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Biharmonic maps into compact Lie groups and integrable systems

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Abstract. In this paper, the formulation of the biharmonic map equation in terms of the Maurer-Cartan form for all smooth maps of a compact Riemannian manifold into a compact Lie group (G, h) with the bi-invariant Riemannian metric h is obtained. Using this, all biharmonic curves into compact Lie groups are determined exactly, and all the biharmonic maps of an open domain of $\mathbb{R}^2$ equipped with a Riemannian metric conformal to the standard Euclidean metric into (G, h) are determined.

Key words: harmonic map, biharmonic map, compact Lie group, integrable system, Maurer-Cartan form.

1. Introduction and statement of results

The theory of harmonic maps of a Riemann surface into Lie groups, symmetric spaces or homogeneous spaces has been extensively studied in connection with the integrable systems ([1], [2], [4], [5], [6], [8], [9], [16]). Let us recall the theory of harmonic maps of a Riemann surface M into a compact Lie group G, briefly. A harmonic map is a critical map of the energy functional defined by

$$E(\psi) := \frac{1}{2}\int_M |d\psi|^2 v_g.$$

For such a map ψ, let α be the pull back of the Maurer-Cartan form θ of G which is decomposed into the sum of the holomorphic part and the antiholomorphic one as $\alpha = \alpha' + \alpha''$. Then, it satisfies $d\alpha = (1/2)[\alpha \wedge \alpha] = 0$ (the integrability condition), and the harmonicity of ψ is equivalent to the condition $\delta\alpha = 0$. Introducing a parameter $\lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ as

$$\alpha_\lambda := \frac{1}{2}(1-\lambda)\alpha' + \frac{1}{2}(1-\lambda^{-1})\alpha'',$$

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both the harmonicity and the integrability condition are equivalent to

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0,$$

which implies that there exists an extended solution $\Phi_\lambda : M \to G$ satisfying ${\Phi_\lambda}^{-1} d\Phi_\lambda = \alpha_\lambda$ ([16]). Guest and Ohnita ([9]) showed that the loop group $\Lambda G^{\mathbb{C}}$ of G acts on the space of all harmonic maps of M into G, and Uhlenbeck ([16]) showed that every harmonic map from the two-sphere into G is a harmonic map of finite uniton number, and Wood ([17]) determined explicitly harmonic maps of finite uniton numbers. On the other hand, the theory of biharmonic maps was initiated by Eells and Lemaire ([6]) and Jiang ([12]). A biharmonic map is a natural extension of harmonic map, and is a critical map of the bienergy functional defined by

$$E_2(\psi) := \frac{1}{2}\int_M |\delta d\psi|^2 v_g = \frac{1}{2}\int_M |\tau(\psi)|^2 v_g,$$

where $\tau(\psi)$ is the tension field of ψ, and, by definition, ψ is harmonic if and only if $\tau(\psi) \equiv 0$.

In this paper, we study biharmonic maps of a compact Riemannian manifold (M, g) into a compact Lie group (G, h) with the bi-invariant Riemannian metric h. For every C^∞ map $\psi : (M, g) \to (G, h)$, let us consider again the pullback α of the Maurer-Cartan form θ. We first will show that the biharmonicity condition for ψ is that

$$\delta d\delta\alpha + \mathrm{Trace}_g([\alpha, d\delta\alpha]) = 0$$

(cf. Corollary 3.5) which is a natural extension of harmonicity. Due to this formula, we can determine all real analytic biharmonic curves into a compact Lie group (G, h) in terms of the initial data $F(0)$, $F'(0)$ and $F''(0)$, where $F(t) = \alpha(\partial/\partial t)$ (cf. Section 4). We give a characterization of biharmonic maps of $(\mathbb{R}^2, \mu^2 g_0)$, where g_0 is the standard Euclidean metric on $\mathbb{R}^2$ and μ is a positive real analytic function on $\mathbb{R}^2$ (cf. Sections 5, 6 and 7).

2. Preliminaries

In this section, we prepare general materials and facts on harmonic maps, biharmonic maps into Riemannian manifolds (cf. [6], [12], [13]). Let

(M, g) be an m-dimensional compact Riemannian manifold, and (N, h), an n-dimensional Riemannian manifold.

The *energy functional* on the space $C^\infty(M, N)$ of all C^∞ maps of M into N is defined by

$$E(\psi) = \frac{1}{2} \int_M |d\psi|^2 v_g,$$

and for a compactly supported C^∞ one parameter deformation $\psi_t \in C^\infty(M, N)$ $(-\epsilon < t < \epsilon)$ of ψ with $\psi_0 = \psi$, the *first variation formula* is given by

$$\left.\frac{d}{dt}\right|_{t=0} E(\psi_t) = -\int_M \langle \tau(\psi), V \rangle v_g,$$

where V is a variation vector field along ψ defined by $V = d/dt|_{t=0}\psi_t$ which belongs to the space $\Gamma(\psi^{-1}TN)$ of sections of the induced bundle of the tangent bundle TN by ψ. The *tension field* $\tau(\psi)$ is defined by

$$\tau(\psi) = -\delta(d\psi), \tag{2.1}$$

where recall the definition $\delta\alpha$ for a $\psi^{-1}TN$-valued 1-form α,

$$\delta\alpha = -\sum_{i=1}^m (\overline{\nabla}_{e_i}\alpha)(e_i) = -\sum_{i=1}^m \big\{\overline{\nabla}(\alpha(e_i)) - \alpha(\nabla_{e_i} e_i)\big\}.$$

Here, ∇, ∇^h and $\overline{\nabla}$ are the Levi-Civita connections of (M, g), (N, h), and the induced connections on the induced bundle $\psi^{-1}TN$ from ∇^h, respectively. For a harmonic map $\psi : (M, g) \to (N, h)$, the *second variation formula* of the energy functional $E(\psi)$ is

$$\left.\frac{d^2}{dt^2}\right|_{t=0} E(\psi_t) = \int_M \langle J(V), V \rangle v_g$$

where

$$
\begin{aligned}
J(V) &= \overline{\Delta} V - \mathcal{R}(V), \\
\overline{\Delta} V &= \overline{\nabla}^* \overline{\nabla} V = -\sum_{i=1}^m \{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V\}, \\
\mathcal{R}(V) &= \sum_{i=1}^m R^h(V, d\psi(e_i)) d\psi(e_i).
\end{aligned}
$$

Here, $\overline{\nabla}$ is the induced connection on the induced bundle $\psi^{-1}TN$, and R^h is the curvature tensor of (N, h) given by $R^h(U, V)W = [\nabla^h_U, \nabla^h V]W - \nabla^h_{[U,V]} W$ $(U, V, W \in \mathfrak{X}(N))$. The *bienergy functional* is defined by

$$E_2(\psi) = \frac{1}{2} \int_M |\delta d\psi|^2 v_g = \frac{1}{2} \int_M |\tau(\psi)|^2 v_g, \tag{2.2}$$

and the *first variation formula* of the bienergy is given ([12]) by

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\psi_t) = -\int_M \langle \tau_2(\psi), V \rangle v_g \tag{2.3}$$

where the *bitension field* $\tau_2(\psi)$ is defined by

$$\tau_2(\psi) = J(\tau(\psi)) = \overline{\Delta} \tau(\psi) - \mathcal{R}(\tau(\psi)), \tag{2.4}$$

and a C^∞ map $\psi : (M, g) \to (N, h)$ is called to be *biharmonic* if

$$\tau_2(\psi) = 0. \tag{2.5}$$

The biharmonic maps are real analytic when both (M, g) and (N, h) are real analytic. This is because the solutions of non-linear elliptic partial differential equations are real analytic.

