}[a^*(k) + a(-k)] \frac{dk}{\sqrt{\omega(k)}}, \quad \text{where } \omega(k) = \sqrt{m_0^2 + k^2},\ m_0 > 0.$$

Let g a nonnegative function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $g(x) = g(-x)$. We define G as the following real-valued polynomial

$$G(z) := \sum_{j=0}^{2n} \alpha_j \int_{\mathbb{R}} \left[\left\langle z, \frac{e^{-ikx}}{\sqrt{\omega(k)}} \right\rangle + \left\langle \frac{e^{-ikx}}{\sqrt{\omega(k)}}, z \right\rangle \right]^j g(x) dx,$$

$$\text{for } \ z \in L^2\left(\mathbb{R}, \frac{dk}{\sqrt{\omega(k)}}\right). \tag{3.21}$$

Lemma 3.20 *The polynomial G given by* (3.21) *has a continuous extension over $\mathcal{Z}$ belonging to the class $\mathcal{K}$.*

Proof. Let $\mathfrak{c}(z) = \overline{z(-k)}$ be a conjugation on $L^2(\mathbb{R})$. For $z \in L^2(\mathbb{R})$ with compact support we can write

$$G(z) = \sum_{j=0}^{2n} \alpha_j \left\langle (z + \mathfrak{c}(z))^{\otimes j}; \int_{\mathbb{R}} \left(\frac{e^{-ikx}\chi(k)}{\sqrt{\omega(k)}} \right)^{\otimes j} g(x)dx \right\rangle,$$

where χ is a smooth cutoff function verifying $\chi(k)z(k) = z(k)$. One can prove that if $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\int_{\mathbb{R}} (e^{-ikx}/\sqrt{\omega(k)})^{\otimes j} g(x)dx$ is a symmetric function belonging to $L^2(\mathbb{R}^j)$ (see [9, Lemma 6.1]). This shows that

$$\begin{aligned} G(z) &= \sum_{j=0}^{2n} \alpha_j \left\langle (z + \mathfrak{c}(z))^{\otimes j}; \chi(k)^{\otimes j} \int_{\mathbb{R}} \left(\frac{e^{-ikx}}{\sqrt{\omega(k)}} \right)^{\otimes j} g(x)dx \right\rangle \\ &= \sum_{j=0}^{2n} \alpha_j \left\langle (z + \mathfrak{c}(z))^{\otimes j}; \int_{\mathbb{R}} \left(\frac{e^{-ikx}}{\sqrt{\omega(k)}} \right)^{\otimes j} g(x)dx \right\rangle. \end{aligned}$$ □

The spatially cutoff Hamiltonian of self-interacting Bose fields in two dimensional space time is given by

$$H = \mathrm{d}\Gamma(\omega) + G(z)^{Wick}$$

on $\mathcal{D}(\mathrm{d}\Gamma(\omega)) \cap \mathcal{D}(G(z)^{Wick})$. Therefore, applying Theorem 3.19 we see that H is essential self-adjointness on $\mathcal{D}(G(z)^{Wick}) \cap \mathcal{D}(\mathrm{d}\Gamma(\omega))$. For further details on how to check the assumptions of Theorem 3.19 we refer the reader to [9], [14], [19], [28]. Thanks to Lemma 3.20 the symbol $G(z)$ has the form

$$G(z) = \sum_{j=0}^{2n} \left\langle \frac{(z + \overline{z})^{\otimes j}}{\sqrt{j!}}; V^{(j)} \right\rangle \quad \text{with}$$

$$V^{(j)} = \sqrt{j!}\alpha_j \int_{\mathbb{R}} \left(\frac{e^{-ikx}}{\sqrt{\omega(k)}} \right)^{\otimes j} g(x)dx \in \otimes_s^j L^2(\mathbb{R})$$

More general, we could consider instead of $G(z)$ a Wick symbol of the form

$$F_V(z) = \sum_{n=0}^{\infty} \left\langle \frac{(z+\bar{z})^{\otimes n}}{\sqrt{n!}}, V^{(n)} \right\rangle.$$

with $V = \bigoplus_{n=0}^{\infty} V^{(n)}$ in the Fock space.

Høegh-Krohn model: This model is due to Høegh-Krohn (see [17], [18]). Let $\varphi(x)$ be the neutral scalar-field on $\mathbb{R}^d$ of mass $m_0 > 0$, *i.e.*,

$$\varphi(x) := \int_{\mathbb{R}^d} e^{-ipx}[a^*(p) + a(-p)] \frac{dp}{\sqrt{\omega(p)}},$$
$$\text{where} \quad \omega(p) = \sqrt{m_0^2 + p^2}, \quad m_0 > 0.$$

Let g be in $C_0^\infty(\mathbb{R}^d)$ such that $g \geq 0$, $g(x) = g(-x)$, $\int g(x)dx = 1$ with support in the open ball of radius 1 centered at the origin. The cut-off field operator is given by

$$\varphi_\kappa(x) = \int_{\mathbb{R}^d} g_\kappa(x-y)\varphi(y)dy, \quad \text{with} \quad g_\kappa(x) = \kappa^{-d} g(\kappa^{-1}x).$$

For every $x \in \mathbb{R}^d$ the operator $\varphi_\kappa(x)$ is self-adjoint. Let V be a bounded continuous real function. We define the Høegh-Krohn Hamiltonian as

$$H = \mathrm{d}\Gamma(\omega) + \int_{|x|\leq r} V(\varphi_\kappa(x))dx. \tag{3.22}$$

It is clearly a self-adjoint operator since the interaction is bounded. Instead of taking V a bounded function we may consider V a real entire function $V(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$. This formally leads to the interaction

$$\sum_{n=0}^{\infty} a_n \int_{|x|\leq r} (\varphi_\kappa(x))^n dx.$$

In order to avoid possible infinities we replace $(\varphi_\kappa(x))^n$ by its normal ordering. This makes indeed the interaction well defined and so it is given by

$$\left(\sum_{n=0}^{\infty} \left\langle (z + \mathfrak{c}(z))^{\otimes n}; a_n \int_{|x|\leq r} \left(e^{ipx} \frac{\hat{g}_\kappa(p)}{\sqrt{\omega(p)}} \right)^{\otimes n} dx \right\rangle \right)^{Wick},$$

whenever $\sum_{n=0}^{\infty} n!a_n^2\|\hat{g}_\kappa/\sqrt{\omega}\|^2_{L^2(\mathbb{R}^d)} < \infty$. So that the modified Hamiltonian has the form

$$H = \mathrm{d}\Gamma(\omega) + F_W^{Wick},$$

with

$$W = \sum_{n=0}^{\infty} \sqrt{n!}a_n \int_{|x|\leq r} \left(e^{ipx}\frac{\hat{g}_\kappa(p)}{\sqrt{\omega(p)}}\right)^{\otimes n} dx \in \Gamma_s(L^2(\mathbb{R}^d)).$$

The raison why we did not stick to the original model is that we are more interested in "analytic" perturbations on the field operators rather than bounded interactions. Moreover, the strategy will be different from the one employed here if the latter is considered.