3. Determination of the bitension field

Now, assume that (N, h) is an n-dimensional compact Lie group with Lie algebra $\mathfrak{g}$, and h, the bi-invariant Riemannian metric on G corresponding to the $\mathrm{Ad}(G)$-invariant inner product $\langle\, ,\, \rangle$ on $\mathfrak{g}$. Let θ be the Maurer-Cartan form on G, i.e., a $\mathfrak{g}$-valued left invariant 1-form on G which is defined by $\theta_y(Z_y) = Z$, $(y \in G, Z \in \mathfrak{g})$. For every C^∞ map ψ of (M, g) into (G, h), let us consider a $\mathfrak{g}$-valued 1-form α on M given by $\alpha = \psi^*\theta$. Then it is well known (see for example, [4]) that

Lemma 3.1 *For every C^∞ map $\psi : (M, g) \to (G, h)$,*

$$\theta(\tau(\psi)) = -\delta\alpha. \tag{3.1}$$

Thus, $\psi : (M, g) \to (G, h)$ is harmonic if and only if $\delta\alpha = 0$.

Let $\{X_s\}_{s=1}^n$ be an orthonormal basis of $\mathfrak{g}$ with respect to the inner product $\langle\, ,\, \rangle$. Then, for every $V \in \Gamma(\psi^{-1}TG)$,

$$\begin{aligned} V(x) &= \sum_{s=1}^n h_{\psi(x)}(V(x), X_{s\,\psi(x)})X_{s\,\psi(x)} \in T_{\psi(x)}G, \\ \theta(V)(x) &= \sum_{s=1}^n h_{\psi(x)}(V(x), X_{s\,\psi(x)})X_s \in \mathfrak{g}, \end{aligned} \tag{3.2}$$

for all $x \in M$. Then, for every $X \in \mathfrak{X}(M)$,

$$\begin{aligned} \theta(\overline{\nabla}_X V) &= \sum_{s=1}^n h(\overline{\nabla}_X V, X_s)\, X_s \\ &= \sum_{s=1}^n \{X\, h(V, X_s) - h(V, \overline{\nabla}_X X_s)\} X_s \\ &= X(\theta(V)) - \sum_{s=1}^n h(V, \overline{\nabla}_X X_s) X_s, \end{aligned} \tag{3.3}$$

where we regarded a vector field $Y \in \mathfrak{X}(G)$ by $Y(x) = Y(\psi(x))$ ($x \in M$) to be an element in the space $\Gamma(\psi^{-1}TG)$ of smooth sections of $\psi^{-1}TG$. Here, let us recall that the Levi-Civita connection ∇^h of (G, h) is given (cf. [13, Vol. II, p. 201, Theorem 3.3]) by

$$\nabla^h_{X_t} X_s = \frac{1}{2}[X_t, X_s] = \frac{1}{2}\sum_{\ell=1}^n C_{ts}^\ell X_\ell, \tag{3.4}$$

where the structure constant C_{ts}^ℓ of $\mathfrak{g}$ is defined by $[X_t, X_s] = \sum_{\ell=1}^n C_{ts}^\ell X_\ell$, and satisfies

$$C_{ts}^{\ell} = \langle [X_t, X_s], X_\ell \rangle = -\langle X_s, [X_t, X_\ell] \rangle = -C_{t\ell}^{s}. \tag{3.5}$$

Thus, we have by (3.4) and (3.5),

$$\begin{aligned}\sum_{s=1}^{n} h(V, \overline{\nabla}_X X_s) X_s &= \frac{1}{2} \sum_{s,t=1}^{n} h\Bigg(V, \sum_{\ell=1}^{n} h(\psi_* X, X_t)\, C_{ts}^{\ell}\, X_\ell\Bigg) X_s \\ &= -\frac{1}{2} \sum_{s,t,\ell=1}^{n} h(V, X_\ell) h(\psi_* X, X_t) C_{t\ell}^{s} X_s \\ &= -\frac{1}{2} \sum_{t,\ell=1}^{n} h(V, X_\ell) h(\psi_* X, X_t) [X_t, X_\ell] \\ &= -\frac{1}{2} \Bigg[\sum_{t=1}^{n} h(\psi_* X, X_t) X_t, \sum_{\ell=1}^{n} h(V, X_\ell) X_\ell \Bigg] \\ &= -\frac{1}{2} [\alpha(X), \theta(V)], \end{aligned} \tag{3.6}$$

which is because we have

$$\alpha(X) = \theta(\psi_* X) = \sum_{t=1}^{n} h(\psi_* X, X_t) X_t, \tag{3.7}$$

and

$$\theta(V) = \sum_{\ell=1}^{n} h(V, X_\ell) \theta(X_\ell) = \sum_{\ell=1}^{n} h(V, X_\ell) X_\ell. \tag{3.8}$$

Therefore, inserting (3.6) into (3.3), we obtain

Lemma 3.2 *For every C^∞ map $\psi : (M, g) \to (G, h)$,*

$$\theta(\overline{\nabla}_X V) = X(\theta(V)) + \frac{1}{2} [\alpha(X), \theta(V)], \tag{3.9}$$

where $V \in \Gamma(\psi^{-1} TG)$ and $X \in \mathfrak{X}(M)$.

We shall show

Theorem 3.3 *For every $\psi \in C^\infty(M, G)$, we have*

$$\begin{aligned}\theta(\tau_2(\psi)) &= \theta(J(\tau(\psi)))\\ &= -\delta\, d\, \delta\alpha - \mathrm{Trace}_g([\alpha, d\,\delta\alpha]), \end{aligned}\tag{3.10}$$

where $\alpha = \psi^*\theta$.

Here, let us recall the definition:

Definition 3.4 For two $\mathfrak{g}$-valued 1-formsff α and β on M, we define a $\mathfrak{g}$-valued symmetric 2-tensor $[\alpha,\beta]$ on M by

$$[\alpha,\beta](X,Y) := \frac{1}{2}\{[\alpha(X),\beta(Y)] + [\alpha(Y),\beta(X)]\}, \quad (X, Y \in \mathfrak{X}(M)) \tag{3.11}$$

and its trace $\mathrm{Trace}_g([\alpha,\beta])$ by

$$\mathrm{Trace}_g([\alpha,\beta]) := \sum_{i=1}^{m} [\alpha,\beta](e_i,e_i). \tag{3.12}$$

Recall that the $\mathfrak{g}$-valued 2-form $[\alpha\wedge\beta]$ on M is given by

$$[\alpha\wedge\beta](X,Y) := \frac{1}{2}\{[\alpha(X),\beta(Y)] - [\alpha(Y),\beta(X)]\} \quad (X, Y \in \mathfrak{X}(M)). \tag{3.13}$$

Then, we have immediately by Theorem 3.3,

Corollary 3.5 *For every* $\psi \in C^\infty(M,G)$*, we have* (1) $\psi : (M,g) \to (G,h)$ *is harmonic if and only if*

$$\delta\alpha = 0. \tag{3.14}$$

(2) $\psi : (M,g) \to (G,h)$ *is biharmonic if and only if*

$$\delta\, d\, \delta\alpha + \mathrm{Trace}_g([\alpha, d\,\delta\alpha]) = 0. \tag{3.15}$$

We give a proof of Theorem 3.3.

Proof. (*The first step*) We first show that, for all $V \in \Gamma(\psi^{-1}TG)$,

$$\theta(\overline{\Delta}V) = \Delta_g\theta(V) - \sum_{i=1}^m \Big\{\frac{1}{2}[e_i(\alpha(e_i)), \theta(V)] + [\alpha(e_i), e_i(\theta(V))] + \frac{1}{4}[\alpha(e_i), [\alpha(e_i), \theta(V)]] - \frac{1}{2}[\alpha(\nabla_{e_i}e_i), \theta(V)]\Big\}, \quad (3.16)$$

where $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M, g), and Δ_g is the (positive) Laplacian of (M, g) acting on $C^\infty(M)$.

Indeed, we have by using Lemma 3.2 twice,

$$\begin{aligned}
\theta(\overline{\Delta}V) &= -\sum_{i=1}^m \{\theta(\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i}V)) - \theta(\overline{\nabla}_{\nabla_{e_i}e_i}V)\} \\
&= -\sum_{i=1}^m \Big\{e_i(\theta(\overline{\nabla}_{e_i}V)) + \frac{1}{2}[\alpha(e_i), \theta(\overline{\nabla}_{e_i}V] \\
&\qquad - \nabla_{e_i}e_i(\theta(V)) - \frac{1}{2}[\alpha(\nabla_{e_i}e_i), \theta(V)]\Big\} \\
&= -\sum_{i=1}^m \Big\{e_i\Big(e_i(\theta(V) + \frac{1}{2}[\alpha(e_i), \theta(V)]\Big) \\
&\qquad + \frac{1}{2}\Big[\alpha(e_i), e_i(\theta(V)) + \frac{1}{2}[\alpha(e_i), \theta(V)]\Big] \\
&\qquad - \nabla_{e_i}e_i(\theta(V)) - \frac{1}{2}[\alpha(\nabla_{e_i}e_i), \theta(V)]\Big\} \\
&= -\sum_{i=1}^m \{e_i(e_i(\theta(V)) - \nabla_{e_i}e_i(\theta(V))\} \\
&\quad - \sum_{i=1}^m \Big\{\frac{1}{2}e_i([\alpha(e_i), \theta(V)]) + \frac{1}{2}[\alpha(e_i), e_i(\theta(\theta(V))] \\
&\qquad + \frac{1}{4}[\alpha(e_i), [\alpha(e_i), \theta(V)]] - \frac{1}{2}[\alpha(\nabla_{e_i}e_i), \theta(V)]\Big\}. \quad (3.17)
\end{aligned}$$

Here, we have

$$e_i([\alpha(e_i), \theta(V)] = [e_i(\alpha(e_i)), \theta(V)] + [\alpha(e_i), e_i(\theta(V))],$$

which we substitute into (3.17), and by definition of Δ_g, we have (3.16).

(*The second step*) On the other hand, we have to consider

$$-\sum_{i=1}^{m} R^h(V,\psi_* e_i)\psi_* e_i = -\sum_{i=1}^{m} R^h\big(L_{\psi(x)\,*}^{-1}V, L_{\psi(x)\,*}^{-1}\psi_* e_i\big)L_{\psi(x)\,*}^{-1}\psi_* e_i. \tag{3.18}$$

Under the identification $T_eG \ni Z_e \leftrightarrow Z \in \mathfrak{g}$, we have

$$T_eG \ni L_{\psi(x)\,*}^{-1}\psi_* e_i \leftrightarrow \alpha(e_i) \in \mathfrak{g}, \tag{3.19}$$

$$T_eG \ni L_{\psi(x)\,*}^{-1}V \leftrightarrow \theta(V) \in \mathfrak{g}, \tag{3.20}$$

respectively. Because, we have

$$L_{\psi(x)\,*}^{-1}\psi_* e_i = \sum_{s=1}^{n} h(\psi_* e_i, X_{s\,\psi(x)})X_{s\,e}$$

and

$$\begin{aligned}\alpha(e_i) = \psi^*\theta(e_i) = \theta(\psi_* e_i) &= \sum_{s=1}^{n} h(\psi_* e_i, X_{s\,\psi(x)})\theta(X_{s\,\psi(x)})\\ &= \sum_{s=1}^{n} h(\psi_* e_i, X_{s\,\psi(x)})X_s,\end{aligned} \tag{3.21}$$

which implies that (3.19). Analogously, we obtain (3.20).

Under this identification, the curvature tensor of (G,h) is given as (see Kobayashi-Nomizu ([13, pp. 203–204])),

$$R^h(X,Y)_e = -\frac{1}{4}\,\mathrm{ad}([X,Y]) \quad (X,Y \in \mathfrak{g}),$$

and then, we have

$$\begin{aligned}\theta\Big(-\sum_{i=1}^{m} R^h(V,\psi_* e_i)\psi_* e_i\Big) &= \frac{1}{4}\sum_{i=1}^{m}[[\theta(V),\alpha(e_i)],\alpha(e_i)]\\ &= \frac{1}{4}\sum_{i=1}^{m}[\alpha(e_i),[\alpha(e_i),\theta(V)]].\end{aligned} \tag{3.22}$$

(*The third step*) By (3.16) and (3.21), for $V \in \Gamma(\psi^{-1}TG)$, we have

$$\theta\Big(\overline{\Delta}V - \sum_{i=1}^m R^h(V, \psi_* e_i)\psi_* e_i\Big)$$
$$= \Delta_g \theta(V) - \sum_{i=1}^m \Big\{ \frac{1}{2}[e_i(\alpha(e_i)), \theta(V)] + [\alpha(e_i), e_i(\theta(V))]$$
$$+ \frac{1}{4}[\alpha(e_i), [\alpha(e_i), \theta(V)]] - \frac{1}{2}[\alpha(\nabla_{e_i} e_i), \theta((V)]\Big\}$$
$$+ \frac{1}{4}\sum_{i=1}^m [\alpha(e_i), [\alpha(e_i), \theta(V)]]$$
$$= \Delta_g \theta(V) - \frac{1}{2}\sum_{i=1}^m e_i(\alpha(e_i)), \theta(V)] + \sum_{i=1}^m [\alpha(e_i), e_i(\theta(V))]$$
$$+ \frac{1}{2}\sum_{i=1}^m [\alpha(\nabla_{e_i} e_i), \theta(V)]$$
$$= \Delta_g \theta(V) - \frac{1}{2}\Big[\sum_{i=1}^m (e_i(\alpha(e_i)) - \alpha(\nabla_{e_i} e_i)), \theta(V)\Big] + \sum_{i=1}^m [\alpha(e_i), e_i(\theta(V))]$$
$$= \Delta_g \theta(V) + \frac{1}{2}[\delta\alpha, \theta(V)] + \sum_{i=1}^m [\alpha(e_i), e_i(\theta(V))]. \tag{3.23}$$

(*The fourth step*) For $V = \tau(\psi)$ in (3.22), since $\theta(\tau(\psi)) = -\delta\alpha$, we have

$$\theta(J(\tau(\psi))) = \Delta_g \theta(\tau(\psi)) + \frac{1}{2}[\delta\alpha, \theta(\tau(\psi))] + \sum_{i=1}^m [\alpha(e_i), e_i(\theta(\tau(\psi))]$$
$$= -\Delta_g \delta\alpha - \frac{1}{2}[\delta\alpha, \delta\alpha] - \sum_{i=1}^m [\alpha(e_i), e_i(\delta\alpha)]$$
$$= -\Delta_g \delta\alpha - \sum_{i=1}^m [\alpha(e_i), e_i(\delta\alpha)]$$
$$= -\Delta_g \delta\alpha - \sum_{i=1}^m [\alpha(e_i), (d\delta\alpha)(e_i)]. \tag{3.24}$$

Then, (3.23) implies the desired (3.10). □

4. Biharmonic curves from $\mathbb{R}$ into compact Lie groups

In this section, we consider the simplest case: $(M, g) = (\mathbb{R}, g_0)$ is the standard 1-dimensional Euclidean space, and (G, h) is an n-dimensional compact Lie group with the bi-invariant Riemannian metric h.

4.1.