4. Propagation of coherent states

4.1. Classical field equation

The classical limit relates models of quantum field theory to classical field equations. For instance the $P(\varphi)_2$ dynamics, in the limit $\varepsilon \to 0$, leads to a nonlinear Klein-Gordon equation. In this subsection we establish global existence and uniqueness of classical dynamics as primary information for the study of propagation of coherent states. Although this relays on standard arguments we provide, for reader convenience, a short proof.

The classical energy functional h associated formally to the quantum Hamiltonian H defined in (3.20) is given by

$$h(z) := \langle z, Az\rangle + F_V(z), \tag{4.1}$$

for $z \in \mathcal{D}(A)$ and $F_V \in \mathcal{K}$. So that we have at hand the nonlinear evolution equation

$$i\partial_t\varphi = A\varphi + \partial_{\bar{z}}F_V(\varphi),$$

with initial data $\varphi_{|t=0} = \varphi_0 \in \mathcal{D}(A)$. In fact we only need to construct mild solutions for (2.9). So we rather focus on the integral equation associated to (2.9), namely

$$\varphi_t = e^{-itA}\varphi_0 - i\int_0^t e^{-i(t-s)A}\partial_{\bar{z}}F_V(\varphi_s)ds. \tag{4.2}$$

A fixed point argument shows the local existence of a unique continuous solution in $C^0(\mathbb{R},\mathcal{Z})$. Then a nonlinear Gronwall inequality allows to prove global existence. We can also apply [24, Theorem 1] or [25, Theorem X.72] in order to show local existence.

Theorem 4.1 *Let A be a self-adjoint operator on $\mathcal{Z}$ and $F_V \in \mathcal{K}$. Then for any $\varphi_0 \in \mathcal{Z}$ the integral equation (4.2) admits a unique solution φ_t in $C^0(\mathbb{R},\mathcal{Z})$. Moreover, the mapping $t \mapsto \tilde{\varphi}_t := e^{itA}\varphi_t \in \mathcal{Z}$ is norm differentiable and satisfies*

$$i\partial_t\tilde{\varphi}_t = \partial_{\bar{z}}F_V(\tilde{\varphi}_t).$$

Proof. The nonlinearity $\partial_{\bar{z}}F_V$ satisfies the explicit estimate

$$\|\partial_{\bar{z}}F_V(\varphi) - \partial_{\bar{z}}F_V(\psi)\| \leq 2\|V\|_{\Gamma_s(\mathcal{Z})}g(\max(\|\varphi\|,\|\psi\|))\|\varphi-\psi\|,$$

where $g(t) = \sqrt{1+\sum_{n=2}^{\infty}(4^{n-2}n(n-1)/(n-2)!)t^{2(n-2)}}$ is an increasing positive function.

For $T>0$, we consider on $C^0([0,T),\mathcal{Z})$ the mapping

$$\mathcal{T}(\varphi)(t) = e^{-itA}\varphi_0 - i\int_0^t e^{-i(t-s)A}\partial_{\bar{z}}F_V(\varphi_s)ds.$$

For any φ and ψ in the closed ball $\mathcal{B}$ of radius $\alpha>0$ and centered at $e^{-itA}\varphi_0$, a direct computation yields

$$\sup_{t\in[0,T)}\|\mathcal{T}(\varphi)(t)-\mathcal{T}(\psi)(t)\| \leq 2\|V\|_{\Gamma_s(\mathcal{Z})}T\,g(\|\varphi_0\|+\alpha)\sup_{t\in[0,T)}\|\varphi(t)-\psi(t)\|.$$

Taking $T < \alpha\, g(\|\varphi_0\|+\alpha)^{-1}/2\|V\|(1+\alpha+\|\varphi_0\|)$ makes $\mathcal{T}$ a contraction on the closed ball $\mathcal{B}$ and hence it admits a unique fixed point. This proves existence and uniqueness of local solutions for (4.2). A similar estimate yields

$$\begin{aligned}\|\varphi_t\| &\leq \|\varphi_0\| + \int_0^t \|\partial_{\bar{z}}F_V(\varphi_s)\|ds \\ &\leq \|\varphi_0\| + \int_0^t 2\|V\|_{\Gamma_s(\mathcal{Z})}g(\|\varphi_s\|)ds.\end{aligned}$$

So applying a nonlinear Gronwall lemma, known as Bihari's inequality [6], we conclude that $\|\varphi_t\|$ is bounded on any finite interval. Therefore the integral equation (4.2) admits a unique global solution φ_t in $C^0(\mathbb{R}, \mathcal{Z})$. $\square$

4.2. Time-dependent quadratic dynamics

We consider in this subsection the dynamics of time-dependent quadratic Hamiltonians. This will be a steep towards the study of the semi-classical approximation of coherent states propagation.

Let Q_t be a real-valued time-dependent quadratic polynomial given by

$$Q_t(z) = \frac{1}{\sqrt{2}} \big\langle (e^{-itA} z + e^{itA} \bar{z})^{\otimes 2}, w_t \big\rangle, \quad t \in \mathbb{R}, \tag{4.3}$$

such that the map $t \mapsto w_t \in \otimes_s^2 \mathcal{Z}$ is norm continuous. Notice that Q_t is no more in $\mathcal{K}$ since the factor e^{itA} has distorted the symmetry of the symbol w_t. However with an appropriate choice of the conjugation $\mathfrak{c}_t z := e^{2itA} \bar{z}$ the symbol $Q_t(z)$ belongs to $\mathcal{K}_{\mathfrak{c}_t}$ with respect to $\mathfrak{c}_t$. As a consequence we have the self-adjointness of the operators Q_t^{Wick} by Lemma 3.11 since

$$\Gamma(\mathfrak{c}_t)(e^{itA} \otimes e^{itA} w_t) = (e^{itA} \otimes e^{itA}) w_t.$$

Next we will use the Hilbert spaces,

$$\mathcal{D}_{+,k} := \mathcal{D}(N^{k/2}), \quad k \geq 1,$$

which are ε-independent vector spaces equipped with the inner product

$$\langle \Psi_1, \Psi_2 \rangle_{\mathcal{D}_{+,k}} := \sum_{n=0}^{\infty} (n^k + 1) \big\langle \Psi_1^{(n)}, \Psi_2^{(n)} \big\rangle_{\otimes_s^n \mathcal{Z}}.$$

We define the Hilbert space $\mathcal{D}_{-,k}$ as the completion of $\Gamma_s(\mathcal{Z})$ with respect to the inner product

$$\langle \Psi_1, \Psi_2 \rangle_{\mathcal{D}_{-,k}} := \sum_{n=0}^{\infty} (n^k + 1)^{-1} \big\langle \Psi_1^{(n)}, \Psi_2^{(n)} \big\rangle_{\otimes_s^n \mathcal{Z}}.$$

Thus, we have a Hilbert rigging

$$\mathcal{D}_{+,k} \subset \Gamma_s(\mathcal{Z}) \subset \mathcal{D}_{-,k}.$$

Lemma 4.2 *Let Q_t be the quadratic polynomial given by* (4.3) *such that $t \mapsto w_t \in \otimes_s^2 \mathcal{Z}$ is norm continuous. Then the mapping*

$$\mathbb{R} \ni t \mapsto Q_t^{Wick} \in \mathcal{L}(\mathcal{D}_{+,k}, \mathcal{D}_{-,k})$$

is strongly continuous.