First, let $\psi : \mathbb{R} \ni t \mapsto \psi(t) \in (G, h)$, a C^∞ curve in G. Then, $\alpha := \psi^*\theta$ is a $\mathfrak{g}$-valued 1-form on $\mathbb{R}$. So, α can be written at $t \in \mathbb{R}$ as

$$\alpha_t = F(t)dt, \tag{4.1}$$

where $F : \mathbb{R} \ni t \mapsto F(t) \in \mathfrak{g}$ is given by

$$F(t) = \alpha\bigg(\frac{\partial}{\partial t}\bigg) = \psi^*\theta\bigg(\frac{\partial}{\partial t}\bigg) = \theta\bigg(\psi_*\bigg(\frac{\partial}{\partial t}\bigg)\bigg). \tag{4.2}$$

Here, since

$$\psi'(t) := \psi_*\bigg(\frac{\partial}{\partial t}\bigg) = \sum_{s=1}^{n} h_{\psi(t)}\bigg(\psi_*\bigg(\frac{\partial}{\partial t}\bigg), X_{s\,\psi(t)}\bigg) X_{s\,\psi(t)}, \tag{4.3}$$

we have

$$F(t) = \sum_{s=1}^{n} h_{\psi(t)}\bigg(\psi_*\bigg(\frac{\partial}{\partial t}\bigg), X_{s\,\psi(t)}\bigg) X_s, \tag{4.4}$$

so that we have the following correspondence:

$$\begin{aligned} T_eG \ni L_{\psi(t)\,*}^{-1}\psi'(t) &= \sum_{s=1}^{n} h_{\psi(t)}(\psi'(t), X_{s\,\psi(t)}) X_{s\,e} \\ &\leftrightarrow F(t) = \theta\bigg(\psi_*\bigg(\frac{\partial}{\partial t}\bigg)\bigg) \in \mathfrak{g}. \end{aligned} \tag{4.5}$$

4.2.

We have that

$$\delta\alpha = -F'(t), \tag{4.6}$$

since we have $\delta\alpha = -e_1(\alpha(e_1)) = -e_1(F(t)) = -F'(t)$.

Therefore, we have $\psi : (\mathbb{R}, g_0) \to (G, h)$ is *harmonic* if and only if

$$\begin{aligned} \delta\alpha = 0 \quad &\Longleftrightarrow \quad F' = 0 \\ &\Longleftrightarrow \quad \alpha = X \otimes dt \quad (\text{for some } X \in \mathfrak{g}) \\ &\Longleftrightarrow \quad \psi : \mathbb{R} \to (G, h), \text{ is a } \textit{geodesic}, \end{aligned} \tag{4.7}$$

since

$$F(t) = \theta(\psi'(t)) = L_{\psi(t)\,*}^{-1}\psi'(t), \tag{4.8}$$

we have

$$\psi'(t) = L_{\psi(t)*}X = X_{\psi(t)}, \tag{4.9}$$

for some $X \in \mathfrak{g}$ which yields that

$$\psi(t) = x\exp(tX).$$

Therefore, *any geodesic through* $\psi(0) = x$ *is given by*

$$\psi(t) = x\exp(tX),\ (t \in \mathbb{R}) \tag{4.10}$$

for some $X \in \mathfrak{g}$.

On the other hand, we want to determine a *biharmonic curve* $\psi : (\mathbb{R}, g_0) \to (G, h)$. By (4.6), we have

$$\delta d\delta\alpha = -\frac{\partial^2}{\partial t^2}(-F'(t)) = F^{(3)}(t), \tag{4.11}$$

and

$$\mathrm{Trace}_g[\alpha, d\delta\alpha] = \left[\alpha\left(\frac{\partial}{\partial t}\right), d\delta\alpha\left(\frac{\partial}{\partial t}\right)\right] = [F(t), F''(t)], \tag{4.12}$$

so by (4.9), (4.10), and (3.16) in Corollary 3.5, $\psi : (\mathbb{R}, g_0) \to (G, h)$ is *biharmonic* if and only if

$$F^{(3)} - [F(t), F''(t)] = 0. \tag{4.13}$$

4.3.

For a C^∞ curve $\psi : \mathbb{R} \to G$, let $\psi(t) := \exp X(t)$, where $X(t) \in \mathfrak{g}$. Then,

$$F(t) = \theta\left(\psi_*\left(\frac{\partial}{\partial t}\right)\right), \quad \psi_*\left(\frac{\partial}{\partial t}\right) \in T_{\psi(t)}G, \tag{4.14}$$

and by the following formula (cf. [10, p. 95])

$$\exp_{*\,X} = L_{\exp X\,*\,e} \circ \frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X} \quad (X \in \mathfrak{g}),$$

we have

$$\begin{aligned} \psi_*\left(\frac{\partial}{\partial t}\right) &= \exp_{*\,X(t)} X'(t) \\ &= L_{\exp X(t)\,*\,e}\left(\sum_{n=0}^{\infty} \frac{(-\operatorname{ad} X(t))^n}{(n+1)!}(X'(t))\right). \end{aligned} \tag{4.15}$$

Since θ is a left invariant 1-form, we have

$$F(t) = \sum_{n=0}^{\infty} \frac{(-\operatorname{ad} X(t))^n}{(n+1)!}(X'(t)). \tag{4.16}$$

4.4.

The initial value problem

$$\begin{cases} F^{(3)}(t) = [F(t), F''(t)], \\ F(0) = B_0, \quad F'(0) = B_1, \quad F''(0) = B_2, \end{cases} \tag{4.17}$$

for every $B_i \in \mathfrak{g}$ $(i = 0, 1, 2)$, has a unique solution $F(t)$. Assume that $X(t)$ is a real analytic curve in t, and $X(0) = 0$. Then, $F(t)$ is also real analytic in t, and we can write as

$$X(t) = \sum_{n=1}^{\infty} A_n\, t^n, \quad F(t) = \sum_{n=0}^{\infty} B_n\, t^n. \tag{4.18}$$

By (4.16), we have

$$F(t) = X'(t) + \frac{1}{2}[-X(t), X'(t)] + \frac{1}{6}[-X(t), [-X(t), X'(t)]] + \sum_{n=3}^{\infty} \frac{(-\operatorname{ad} X(t))^n}{(n+1)!}(X'(t)). \tag{4.19}$$

Since $X'(t) = \sum_{m=0}^{\infty} A_{m+1}(m+1)\, t^m$, we have

$$\frac{1}{2}[-X(t), X'(t)] = -\frac{1}{2}[A_1, A_2]t^2 + O(t^3),$$

and

$$\frac{1}{6}[-X(t), [-X(t), X'(t)]] = O(t^3),$$

so that we have

$$F(t) = A_1 + 2A_2 t + \left(3A_3 - \frac{1}{2}[A_1, A_2]\right)t^2 + O(t^3).$$

Continuing this process, we have

$$\begin{cases} B_0 = A_1, \\ B_1 = 2A_2, \\ B_2 = 3A_3 - \frac{1}{2}[A_1, A_2], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ B_n = (n+1)A_{n+1} + G_n(A_1, \ldots, A_n), \end{cases} \tag{4.20}$$

where $G_n(x_1, \ldots, x_n)$ is a polynomial in $(x_1, \ldots, x_n)$. Notice that for arbitrary given data (B_0, B_1, B_2), all B_n $(n = 0, 1, \ldots)$ are determined, and by using (4.20), one can determine all A_n $(n = 1, 2, \ldots)$, uniquely. Therefore, by summarizing the above, we obtain

Theorem 4.1 *For every C^∞ curve $\psi : \mathbb{R} \to G$, $\psi(t) = \exp X(t)$ $(X(t) \in \mathfrak{g})$, and*

$$\alpha\left(\frac{\partial}{\partial t}\right) = F(t) = \sum_{n=0}^{\infty} \frac{(-\operatorname{ad} X(t))^n}{(n+1)!}(X'(t)). \tag{4.21}$$

(1) *$\psi : (\mathbb{R}, g_0) \to (G, h)$ is biharmonic if and only if*

$$F^{(3)}(t) = [F(t), F''(t)]. \tag{4.22}$$

(2) *The initial value problem*

$$\begin{cases} F^{(3)}(t) = [F(t), F''(t)], \\ F(0) = B_0,\ F'(0) = B_1,\ F''(0) = B_2, \end{cases} \tag{4.23}$$

has a unique solution $F(t)$ for arbitrary given data (B_0, B_1, B_2) in $\mathfrak{g}$.

(3) *Assume that $\psi : (\mathbb{R}, g_0) \to (G, h)$ is a real analytic biharmonic curve with $\psi(0) = e$. Then, $\psi(t)$ is uniquely determined by $F(0) = B_0$, $F'(0) = B_1$, and $F''(0) = B_2$.*

Example If G is abelian, let us consider a C^∞ curve $\psi : \mathbb{R} \to G$ given by $\psi(t) = \exp X(t)$. Then, $F(t) = X'(t)$, and $\psi : (\mathbb{R}, g_0) \to (G, h)$ is biharmonic if and only if $F^{(3)}(t) = X^{(4)}(t) = 0$. Then, $X(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3$. Thus, every biharmonic curve $\psi : (\mathbb{R}, g_0) \to (G, h)$ with $\psi(0) = e$ is given by

$$\psi(t) = \exp(A_1 t + A_2 t^2 + A_3 t^3).$$

4.5.

Now we will solve the ODE (4:22) for a biharmonic isometric immersion $\psi : (\mathbb{R}, g_0) \to G$ and a $\mathfrak{g}$-valued curve $F(t)$ in the case of $\mathfrak{g} = \mathfrak{s}u(2)$. Let $G = SU(2)$ with the bi-invariant Riemannian metric h which corresponds to the following $\operatorname{Ad}(SU(2))$-invariant inner product $\langle\ ,\ \rangle$ on

$$\begin{aligned} &\mathfrak{g} = \mathfrak{s}u(2) = \{X \in M(2, \mathbb{C}); X + {}^{t}\overline{X} = 0, \operatorname{Tr}(X) = 0\}, \\ &\langle X, Y\rangle = -2\operatorname{Tr}(XY) \quad (X; Y \in \mathfrak{s}u(2)). \end{aligned}$$

If we choose

$$X_1 = \begin{pmatrix} \frac{\sqrt{-1}}{2} & 0 \\ 0 & -\frac{\sqrt{-1}}{2} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & \frac{\sqrt{-1}}{2} \\ \frac{\sqrt{-1}}{2} & 0 \end{pmatrix},$$

then $\{X_1, X_2, X_3\}$ is an orthonormal basis of $(\mathfrak{su}(2), \langle\ ,\ \rangle)$, and satisfies the Lie bracket relations:

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.$$

Thus, the ODE (4.22) becomes

$$\begin{cases} {y_1}^{(3)} = y_2\, {y_3}'' - y_3\, {y_2}'', \\ {y_2}^{(3)} = y_3\, {y_1}'' - y_1\, {y_3}'', \\ {y_3}^{(3)} = y_1\, {y_2}'' - y_2\, {y_1}'', \end{cases} \tag{4.24}$$

which is equivalent to

$$\mathbf{y}^{(3)} = \mathbf{y} \times \mathbf{y}'', \tag{4.25}$$

where $\mathbf{y} := {}^{t}(y_1, y_2, y_3) \in \mathbb{R}^3$, and $\mathbf{a} \times \mathbf{b}$ stands for the vector cross product in $\mathbb{R}^3$. Notice here that $\mathfrak{g}$ is non-abelian, but our equation (4.22) turns to the vector equation (4.26) depending on the time t of the Euclidean space $\mathbb{R}^3$ by identifying $\mathfrak{g} \ni \sum_{i=1}^{3} y_i\, X_i \mapsto (y_1, y_2, y_3) \in \mathbb{R}^3$.

Then, the ODE (4.25) can be solved as follows:

Let $\mathbf{x}(s) = {}^{t}(x_1(s), x_2(s), x_3(s))$ be a C^∞ curve in $\mathbb{R}^3$ with arc length parameter s, and then

$$\mathbf{y}(s) = \mathbf{x}'(s) = \mathbf{e}_1(s).$$

Let $\{\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s))\}$ be the Frenet frame field along $\mathbf{x}(s)$. Recall the Frenet-Serret formula:

$$\begin{cases} {\mathbf{e}_1}' = & \kappa\, \mathbf{e}_2 & \\ {\mathbf{e}_2}' = -\kappa\, \mathbf{e}_1 & & +\, \tau\, \mathbf{e}_3 \\ {\mathbf{e}_3}' = & -\tau\, \mathbf{e}_2 & \end{cases}$$

where κ and τ are the curvature and torsion of $\mathbf{x}(s)$, respectively. Then, we have

$$\begin{cases} \mathbf{y}' = \kappa\, \mathbf{e}_2 \\ \mathbf{y}'' = -\kappa^2\, \mathbf{e}_1 + \kappa'\, \mathbf{e}_2 + \kappa\, \tau\, \mathbf{e}_3 \\ \mathbf{y}''' = -3\kappa\kappa'\, \mathbf{e}_1 + (\kappa'' - \kappa^3 - \kappa\tau^2)\, \mathbf{e}_2 + (2\kappa'\tau + \kappa\tau')\, \mathbf{e}_3. \end{cases} \tag{4.26}$$

Thus, (4.24) is equivalent to

$$\begin{aligned} &-3\kappa\kappa'\, \mathbf{e}_1 + (\kappa'' - \kappa^3 - \kappa\tau^2)\, \mathbf{e}_2 + (2\kappa'\tau + \kappa\tau')\, \mathbf{e}_3 \\ &\quad = \mathbf{e}_1 \times (-\kappa^2\, \mathbf{e}_1 + \kappa'\, \mathbf{e}_2 + \kappa\tau\, \mathbf{e}_3) \\ &\quad = -\kappa\tau\, \mathbf{e}_2 + \kappa'\, \mathbf{e}_3 \end{aligned} \tag{4.27}$$

which is equivalent to

$$\begin{cases} -3\kappa\kappa' = 0 \\ \kappa'' - \kappa^3 - \kappa\tau^2 = -\kappa\tau \\ 2\kappa'\tau + \kappa\tau' = \kappa'. \end{cases} \tag{4.28}$$

Then, the first equation of (4.28) turns out that $(\kappa^2)' = 0$, that is, κ^2 is constant, i.e., $\kappa \equiv 0$, or $\kappa \equiv \kappa_0 \neq 0$. In the case that $\kappa \equiv 0$, the solution of (4.28), $\mathbf{x}(s)$, is a line in $\mathbb{R}^3$.