Proof. By the number estimate (3.8), it follows that Q_t^{Wick} is a bounded operator in $\mathcal{L}(\mathcal{D}_{+,k}, \mathcal{D}_{-,k})$ for each $t \in \mathbb{R}$ and k positive integer. More explicitly, we have

$$\begin{aligned}
&\left\| \left\langle \frac{(e^{-itA}z + e^{itA}\bar{z})^{\otimes 2}}{\sqrt{2}}, w_t \right\rangle^{Wick} \Psi \right\|_{\mathcal{D}_{-,k}}^2 \\
&\quad \leq \frac{3}{2} \Big[\big\| \langle (e^{-itA}z)^{\otimes 2}, w_t \rangle^{Wick} \Psi \big\|_{\mathcal{D}_{-,k}}^2 + \big\| \langle (e^{itA}\bar{z})^{\otimes 2}, w_t \rangle^{Wick} \Psi \big\|_{\mathcal{D}_{-,k}}^2 \\
&\qquad + 4 \big\| \langle e^{-itA}z \otimes e^{itA}\bar{z}, w_t \rangle^{Wick} \Psi \big\|_{\mathcal{D}_{-,k}}^2 \Big].
\end{aligned}$$

We estimate each of the three terms in the r.h.s in the same way as for the following one

$$\begin{aligned}
&\big\| \langle z^{\otimes 2}, e^{itA} \otimes e^{itA} w_t \rangle^{Wick} \Psi \big\|_{\mathcal{D}_{-,k}}^2 \\
&\quad \leq \varepsilon^2 \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{((n+2)^k + 1)} \big\| \mathcal{S}_{n+2}(e^{itA} \otimes e^{itA}\, w_t) \otimes \Psi^{(n)} \big\|_{\Gamma_s(\mathcal{Z})}^2 \\
&\quad \leq \varepsilon^2 \|w_t\|_{\otimes_s^2 \mathcal{Z}}^2 \|\Psi\|_{\mathcal{D}_{+,k}}^2 .
\end{aligned}$$

Putting $\tilde{w}_t = e^{itA} \otimes e^{itA} w_t$, we obtain

$$\big\| \langle z^{\otimes 2}, \tilde{w}_t - \tilde{w}_s \rangle^{Wick} \Psi \big\|_{\mathcal{D}_{-,k}} \leq \varepsilon \|w_t - w_s\|_{\otimes_s^2 \mathcal{Z}} \|\Psi\|_{\mathcal{D}_{+,k}} .$$

□

Lemma 4.3 *Let Q_t be the quadratic polynomial given by* (4.3) *such that $t \mapsto w_t \in \otimes_s^2 \mathcal{Z}$ is norm continuous. Then for any $k \geq 1$ there exist $C_k > 0$ such that for any $\Psi, \Phi \in \mathcal{D}_f$*

$$\left| \left\langle \left(\frac{N}{\varepsilon}\right)^k \Psi, Q_t^{Wick} \Phi \right\rangle - \left\langle Q_t^{Wick} \Psi, \left(\frac{N}{\varepsilon}\right)^k \Phi \right\rangle \right| \leq C_k \varepsilon \|w_t\|_{\otimes_s^2 \mathcal{Z}} \|\Psi\|_{\mathcal{D}_{+,k}} \|\Phi\|_{\mathcal{D}_{+,k}} .$$

Proof. We give the proof only for $k = 1$, the case $k > 1$ is similar. Actually, we will consider more carefully such an estimate in the proof of Proposition 4.5, where we need to explicit the dependence in k of the bound C_k. A simple computation yields

$$\begin{aligned}
&\left\langle N\Psi, Q_t^{Wick}\Phi\right\rangle - \left\langle Q_t^{Wick}\Psi, N\Phi\right\rangle \\
&= \varepsilon^2\Bigg[\sum_{n=0}^{\infty}\sqrt{2(n+1)(n+2)}\left\langle \Psi^{(n+2)}, (e^{itA}\otimes e^{itA}w_t)\otimes\Phi^{(n)}\right\rangle \\
&\qquad - \sum_{n=0}^{\infty}\sqrt{2(n+1)(n+2)}\left\langle (e^{itA}\otimes e^{itA}w_t)\otimes\Psi^{(n)}, \Phi^{(n+2)}\right\rangle\Bigg].
\end{aligned}$$

By Cauchy-Schwarz inequality we get

$$\begin{aligned}
&\left|\langle N\Psi, Q_t^{Wick}\Phi\rangle - \langle Q_t^{Wick}\Psi, N\Phi\rangle\right| \\
&\leq 2\sqrt{2}\varepsilon^2\|w_t\|\left[\sum_{n=0}^{\infty}(n+1)\|\Phi^{(n)}\|^2\right]^{1/2}\left[\sum_{n=0}^{\infty}(n+1)\|\Psi^{(n)}\|^2\right]^{1/2}. \qquad \square
\end{aligned}$$

There exist several results on non-autonomous abstract linear Schrödinger equations (see *e.g.* [21], [22], [29] and also [13]). We will use a result in [2, Corollary C.4] which is quite adapted to quadratic Hamiltonians of quantum field theory.

We say that the map $\mathbb{R}\times\mathbb{R}\ni(t,s)\mapsto U(t,s)$ is a unitary propagator of the non-autonomous Schrödinger equation

$$\begin{cases} i\varepsilon\partial_t u = Q_t^{Wick}u, & t\in\mathbb{R} \\ u(t=0) = u_0\in\mathcal{D}_{+,1}, \end{cases} \tag{4.4}$$

if and only if

(a) $U(t,s)$ is unitary on $\Gamma_s(\mathcal{Z})$,
(b) $U(t,t) = 1$ and $U(t,s)U(s,r) = U(t,r)$ for all $t,s,r\in\mathbb{R}$,
(c) The map $t\in\mathbb{R}\mapsto U(t,s)$ belongs to $C^0(\mathbb{R},\mathcal{L}(\mathcal{D}_{+,1}))\cap C^1(\mathbb{R},\mathcal{L}(\mathcal{D}_{+,1},\mathcal{D}_{-,1}))$, and satisfies

$$i\varepsilon\partial_t U(t,s)\psi = Q_t^{Wick}U(t,s)\psi, \quad \forall\psi\in\mathcal{D}_{+,1},\ \forall t,s\in\mathbb{R}.$$

Here $C^k(I,\mathfrak{B})$ denotes the space of k-continuously differentiable $\mathfrak{B}$-valued functions where $\mathfrak{B}$ is endowed with the strong operator topology.

Theorem 4.4 *Let Q_t be the quadratic polynomial given by* (4.3) *such that*:

- $t\in\mathbb{R}\mapsto w_t\in\otimes_s^2\mathcal{Z}$ *is norm continuous,*
- $\Gamma(\mathfrak{c})w_t=w_t$ *for any* $t\in\mathbb{R}$.

Then the non-autonomous Cauchy problem (4.4) *admits a unique unitary propagator $U(t,s)$. Moreover for every $k\geq 1$ there exists $C_k>0$ such that for all $s,t\in\mathbb{R}$*

$$\|U(t,s)\|_{\mathcal{L}(\mathcal{D}_{+,k})}\leq e^{C_k|\int_s^t\|w_\tau\|\mathrm{d}\tau|}. \tag{4.5}$$

Proof. It follows by direct application of [2, Corollary C.4] and using Lemma 4.2–4.3. □

The regularity property (4.5) of the propagator $U(t,s)$ contains the bound C_k which we need to explicit its dependence in k. Actually, this can be done using [2, Corollary C.4]. However, we prefer to give such an inequality with a direct proof.

Proposition 4.5 *Assume the same hypothesis as in Theorem* 4.4. *Then for any $\lambda>1$ there exists $c>0$ such that for any integer k*

$$\left\|\left(\frac{N}{\varepsilon}+1\right)^{k/2}U(t,0)\Psi\right\|_{\Gamma_s(\mathcal{Z})}$$
$$\leq e^{\sqrt{2}k\lambda^k|\int_0^t\|w_s\|ds|}\left[ck\left|\int_0^t\|w_s\|ds\right|\|\Psi\|^2+\left\|\left(\frac{N}{\varepsilon}+1\right)^{k/2}\Psi\right\|^2_{\Gamma_s(\mathcal{Z})}\right]^{1/2}.$$

Proof. By Theorem 4.4 we know that $U(t,s)$ preserves the domains $\mathcal{D}(N^{k/2})$ for any $k\geq 1$. Differentiating the function $u(t)=\|(N/\varepsilon+1)^{k/2}U(t,0)\Psi\|^2_{\Gamma_s(\mathcal{Z})}$ for $\Psi\in\mathcal{D}_f$, we get

$$u'(t)=\left\langle U(t,0)\Psi,\frac{i}{\varepsilon}\left[Q_t^{Wick},\left(\frac{N}{\varepsilon}+1\right)^k\right]U(t,0)\Psi\right\rangle.$$

We decompose $Q_t^{Wick}=B_1^{Wick}+C^{Wick}+B_2^{Wick}$ with $B_1(z)=(1/\sqrt{2})\cdot\langle e^{-itA}z\otimes e^{-itA}z,w_t\rangle$, $C(z)=\sqrt{2}\langle e^{-itA}z\otimes e^{itA}\bar{z},w_t\rangle$ and $B_2(z)=\overline{B_1(z)}$.

The polynomial Wick calculus yields

$$\left[Q_t^{Wick}, \left(\frac{N}{\varepsilon}+1\right)^k\right] = \left(\left(\frac{N}{\varepsilon}-1\right)^k - \left(\frac{N}{\varepsilon}+1\right)^k\right) B_1^{Wick} + B_2^{Wick}\left(\left(\frac{N}{\varepsilon}-1\right)^k - \left(\frac{N}{\varepsilon}+1\right)^k\right).$$

Let $\Phi = U(t,0)\Psi$ then by explicit computations and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\pm \left\langle \Phi, \frac{i}{\varepsilon}\left[Q_t^{Wick}, (N+1)^k\right]\Phi \right\rangle \\
&\leq 4\left|\sum_{n=2}^{\infty} \left\langle \Phi^{(n)}, [(n+1)^{k-1} + \cdots + (n-1)^{k-1}](B_1^{Wick}\Phi)^{(n)} \right\rangle\right| \\
&\leq 4\frac{\|w_t\|}{\sqrt{2}} \sum_{n=2}^{\infty} k(n+1)^k \|\Phi^{(n)}\| \|\Phi^{(n-2)}\| \\
&\leq 4k\frac{\|w_t\|}{\sqrt{2}} \sqrt{\sum_{n=2}^{\infty}(n+1)^k\|\Phi^{(n)}\|^2}\sqrt{\sum_{n=0}^{\infty}(n+3)^k\|\Phi^{(n-2)}\|^2}.
\end{aligned}$$

For $\lambda > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=0}^{\infty}(n+3)^k\|\Phi^{(n-2)}\|^2 \leq (n_0+3)\|\Psi\|^2 + \lambda^k \sum_{n=n_0+1}^{\infty}(n+1)^k\|\Phi^{(n-2)}\|^2.$$

So that we obtain

$$\pm u'(t) \leq 4k\lambda^k \frac{\|w_t\|}{\sqrt{2}} u(t) + 4k\frac{\|w_t\|}{\sqrt{2}}(n_0+3)\|\Psi\|^2.$$

The Gronwall lemma ends up the proof. □

We will use the previous result with a specific choice of $w_t \in \otimes_s^2 \mathcal{Z}$ related to the problem at hand. Consider a symbol $F_V \in \mathcal{K}$ and let φ_t be a solution of the field equation (2.9) with an initial data $\varphi_0 \in \mathcal{Z}$. We define a real-valued polynomial symbol $F_{V_2(t)} \in \mathcal{K}$ by

$$F_{V_2(t)}[z] := \sum_{n=2}^{\infty} \frac{n(n-1)}{\sqrt{n!}} \langle (\varphi_t + \bar{\varphi}_t)^{\otimes(n-2)} \otimes (z+\bar{z})^{\otimes 2}, V^{(n)} \rangle$$
$$= \left\langle \frac{(z+\bar{z})^{\otimes 2}}{\sqrt{2}}, V_2(t) \right\rangle \in \mathcal{P} \cap \mathcal{K}.$$

We check by direct computation that

$$V_2(t) = \sqrt{2} \sum_{n=2}^{\infty} \frac{n(n-1)}{\sqrt{n!}} \mathcal{S}_2 \langle (\varphi_t + \bar{\varphi}_t)^{\otimes(n-2)} | \otimes 1^{(2)} V^{(n)} \in \otimes_s^2 \mathcal{Z}. \qquad (4.6)$$

Corollary 4.6 *Let $\varphi_0 \in \mathcal{Z}$, $F_V \in \mathcal{K}$ and $V_2(t)$ given by* (4.6)*. Consider the family of polynomials*

$$F^{\mathfrak{c}_t}_{\tilde{V}_2(t)}(z) = \frac{1}{\sqrt{2}} \langle (e^{-itA} z + e^{itA} \bar{z})^{\otimes 2}, V_2(t) \rangle \in \mathcal{K}_{\mathfrak{c}_t}.$$