For the case that $\kappa \equiv \kappa_0 \neq 0$, the only solution of (4.24) is

$$\begin{cases} \kappa \equiv \kappa_0 \neq 0, \\ \tau \equiv \tau_0, \quad \text{and} \\ {\kappa_0}^2 = \tau_0(1 - \tau_0), \end{cases} \tag{4.29}$$

and the unique solution of (4.25) is given by

$$\mathbf{x}(s) = \begin{pmatrix} x_1(s) \\ x_2(s) \\ x_3(s) \end{pmatrix} = \begin{pmatrix} a\, \cos \frac{s}{\sqrt{a^2+1}} + b \\ a\, \sin \frac{s}{a^2+1} + b \\ \frac{s}{\sqrt{a^2+1}} + b \end{pmatrix} \tag{4.30}$$

for some positive constant $a > 0$ and some constant b. Thus, $F(s)$ is given as follows:

$$\begin{aligned}F(s) = \mathbf{x}'(s) &= \sum_{i=1}^{3} x_i{}'(s)\, X_i \\ &= \Big(- \frac{a}{\sqrt{a^2+1}} \sin \frac{s}{\sqrt{a^2+1}} \Big) X_1 + \Big(\frac{a}{\sqrt{a^2+1}} \cos \frac{s}{\sqrt{a^2+1}} \Big) X_2 \\ &\quad + \Big(\frac{1}{\sqrt{a^2+1}} \Big) X_3, \end{aligned} \tag{4.31}$$

for any constant $a > 0$. Conversely, it is easy to see that every such $F(s)$ in (4.31) is a solution of (4.22): $F^{(3)} = [F(s), F''(s)]$.

Remark It is still difficult to determine $X(t)$ to satisfy (4.21):

$$F(t) = \sum_{n=0}^{\infty} \frac{(-\operatorname{ad} X(t))^n}{(n+1)!} (X'(t)),$$

in the case of $\mathfrak{su}(2)$.

5. Biharmonic maps from an open domain in $\mathbb{R}^2$

In this section, we consider a biharmonic map $\psi : (\mathbb{R}^2, g) \supset \Omega \to (G, h)$. Here, we assume that G is a linear compact Lie group, i.e., G is a subgroup of the unitary group $U(N)(\subset GL(N, \mathbb{C}))$ of degree N with a bi-invariant Riemannian metric h on G. Let $\mathfrak{g}$ be the Lie algebra of G which is a Lie subalgebra of the Lie algebra $\mathfrak{u}(N)$ of $U(N)$. The Riemannian metric g on $\mathbb{R}^2$ is a conformal metric which is given by $g = \mu^2 g_0$ with a C^∞ positive function μ on Ω and $g_0 = dx \cdot dx + dy \cdot dy$, where (x, y) is the standard coordinate on $\mathbb{R}^2$.

Let $\psi : \Omega \ni (x, y) \mapsto \psi(x, y) = (\psi_{ij}(x, y)) \in U(N)$ a C^∞ map. Let us consider

$$\frac{\partial \psi}{\partial x} := \Big(\frac{\partial \psi_{ij}}{\partial x} \Big), \qquad \frac{\partial \psi}{\partial y} := \Big(\frac{\partial \psi_{ij}}{\partial y} \Big).$$

Then,

$$A_x := \psi^{-1} \frac{\partial \psi}{\partial x}, \qquad A_y := \psi^{-1} \frac{\partial \psi}{\partial y} \tag{5.1}$$

are $\mathfrak{g}$-valued C^∞ functions on Ω. It is known that, for two given $\mathfrak{g}$-valued 1-forms A_x and A_y on Ω, there exists a C^∞ mapping $\psi : \Omega \to G$ satisfying the equations (5.1) if the *integrability condition* holds:

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} + [A_x, A_y] = 0. \tag{5.2}$$

The pull back of the Maurer-Cartan form θ by ψ is given by

$$\begin{aligned}\alpha := \psi^*\theta = \psi^{-1}d\psi &= \psi^{-1}\frac{\partial \psi}{\partial x}dx + \psi^{-1}\frac{\partial \psi}{\partial y}dy \\ &= A_x\,dx + A_y\,dy,\end{aligned} \tag{5.3}$$

which is a $\mathfrak{g}$-valued 1-form on Ω.

Recall that the codifferential $\delta\alpha$ of a $\mathfrak{g}$-valued 1-form $\alpha = A_x\,dx + A_y\,dy$, where $A_x = \psi^{-1}(\partial\psi/\partial x)$ and $A_y = \psi^{-1}(\partial\psi/\partial y)$, is given by

$$\delta\alpha = -\mu^{-2}\left\{\frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y\right\}. \tag{5.4}$$

Then, we have the following well known facts:

Lemma 5.1 *We have*

$$\delta\alpha = -\mu^{-2}\left\{\frac{\partial}{\partial x}\left(\psi^{-1}\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\psi^{-1}\frac{\partial\psi}{\partial y}\right)\right\} \tag{5.5}$$

$$= -\mu^{-2}\left\{\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right\}. \tag{5.6}$$

Therefore, the following three statements are equivalent:

(i) $\psi : (\Omega, g) \to (G, h)$ *is harmonic*,

(ii) $\delta\alpha = 0,$ (5.7)

(iii) $\dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} = 0.$ (5.8)

Next, calculate the Laplacian Δ_g of $(\mathbb{R}^2, g)$ for $g = \mu^2\,g_0$. We obtain

$$\begin{aligned}\Delta_g &= -\sum_{i,j=1}^{2} g^{ij}\left(\frac{\partial^2}{\partial x^i\,\partial x^j} - \sum_{k=1}^{2}\Gamma_{ij}^k \frac{\partial}{\partial x^k}\right)\\ &= -\mu^{-2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \end{aligned} \tag{5.9}$$

Thus we have

$$\begin{aligned}\delta d\delta\alpha &= \Delta_g(\delta\alpha)\\ &= \mu^{-2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[\mu^{-2}\left\{\frac{\partial}{\partial x}\left(\psi^{-1}\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\psi^{-1}\frac{\partial\psi}{\partial y}\right)\right\}\right]\\ &= \mu^{-2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[\mu^{-2}\left\{\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right\}\right]\\ &= -\mu^{-2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\delta\alpha). \end{aligned} \tag{5.10}$$

On the other hand, by taking an orthonormal local frame field $\{e_1, e_2\}$ of $(\mathbb{R}^2, g)$, as $e_1 = \mu^{-1}(\partial/\partial x)$, $e_2 = \mu^{-1}(\partial/\partial y)$, we have

$$\begin{aligned}\mathrm{Trace}_g([\alpha, d\delta\alpha]) &= [\alpha(e_1), d\delta\alpha(e_1)] + [\alpha(e_2), d\delta\alpha(e_2)]\\ &= -\mu^{-2}\left[A_x, \frac{\partial}{\partial x}\left(\mu^{-2}\left\{\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right\}\right)\right]\\ &\quad - \mu^{-2}\left[A_y, \frac{\partial}{\partial y}\left(\mu^{-2}\left\{\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right\}\right)\right]\\ &= \mu^{-2}\left[A_x, \frac{\partial}{\partial x}(\delta\alpha)\right] + \mu^{-2}\left[A_y, \frac{\partial}{\partial y}(\delta\alpha)\right]. \end{aligned} \tag{5.11}$$

By (5.10) and (5.11), we obtain

$$\begin{aligned}&\delta d\delta\alpha + \mathrm{Trace}_g([\alpha, d\delta\alpha])\\ &\quad = -\mu^{-2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\delta\alpha) + \mu^{-2}\left[A_x, \frac{\partial}{\partial x}(\delta\alpha)\right] + \mu^{-2}\left[A_y, \frac{\partial}{\partial y}(\delta\alpha)\right]\\ &\quad = -\mu^{-2}\left\{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\delta\alpha) - \frac{\partial}{\partial x}[A_x, \delta\alpha] - \frac{\partial}{\partial y}[A_y, \delta\alpha]\right\}, \end{aligned} \tag{5.12}$$

where in the last equation in (5.11), we only notice that

$$\begin{aligned}
&\frac{\partial}{\partial x}[A_x,\delta\alpha]+\frac{\partial}{\partial y}[A_y,\delta\alpha]\\
&=\left[\frac{\partial}{\partial x}A_x,\delta\alpha\right]+\left[A_x,\frac{\partial}{\partial x}(\delta\alpha)\right]+\left[\frac{\partial}{\partial y}A_y,\delta\alpha\right]+\left[A_y,\frac{\partial}{\partial y}(\delta\alpha)\right]\\
&=\left[\frac{\partial}{\partial x}A_x+\frac{\partial}{\partial y}A_y,\delta\alpha\right]+\left[A_x,\frac{\partial}{\partial x}(\delta\alpha)\right]+\left[A_y,\frac{\partial}{\partial y}(\delta\alpha)\right]\\
&=[-\mu^{-2}\delta\alpha,\delta\alpha]+\left[A_x,\frac{\partial}{\partial x}(\delta\alpha)\right]+\left[A_y,\frac{\partial}{\partial y}(\delta\alpha)\right]\\
&=\left[A_x,\frac{\partial}{\partial x}(\delta\alpha)\right]+\left[A_y,\frac{\partial}{\partial y}(\delta\alpha)\right].
\end{aligned}$$