Then the non-autonomous Cauchy problem

$$\begin{cases} i\varepsilon \partial_t u = \left(F^{\mathfrak{c}_t}_{\tilde{V}_2(t)}\right)^{Wick} u, \quad t \in \mathbb{R}, \\ u(t=s) = u_s \in \mathcal{D}_{+,1}, \end{cases} \qquad (4.7)$$

admits a unique unitary propagator $\tilde{U}_2(t,s)$ on $\Gamma_s(\mathcal{Z})$. Furthermore, there exists $c > 0$ depending only on φ_0 such that

$$\|\tilde{U}_2(t,s)\|_{\mathcal{L}(\mathcal{D}_{+,1})} \leq \exp\left(c \int_s^t dr \|V_2(r)\|_{\otimes_s^2 \mathcal{Z}} \right).$$

Moreover $U_2(t,s) = e^{-i(t/\varepsilon)\mathrm{d}\Gamma(A)} \tilde{U}_2(t,s) e^{i(s/\varepsilon)\mathrm{d}\Gamma(A)}$ is a mild solution of the Cauchy problem

$$\begin{cases} i\varepsilon \partial_t u = \left(\mathrm{d}\Gamma(A) + F^{Wick}_{V_2(t)}\right) u, \quad t \in \mathbb{R}, \\ u(t=s) = u_s \in \mathcal{D}_{+,1}, \end{cases} \qquad (4.8)$$

The quantum quadratic dynamics $U_2(t,s)$ can be interpreted as a time dependent Bogoliubov transform of the Weyl commutation relations (2.4).

Proposition 4.7 *Let $\varphi_0 \in \mathcal{Z}$ and consider the propagator $\tilde{U}_2(t,0)$ given in Corollary* 4.6*. For a given $\xi_0 \in \mathcal{Z}$, we have*

$$\tilde{U}_2(t,0)W(i\xi_0)\tilde{U}_2(0,t) = W(i\beta(t,0)\xi_0)$$

where $\beta(t,0)$ is the symplectic propagator on $\mathcal{Z}$ solving the equation

$$\begin{cases} i\partial_t \xi_t(x) = e^{itA}\partial_{\bar{z}} F^{\mathfrak{c}_t}_{\tilde{V}_2(t)}[\xi_t], \\ \xi_{|t=0} = \xi_0, \end{cases} \tag{4.9}$$

such that $\beta(t,0)\xi_0 = \xi_t$.

Proof. The Cauchy problem (4.9) admits a unique solution $\xi_t \in C^0(\mathbb{R},\mathcal{Z})$ given by a time-ordered Dyson series since the mappings $L_t : u \mapsto e^{itA}\partial_{\bar{z}} F^{\mathfrak{c}_t}_{\tilde{V}_2(t)}[u]$ re bounded $\mathbb{R}$-linear operators on $\mathcal{Z}$ with the estimate

$$\|L_t(u)\| \leq c\|u\|\|V_2(t)\|_{\otimes_s^2 \mathcal{Z}}$$

satisfied for all times. Therefore we have a well defined non-autonomous dynamical system $\beta(t,s)$ such that $\beta(t,s)\xi_s = \xi_t$ verifying

$$\beta(s,s) = 1, \quad \beta(t,s)\beta(s,r) = \beta(t,r) \quad \text{for all} \quad t,r,s \in \mathbb{R}.$$

Moreover $\beta(t,s)$ are symplectic transforms on $\mathcal{Z}$ for any $t,s \in \mathbb{R}$ which can be checked by differentiating $\mathrm{Im}\langle \beta(t,s)\xi, \beta(t,s)\eta\rangle$ with respect to t for $\xi,\eta \in \mathcal{Z}$.

Differentiate with respect to t the quantity

$$\tilde{U}_2(0,t)W\Big(-i\frac{\sqrt{2}}{\varepsilon}\xi_t\Big)\tilde{U}_2(t,0)$$

in the sense of quadratic forms on $\mathcal{D}_{+,1}$, we get

$$\begin{aligned} &i\varepsilon\partial_t\Big[\tilde{U}_2(0,t)W\Big(-i\frac{\sqrt{2}}{\varepsilon}\xi_t\Big)\tilde{U}_2(t,0)\Big] \\ &\quad= \tilde{U}_2(0,t)W\Big(-i\frac{\sqrt{2}}{\varepsilon}\xi_t\Big)\Big[\big(F^{\mathfrak{c}_t}_{\tilde{V}_2(t)}\big)^{Wick} - W\Big(-i\frac{\sqrt{2}}{\varepsilon}\xi_t\Big)^*\big(F^{\mathfrak{c}_t}_{\tilde{V}_2(t)}\big)^{Wick} \\ &\qquad\times W\Big(-i\frac{\sqrt{2}}{\varepsilon}\xi_t\Big) + \big(\mathrm{Re}\langle \xi_t, i\partial_t\xi_t\rangle + 2\mathrm{Re}\langle z, i\partial_t\xi_t\rangle^{Wick}\big)\Big]\tilde{U}_2(t,s). \end{aligned} \tag{4.10}$$

Using [3, Lemma 2.10], we see that

$$W\Big(-i\frac{\sqrt{2}}{\varepsilon}\xi_t\Big)^*\big(F^{\mathfrak{c}_t}_{\tilde V_2(t)}\big)^{Wick}W\Big(-i\frac{\sqrt{2}}{\varepsilon}\xi_t\Big)=\big(F^{\mathfrak{c}_t}_{\tilde V_2(t)}[z+\xi_t]\big)^{Wick}.$$

Hence the right hand side of (4.10) vanishes since

$$F^{\mathfrak{c}_t}_{\tilde V_2(t)}[z+\xi_t]-F^{\mathfrak{c}_t}_{\tilde V_2(t)}-\mathrm{Re}\langle\xi_t,i\partial_t\xi_t\rangle-2\mathrm{Re}\langle z,i\partial_t\xi_t\rangle=0. \qquad \square$$

4.3. Proof of the main result

In this subsection we give the proof of our main result (Theorem 2.1). Let F_V be a real-valued function in $\mathcal{K}$ with $V\in\Gamma_s(\mathcal{Z})$, i.e.,

$$F_V(z)=\sum_{n=0}^{\infty}\Big\langle\frac{(z+\bar z)^{\otimes n}}{\sqrt{n!}},V^{(n)}\Big\rangle \quad\text{and}\quad V=\bigoplus_{n=0}^{\infty}V^{(n)} \quad\text{with}\quad V^{(n)}\in\otimes_s^n\mathcal{Z}.$$

We consider the symbol $F^{\mathfrak{c}_t}_{V(t)}\in\mathcal{K}_{\mathfrak{c}_t}$, with respect to the conjugation $\mathfrak{c}_t z:=e^{2itA}\bar z$, obtained from $V\in\Gamma_s(\mathcal{Z})$ as follows

$$F^{\mathfrak{c}_t}_{V(t)}(z)=\sum_{n=0}^{\infty}\Big\langle\frac{(e^{-itA}z+e^{itA}\bar z)^{\otimes n}}{\sqrt{n!}},V^{(n)}\Big\rangle. \tag{4.11}$$

We first prove some preliminary lemmas.

Lemma 4.8 *The map $\mathbb{R}\ni t\mapsto e^{i(t/\varepsilon)H}e^{-i(t/\varepsilon)\mathrm{d}\Gamma(A)}\Psi$ is norm differentiable in $\Gamma_s(\mathcal{Z})$ for any $\Psi\in\mathcal{D}(\Gamma(\lambda))$, with $\lambda\geq\sqrt{3}$ and $\varepsilon\in(0,1/3]$. Moreover, the following identity holds*

$$i\varepsilon\partial_t e^{i(t/\varepsilon)H}e^{-i(t/\varepsilon)\mathrm{d}\Gamma(A)}\Psi=-e^{i(t/\varepsilon)H}e^{-i(t/\varepsilon)\mathrm{d}\Gamma(A)}\big(F^{\mathfrak{c}_t}_{V(t)}\big)^{Wick}\Psi.$$

Proof. By Lemma 3.14, we know that if $\lambda\geq\sqrt{3}$ then $\mathcal{D}(\Gamma(\lambda))\subset\mathcal{D}(F_V^{Wick})$. Hence for $\Psi\in\mathcal{D}(\mathrm{d}\Gamma(A))\cap\mathcal{D}(\Gamma(\lambda))$ with $\lambda\geq\sqrt{3}$, we have

$$\begin{aligned}-i\varepsilon e^{i(t/\varepsilon)H}e^{-i(t/\varepsilon)\mathrm{d}\Gamma(A)}\Psi&=e^{i(t/\varepsilon)H}(H-\mathrm{d}\Gamma(A))e^{-i(t/\varepsilon)\mathrm{d}\Gamma(A)}\Psi\\&=e^{i(t/\varepsilon)H}F_V^{Wick}e^{-i(t/\varepsilon)\mathrm{d}\Gamma(A)}\Psi\\&=e^{i(t/\varepsilon)H}e^{-i(t/\varepsilon)\mathrm{d}\Gamma(A)}\big(F^{\mathfrak{c}_t}_{V(t)}\big)^{Wick}\Psi.\end{aligned}$$

The two last equalities hold using the fact that $H = \mathrm{d}\Gamma(A) + F_V^{Wick}$ when restricted to $\mathcal{D}(\mathrm{d}\Gamma(A)) \cap \mathcal{D}(\lambda^N)$. Now, for $\Psi \in \mathcal{D}(\Gamma(\lambda))$ we take a sequence $\Psi_\kappa \in \mathcal{D}(\mathrm{d}\Gamma(A)) \cap \mathcal{D}(\Gamma(\lambda))$ such that $\Psi_\kappa \to \Psi$ when $\kappa \to \infty$. Therefore, we can write

$$\begin{aligned} & e^{i(t/\varepsilon)H} e^{-i(t/\varepsilon)\mathrm{d}\Gamma(A)} \Psi_\kappa \\ & \quad = \Psi_\kappa + \frac{i}{\varepsilon} \int_0^t e^{i(s/\varepsilon)H} e^{-i(s/\varepsilon)\mathrm{d}\Gamma(A)} \big(F_{V(s)}^{\mathfrak{c}_t}\big)^{Wick} \Psi_\kappa ds. \qquad (4.12) \end{aligned}$$

Letting $\kappa \to \infty$ that the same identity holds for Ψ instead of Ψ_κ. This yields that

$$t \mapsto e^{i(t/\varepsilon)H} e^{-i(t/\varepsilon)\mathrm{d}\Gamma(A)} \Psi$$

is norm differentiable in $\Gamma_s(\mathcal{Z})$ for any $\Psi \in \mathcal{D}(\Gamma(\lambda))$, $\lambda \geq \sqrt{3}$. $\square$

Lemma 4.9 *For any $V \in \mathcal{D}(e^{\alpha\Gamma(\lambda)})$ with $\lambda > 1$, $\alpha > 0$ and any $\varphi \in \mathcal{Z}$ it holds for $\varepsilon \in (0, 1/3]$:*

(i) *$W(-i(\sqrt{2}/\varepsilon)\varphi) U_2(t,0)\mathcal{D}_f \subset \mathcal{D}(F_V^{Wick})$.*

(ii) *There exist a (ε, V, t)-independent constant $C > 0$ such that for any $\Psi \in \mathcal{D}_f$*

$$\begin{aligned} \big\|F_V^{Wick} U_2(t,0)\Psi\big\| \leq C \big\|e^{\alpha\lambda^{N/\varepsilon}} \Gamma(\sqrt{\varepsilon}) V\big\|_{\Gamma_s(\mathcal{Z})} & \\ \times \left[\left\| \sqrt{g_t\left(\frac{N}{\varepsilon}\right)} \Psi \right\|^2 + g_t'(0) \left| \int_0^t \|V_2(s)\| ds \right| \|\Psi\|^2 \right]^{1/2}, & \end{aligned}$$

where $g_t(r) = \sum_{k=0}^\infty e^{-\alpha_0 \lambda^k} e^{2\sqrt{2}\lambda_0^k \int_0^t \|V_2(s)\| ds} (r+1)^k$ for $1 < \lambda_0 < \lambda$, $0 < \alpha_0 \lambda^2 < \alpha$ and $g_t'(r) = (d/dr) g_t(r)$.

Proof. We observe that Lemma 3.18 gives

$$W\left(-i\frac{\sqrt{2}}{\varepsilon}\varphi\right)^* F_V^{Wick} W\left(-i\frac{\sqrt{2}}{\varepsilon}\varphi\right) = F_{V_\varphi}^{Wick},$$

with V_φ satisfying the inequality $\|V_\varphi\| \leq c\|\Gamma(\sqrt{2})V\|$. Hence $V_\varphi \in \mathcal{D}(e^{\beta\Gamma(\lambda)})$ with $0 < \beta < \alpha$ and therefore it is enough to show that $U_2(t,0)\mathcal{D}_f \subset \mathcal{D}(F_V^{Wick})$ which follows by (ii).

Let $V, \Psi \in \mathcal{D}_f$ and $S(x) = \sum_{k=0}^{\infty} a_k x^k$ be the entire series with $a_k = e^{-\alpha_0 \lambda^k}$ such that $\alpha_0 > 0$ and $\alpha_0 \lambda^2 < \alpha$. Writing

$$\left\| F_V^{Wick} U_2(t,0)\Psi \right\| \leq \left\| F_V^{Wick} S\Big(\frac{N}{\varepsilon}\Big)^{-1/2} \right\| \left\| S\Big(\frac{N}{\varepsilon}\Big)^{1/2} U_2(t,0)\Psi \right\|,$$

then applying Proposition 4.5 with $1 < \lambda_0 < \lambda$, we get

$$\begin{aligned} &\left\| S\Big(\frac{N}{\varepsilon}\Big)^{1/2} U_2(t,0)\Psi \right\| \\ &\quad \leq c \left[\left\| \sqrt{\sum_{k=0}^{\infty} a_k(t) \Big(\frac{N}{\varepsilon}+1\Big)^k} \Psi \right\|^2 + \sum_{k=0}^{\infty} k a_k(t) \left| \int_0^t \|V_2(s)\| ds \right| \|\Psi\|^2 \right]^{1/2}, \end{aligned}$$