Thus, we have

Theorem 5.2 *Let Ω be an open subset of $\mathbb{R}^2$, $g=\mu^2 g_0$, a Riemannian metric conformal to the standard metric g_0 on Ω with a C^∞ positive function μ on Ω, and $\psi:\Omega\to G$, a C^∞ map of Ω into a compact linear Lie group (G,h) with bi-invariant Riemannian metric h. Then,*

(1) *The 1-form α satisfies $d\alpha+(1/2)[\alpha\wedge\alpha]=0$ which is equivalent to*

$$\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}+[A_x,A_y]=0. \tag{5.13}$$

(2) *The following three are equivalent*:

(i) $\psi:(\Omega,g)\to(G,h)$ *is harmonic,*

(ii) $\delta\alpha=0,$ (5.14)

(iii) $\frac{\partial}{\partial x}A_x+\frac{\partial}{\partial y}A_y=0.$ (5.15)

(3) *The following three are equivalent*:

(i) $\psi:(\Omega,g)\to(G,h)$ *is biharmonic,*

(ii) $\delta d\delta\alpha+\mathrm{Trace}_g([\alpha,d\delta\alpha])=0,$ (5.16)

$$\text{(iii)} \qquad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\delta\alpha)-\frac{\partial}{\partial x}[A_x,\delta\alpha]-\frac{\partial}{\partial y}[A_y,\delta\alpha]=0. \tag{5.17}$$

(4) *Let us consider two* $\mathfrak{g}$*-valued* 1*-forms* β *and* Θ *on* Ω*, defined by*

$$\beta := [A_x,\delta\alpha]dx+[A_y,\delta\alpha]dy, \tag{5.18}$$

$$\Theta := d\delta\alpha-\beta, \tag{5.19}$$

respectively. Then, $\psi:(\Omega,g)\to(G,h)$ *is biharmonic if and only if*

$$\delta\Theta = 0. \tag{5.20}$$

Proof. (1) is clear. We see already (2) and (3). For (4), we only have to see that (5.17) is equivalent to

$$0=-\Delta_g(\delta\alpha)+\delta\beta=-\delta(d\delta\alpha-\beta)=-\delta\Theta \tag{5.21}$$

where

$$\begin{aligned}\Theta &:= d\delta\alpha-\beta\\ &=\frac{\partial}{\partial x}(\delta\alpha)dx+\frac{\partial}{\partial y}(\delta\alpha)dy-[A_x,\delta\alpha]dx-[A_y,\delta\alpha]dy\\ &=\left\{\frac{\partial}{\partial x}(\delta\alpha)-[A_x,\delta\alpha]\right\}dx+\left\{\frac{\partial}{\partial y}(\delta\alpha)-[A_y,\delta\alpha]\right\}dy.\end{aligned} \tag{5.22}$$

□

6. Complexification of the biharmonic map equation

We use the complex coordinate $z=x+iy$ ($i=\sqrt{-1}$) in Ω, and we put $A_z=(1/2)(A_x-i\,A_y)$ and $A_{\overline{z}}=(1/2)(A_x+i\,A_y)$ which are $\mathfrak{g}^{\mathbb{C}}$-valued functions with $A_{\overline{z}}=\overline{A_z}$. Then, it is well known that

$$\begin{aligned}\frac{\partial}{\partial\overline{z}}A_z+\frac{\partial}{\partial z}A_{\overline{z}}&=\frac{1}{2}\left\{\frac{\partial}{\partial x}A_x+\frac{\partial}{\partial y}A_y\right\},\\ \frac{\partial}{\partial z}A_{\overline{z}}-\frac{\partial}{\partial\overline{z}}A_z+[A_z,A_{\overline{z}}]&=\frac{i}{2}\left\{\frac{\partial}{\partial x}A_y-\frac{\partial}{\partial y}A_x+[A_x,A_y]\right\},\end{aligned} \tag{6.1}$$

and also

$$\alpha = A_x dx + A_y dy = A_z dz + A_{\overline{z}} d\overline{z},$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}},$$

$$\delta\alpha = -\mu^{-2}\left(\frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y\right) = -2\mu^{-2}\left(\frac{\partial}{\partial \overline{z}}A_z + \frac{\partial}{\partial z}A_{\overline{z}}\right). \tag{6.2}$$

Then, the condition (5.20) is equivalent to

$$\delta\widetilde{\Theta} = 0, \tag{6.3}$$

where

$$\widetilde{\Theta} := \left\{\frac{\partial}{\partial z}(\delta\alpha) - [A_z, \delta\alpha]\right\}dz + \left\{\frac{\partial}{\partial \overline{z}}(\delta\alpha) - [A_{\overline{z}}, \delta\alpha]\right\}dz. \tag{6.4}$$

The integrability condition (5.13) is equivalent to

$$\frac{\partial}{\partial z}A_{\overline{z}} - \frac{\partial}{\partial \overline{z}}A_z + [A_z, A_{\overline{z}}] = 0 \tag{6.5}$$

7. Determination of biharmonic maps

In this section, we want to show how to determine all the biharmonic maps of (Ω, g) into a compact Lie group (G, h) where $g = \mu^2 g_0$ with a positive C^∞ function on Ω and h is a bi-invariant Riemannian metric on G. Our method to obtain all the biharmonic maps can be divided into three steps:

(*The first step*) We first solve the equation:

$$\frac{\partial}{\partial \overline{z}}B_z + \frac{\partial}{\partial z}B_{\overline{z}} = 0 \tag{7.1}$$

Notice that, if these B_z and $B_{\overline{z}}$ satisfy furthermore, the integrability condition

$$\frac{\partial}{\partial z}B_{\overline{z}} - \frac{\partial}{\partial \overline{z}}B_z + [B_z, B_{\overline{z}}] = 0, \tag{7.2}$$

then, there exists a harmonic map $\Psi : (\Omega, g) \to (G, h)$ such that

$$\begin{cases} \Phi^{-1}\dfrac{\partial \Psi}{\partial z} = B_z, \\ \Phi^{-1}\dfrac{\partial \Phi}{\partial \overline{z}} = B_{\overline{z}}, \end{cases} \tag{7.3}$$

and the converse is true.