where $a_k(t) = a_k e^{\sqrt{2} k \lambda_0^k | \int_0^t \|V_2(s)\| ds|}$. Since $\Psi \in \mathcal{D}_f$ there exists $m \in \mathbb{N}$ such that $\Psi \in \bigoplus_{n=0}^{m} \otimes_s^n \mathcal{Z}$ and therefore the inequality

$$\left\| \sqrt{\sum_{k=0}^{\infty} a_k(t) \Big(\frac{N}{\varepsilon}+1\Big)^k} \Psi \right\| \leq \sqrt{\sum_{k=0}^{\infty} a_k(t)(m+1)^k} \|\Psi\|$$

holds with a finite right hand side. Using Proposition 3.17 we see that for $\lambda_1 > 8e$

$$\left\| F_V^{Wick} S\Big(\frac{N}{\varepsilon}\Big)^{-1/2} \right\| \leq C \sqrt{\sum_{k\geq 0} a_{k+2}^{-1} (\lambda_1 \varepsilon)^n \|V^{(k)}\|^2} \leq C' \left\| e^{\alpha \lambda^{N/\varepsilon}} \Gamma(\sqrt{\varepsilon}) V \right\|.$$

□

Proof of Theorem 2.1. Since the main quantity to be estimated in Theorem 2.1 is bounded by 2, we can assume without restriction that ε is sufficiently small. Let $F_{\tilde{V}_2(t)}^{\mathfrak{c}_t}$ be a symbol in $\mathcal{K}_{\mathfrak{c}_t}$, with respect to the conjugation $\mathfrak{c}_t z := e^{2itA} \bar{z}$, given by

$$F_{\tilde{V}_2(t)}^{\mathfrak{c}_t} := \left\langle \frac{(e^{-itA} z + e^{itA} \bar{z})^{\otimes 2}}{\sqrt{2}}, V_2(t) \right\rangle \in \mathcal{P} \cap \mathcal{K}, \tag{4.13}$$

where $V_2(t)$ is defined by (4.6). For $\Psi \in \mathcal{D}_f$, we write

$$\begin{aligned}\Theta(t) &:= e^{i(t/\varepsilon)H} e^{i(\omega(t)/\varepsilon)} W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_t\Big) U_2(t,0)\Psi \\ &= e^{i(t/\varepsilon)H} e^{-i(t/\varepsilon)\mathrm{d}\Gamma(A)} e^{i(\omega(t)/\varepsilon)} W\Big(-i\frac{\sqrt{2}}{\varepsilon}\tilde{\varphi}_t\Big) e^{i(t/\varepsilon)\mathrm{d}\Gamma(A)} U_2(t,0)\Psi.\end{aligned}$$

We differentiate the above quantity with respect to time t. We recall the formula

$$\begin{aligned} & i\varepsilon\partial_t W\Big(-i\frac{\sqrt{2}}{\varepsilon}\tilde{\varphi}_t\Big)\Psi \\ &\quad = W\Big(-i\frac{\sqrt{2}}{\varepsilon}\tilde{\varphi}_t\Big)\big[\mathrm{Re}\langle\tilde{\varphi}_t, i\partial_t\tilde{\varphi}_t\rangle + 2\mathrm{Re}\langle z, i\partial_t\tilde{\varphi}_t\rangle^{Wick}\big]\Psi, \end{aligned} \tag{4.14}$$

where $\Psi \in \mathcal{D}_f$ (see [2], [13]). Since $e^{i(t/\varepsilon)d\Gamma(A)}$ commutes with the number operator N we know, by Lemma 4.9, that the vector $W(-i(\sqrt{2}/\varepsilon)\tilde{\varphi}_t)$ $\cdot e^{i(t/\varepsilon)d\Gamma(A)}U_2(t,0)\Psi$ belongs to $\mathcal{D}(e^{\alpha\lambda^{N/\varepsilon}})$. Therefore, using Lemma 4.8, we can differentiate $e^{i(t/\varepsilon)H}e^{-i(t/\varepsilon)d\Gamma(A)}$ then differentiate $e^{i(\omega(t)/\varepsilon)}W$ $\cdot(-i(\sqrt{2}/\varepsilon)\tilde{\varphi}_t)$ using (4.14) and finally differentiate $e^{i(t/\varepsilon)\mathrm{d}\Gamma(A)}U_2(t,0)\Psi$ using Corollary 4.6. So that, we get

$$\begin{aligned}\Theta'(t) &= \frac{i}{\varepsilon} e^{i(t/\varepsilon)H} e^{-i(t/\varepsilon)\mathrm{d}\Gamma(A)} e^{i(\omega(t)/\varepsilon)} W\Big(-i\frac{\sqrt{2}}{\varepsilon}\tilde{\varphi}_t\Big) \\ &\quad \times \Big[W\Big(-i\frac{\sqrt{2}}{\varepsilon}\tilde{\varphi}_t\Big)^* \big(F^{\mathfrak{c}_t}_{V(t)}\big)^{Wick} W\Big(-i\frac{\sqrt{2}}{\varepsilon}\tilde{\varphi}_t\Big) + \partial_t\omega(t) \\ &\qquad - \mathrm{Re}\langle\tilde{\varphi}_t, i\partial_t\tilde{\varphi}_t\rangle - 2\mathrm{Re}\langle z, i\partial_t\tilde{\varphi}_t\rangle^{Wick} - \big(F^{\mathfrak{c}_t}_{\tilde{V}_2(t)}\big)^{Wick}\Big] \\ &\quad \times e^{i(t/\varepsilon)\mathrm{d}\Gamma(A)} U_2(t,0)\Psi \\ &= \frac{i}{\varepsilon} e^{i(t/\varepsilon)H} e^{i(\omega(t)/\varepsilon)} W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_t\Big) \\ &\quad \times \Big[W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_t\Big)^* F_V^{Wick} W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_t\Big) + \partial_t\omega(t) \\ &\qquad - \mathrm{Re}\langle\varphi_t, \partial_{\bar{z}}F_V(\varphi_t)\rangle - 2\mathrm{Re}\langle z, \partial_{\bar{z}}F_V(\varphi_t)\rangle^{Wick} - F^{Wick}_{V_2(t)}\Big] U_2(t,0)\Psi\end{aligned}$$

where $F^{\mathfrak{c}_t}_{V(t)}$ and $F^{\mathfrak{c}_t}_{\tilde{V}_2(t)}$ are given respectively by (4.11)–(4.13). By Lemma 3.18 we know that

$$W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_t\Big)^* F_V^{Wick} W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_t\Big) = F_V(.+\varphi_t)^{Wick}.$$

We define for any $n, k \in \mathbb{N}$, $k \geq n$

$$V_n^{(k)} := \mathcal{S}_n \big\langle (\varphi_t + \overline{\varphi}_t)^{\otimes (k-n)} \big| \otimes 1\!\!1^{(n)} V^{(k)} \in \otimes_s^n \mathcal{Z}.$$

One can check by direct computation that $\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\sqrt{k!/n!}(1/(k-n)!)$ $\cdot V_n^{(k)} \in \Gamma_s(\mathcal{Z})$. Hence expanding $F_V(.+\varphi_t)$ around φ_t we obtain

$$F_V(z+\varphi_t) = \sum_{n=0}^{\infty}\Big\langle \frac{(z+\bar{z})^{\otimes n}}{\sqrt{n!}}, \sum_{k=n}^{\infty}\sqrt{\frac{n!}{k!}}C_k^n V_n^{(k)}\Big\rangle,$$

where $C_k^n = k!/n!(k-n)!$. Using the fact that $\partial_{\bar{z}}F_V(\varphi_t) = \sum_{k=1}^{\infty}(k/\sqrt{k!})$ $\cdot\langle(\varphi_t+\bar{\varphi}_t)^{\otimes k-1}| \otimes 1\!\!1\, V^{(k)} \in \mathcal{Z}$, we get

$$\begin{aligned} F_V(z+\varphi_t) &= F_{R(t)}(z) + F_V(\varphi_t) + \sum_{k=1}^{\infty}\frac{k}{\sqrt{k!}}\langle (z+\bar{z})\otimes(\varphi_t+\bar{\varphi}_t)^{\otimes k-1}, V^{(k)}\rangle \\ &\quad + \sum_{k=2}^{\infty}\frac{C_k^2}{\sqrt{k!}}\langle (z+\bar{z})^{\otimes 2}\otimes(\varphi_t+\bar{\varphi}_t)^{\otimes k-2}, V^{(k)}\rangle \\ &= F_{R(t)}(z) + F_V(\varphi_t) + 2\mathrm{Re}\langle z, \partial_{\bar{z}}F_V(\varphi_t)\rangle + F_{V_2(t)}(z), \end{aligned}$$

where $F_{R(t)}$ is $\sum_{n=3}^{\infty}\big\langle (z+\bar{z})^{\otimes n}/\sqrt{n!}, \sum_{k=n}^{\infty}\sqrt{n!/k!}C_k^n V_k^{(n)}\big\rangle$. This implies that the t-vector $\Theta'(t)$ is given by $\Theta'(t) = (i/\varepsilon)e^{i(t/\varepsilon)H}e^{i(\omega(t)/\varepsilon)}W$ $\cdot(-i(\sqrt{2}/\varepsilon)\varphi_t)F_{R(t)}^{Wick}U_2(t,0)\Psi$, if we choose ω such that

$$\partial_t\omega = \mathrm{Re}\langle\varphi_t, \partial_{\bar{z}}F_V(\varphi_t)\rangle - F_V(\varphi_t).$$

This holds with

$$\omega(t) = \int_0^t \sum_{k=0}^{\infty}\frac{(k-2)}{2}\Big\langle\frac{(\varphi_s+\bar{\varphi}_s)^{\otimes k}}{\sqrt{k!}}, V^{(k)}\Big\rangle ds.$$

Hence we conclude that

$$e^{i(t/\varepsilon)H}e^{i(\omega(t)/\varepsilon)}W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_t\Big)U_2(t,0)\Psi - W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_0\Big)\Psi$$
$$= \frac{i}{\varepsilon}\int_0^t e^{i(s/\varepsilon)H}e^{i(\omega(s)/\varepsilon)}W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_s\Big)F^{Wick}_{R(s)}U_2(s,0)\Psi\,ds.$$

We observe that $R(t) \in \bigoplus_{n\geq 3}\otimes_s^n\mathcal{Z}$ and proceed to estimate the right hand side. So we have (for $t > 0$)

$$\Big\|e^{-i(t/\varepsilon)H}W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_0\Big)\Psi - e^{i(\omega(t)/\varepsilon)}W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_t\Big)U_2(t,0)\Psi\Big\|_{\Gamma_s(\mathcal{Z})}$$
$$\leq \frac{1}{\varepsilon}\int_0^t \big\|F^{Wick}_{R(s)}U_2(s,0)\Psi\big\|_{\Gamma_s(\mathcal{Z})}ds,$$

Now using the estimate (ii) of Lemma 4.9, with $0 < \gamma < \alpha$, we obtain

$$\big\|F^{Wick}_{R(s)}U_2(s,0)\Psi\big\|_{\Gamma_s(\mathcal{Z})} \leq C(s)\varepsilon^{3/2}\big\|e^{\gamma\lambda^{N/\varepsilon}}R(s)\big\|_{\Gamma_s(\mathcal{Z})},$$

such that $C > 0$ depending only on (α,λ) and

$$C(s) = C\bigg[\bigg\|\sqrt{g_s\Big(\frac{N}{\varepsilon}\Big)}\Psi\bigg\|^2 + g_s'(0)\bigg|\int_0^s \|V_2(r)\|dr\bigg|\|\Psi\|^2\bigg]^{1/2},$$

where $g_t(r) = \sum_{k=0}^{\infty} e^{-\alpha_0\lambda^k}e^{2\sqrt{2}\lambda_0^k\int_0^t\|V_2(s)\|ds}(r+1)^k$ for $1 < \lambda_0 < \lambda$, $0 < \alpha_0\lambda^2 < \alpha$ and $g_t'(r) = (d/dr)g_t(r)$.

A similar estimate as in the proof of Lemma 3.18 yields

$$\big\|e^{\gamma\lambda^{N/\varepsilon}}R(s)\big\|_{\Gamma_s(\mathcal{Z})} \leq \sqrt{2}e^{4\|\varphi_s\|^2_{\mathcal{Z}}}\big\|\sqrt{2}^{N/\varepsilon}e^{\gamma\lambda^{N/\varepsilon}}V\big\|_{\Gamma_s(\mathcal{Z})}$$
$$\leq c\,e^{4\|\varphi_s\|^2_{\mathcal{Z}}}\big\|e^{\alpha\lambda^{N/\varepsilon}}V\big\|_{\Gamma_s(\mathcal{Z})}.$$

Hence there exist a (ε, V)-independent constant $c(t) > 0$ such that

$$
\begin{aligned}
&\Big\| e^{-i(t/\varepsilon)H} W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_0\Big)\Psi - e^{i(\omega(t)/\varepsilon)} W\Big(-i\frac{\sqrt{2}}{\varepsilon}\varphi_t\Big) U_2(t,0)\Psi \Big\| \\
&\quad \leq c(t)\sqrt{\varepsilon}\big\| e^{\alpha\lambda^{N/\varepsilon}} V \big\|,
\end{aligned}
$$

with $C > 0$ depending only on (α, λ) and

$$
c(t) = C\int_0^t e^{4\|\varphi_s\|_{\mathcal{Z}}^2} \bigg[\bigg\| \sqrt{g_s\Big(\frac{N}{\varepsilon}\Big)}\Psi \bigg\|^2 + g_s'(0) \bigg| \int_0^s \|V_2(r)\| dr \bigg| \|\Psi\|^2 \bigg]^{1/2} ds.
$$

Since $\Psi \in \mathcal{D}_f$ and $\|\varphi_t\|_{\mathcal{Z}}$ is bounded on compact intervals we see that the r.h.s is finite. □

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