(*The second step*) For such two $\mathfrak{g}^{\mathbb{C}}$-valued functions B_z and $B_{\overline{z}}$ on Ω satisfying (7.1) not necessarily satisfying (7.2), we should detect two $\mathfrak{g}^{\mathbb{C}}$-valued functions A_z and $A_{\overline{z}}$ on Ω satisfying that

$$\begin{cases} \dfrac{\partial}{\partial z}\Big(-2\mu^{-2}\Big(\dfrac{\partial A_z}{\partial \overline{z}} + \dfrac{\partial A_{\overline{z}}}{\partial z}\Big)\Big) - \Big[A_z, -2\mu^{-2}\Big(\dfrac{\partial A_z}{\partial \overline{z}} + \dfrac{\partial A_{\overline{z}}}{\partial z}\Big)\Big] = B_z, \\ \dfrac{\partial}{\partial \overline{z}}\Big(-2\mu^{-2}\Big(\dfrac{\partial A_z}{\partial \overline{z}} + \dfrac{\partial A_{\overline{z}}}{\partial z}\Big)\Big) - \Big[A_{\overline{z}}, -2\mu^{-2}\Big(\dfrac{\partial A_z}{\partial \overline{z}} + \dfrac{\partial A_{\overline{z}}}{\partial z}\Big)\Big] = B_{\overline{z}}, \\ \dfrac{\partial}{\partial z}A_{\overline{z}} - \dfrac{\partial}{\partial \overline{z}}A_z + [A_z, A_{\overline{z}}] = 0. \end{cases} \tag{7.4}$$

(*The third step*) Finally, for the above $\mathfrak{g}^{\mathbb{C}}$-valued functions A_z and $A_{\overline{z}}$ on Ω satisfying (7.4) and $a \in G$, there exists a C^∞ mapping $\psi : \Omega \to G$ satisfying that

$$\begin{cases} \psi(x_0, y_0) = a, \\ \psi^{-1}\dfrac{\partial \psi}{\partial z} = A_z, \\ \psi^{-1}\dfrac{\partial \psi}{\partial \overline{z}} = A_{\overline{z}}. \end{cases} \tag{7.5}$$

Then, $\psi : (\Omega, g) \to (G, h)$ is a *biharmonic map* due to (5.20), (6.1) and (7.4), and conversely, every biharmonic map $\psi : (\Omega, g) \to (G, h)$ could be obtained in this way. To do the these procedures rigorously, let us define

Definition 7.1

(1) Let us define the four sets Λ, Λ_1, Λ_2, and Λ_0 :

- Let Λ be the set of all $\mathfrak{g}$-valued two functions (A_x, A_y) on Ω, (or all $\mathfrak{g}^{\mathbb{C}}$-valued two functions $(A_z, A_{\overline{z}})$ on Ω with $A_{\overline{z}} = \overline{A_z}$,

- let Λ_1, the set of $(A_x, A_y) \in \Lambda$ which satisfy the harmonic map equation (5.12) (or (7.1)),
- let Λ_2, the set of $(A_x, A_y) \in \Lambda$ which satisfy the biharmonic map equation (5.17) (or (6.1)), and
- let Λ_0, the set of $(A_x, A_y) \in \Lambda$ which satisfy the integrability condition (5.13), (or (6.3)), respectively.

(2) Let us define two sets Ξ and Ξ_1 :

- Let Ξ be the set of all $\mathfrak{g}$-valued two real analytic functions (B_x, B_y) on Ω (or $\mathfrak{g}^{\mathbb{C}}$-valued two real analytic functions $(B_z, B_{\overline{z}})$ on Ω with $B_{\overline{z}} = \overline{B_z}$), and
- let Ξ_1, the set of all $(B_x, B_y) = (B_z, B_{\overline{z}}) \in \Xi$ satisfying the harmonic map equation (7.1), respectively.

Definition 7.2 Let us define two C^∞ mappings Φ_i $(i = 1, 2)$ of Λ into Ξ by

$$\begin{aligned}
&\Phi_1(A_x, A_y) \\
&:= \left(\frac{\partial}{\partial x}\left(-\mu^{-2}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right)\right) - \left[A_x, -\mu^{-2}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right)\right], \right. \\
&\qquad \left. \frac{\partial}{\partial y}\left(-\mu^{-2}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right)\right) - \left[A_y, -\mu^{-2}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right)\right] \right),
\end{aligned} \tag{7.6}$$

and also

$$\begin{aligned}
&\Phi_2(A_x, A_y) \\
&:= \left(-\mu^{-2}\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial}{\partial y}[A_x, A_y] \right) \right. \\
&\qquad - \frac{\partial \mu^{-2}}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) - \left[A_x, -\mu^{-2}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right)\right], \\
&\qquad -\mu^{-2}\left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} - \frac{\partial}{\partial x}[A_x, A_y] \right) \\
&\qquad \left. - \frac{\partial \mu^{-2}}{\partial y}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) - \left[A_y, -\mu^{-2}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right)\right] \right),
\end{aligned} \tag{7.7}$$

respectively.

Then, we obtain

Theorem 7.3 *Assume that Ω be a simply connected open domain in $\mathbb{R}^2$, and μ is a positive real analytic function on Ω. Then, we have*:

(1) *For every $(B_x, B_y) = (B_z, B_{\overline{z}}) \in \Xi$ there exists $(A_x, A_y) = (A_z, A_{\overline{z}}) \in \Lambda$ such that $\Phi_2(A_x, A_y) = (B_x, B_y)$ (or $\Phi_2(A_z, A_{\overline{z}}) = (B_z, B_{\overline{z}})$). The solution $(A_x, A_y) = (A_z, A_{\overline{z}})$ is uniquely determined by the initial data $A_x(x_0, y)$, $A_y(x_0, y)$, $(\partial A_x/\partial x)(x_0, y)$ and $(\partial A_y/\partial x)(x_0, y)$, $(x_0, y) \in \Omega$.*
(2) *$\Phi_1 = \Phi_2$ on Λ_0,*
(3) *${\Phi_1}^{-1}(\Xi_1) = \Lambda_2$, and $\Phi_1(\Lambda_2 \cap \Lambda_0) = \Phi_2(\Lambda_2 \cap \Lambda_0) = \Xi_1$.*

Proof. For (1), by definition of Φ_2, that $\Phi_2(A_x, A_y) = (B_x, B_y)$ is equivalent to the following two equations:

$$\begin{aligned}\frac{\partial^2 A_x}{\partial x^2} = &-\frac{\partial^2 A_x}{\partial y^2} + \frac{\partial}{\partial y}[A_x, A_y] \\ &- \mu^2 \frac{\partial \mu^{-2}}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right) - \mu^2\left[A_x, -\mu^{-2}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right)\right] \\ &- \mu^2 B_x, \qquad (7.8)\end{aligned}$$

and also

$$\begin{aligned}\frac{\partial^2 A_y}{\partial x^2} = &-\frac{\partial^2 A_y}{\partial y^2} + \frac{\partial}{\partial x}[A_x, A_y] \\ &- \mu^2 \frac{\partial \mu^{-2}}{\partial y}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right) - \mu^2\left[A_y, -\mu^{-2}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right)\right] \\ &- \mu^2 B_y. \qquad (7.9)\end{aligned}$$

Notice that the system of (7.8) and (7.9) satisfies all the conditions of the theorem of Cauchy-Kovalevskaya when $n_i = 2$ $(i = 1, 2)$ (cf. [7, p. 1305, 429 B], [14, p. 224], [11, p. 181])

Theorem 7.4 (Cauchy-Kovalevskaya) *Let us consider the following Cauchy problem of unknown N functions $u_i(t, x)$ $(i = 1, \ldots, N)$ in t and $x = (x_1, \ldots, x_m)$,*

$$\begin{cases} \dfrac{\partial^{n_i} u_i}{\partial t^{n_i}} = F_i(t, x, D_t^k D_x^p u_j) \quad (i = 1, \ldots, N), \\ \dfrac{\partial^k u_i}{\partial t^k}(t_0, x) = \varphi_i^k(x) \qquad (0 \leq k \leq n_i - 1; i = 1, \ldots, N), \end{cases} \tag{7.10}$$

where, for $p = (p_1, \ldots, p_m)$, $|p| = p_1 + \cdots + p_m$, $D_t^k D_x^p := (\partial^k/\partial t^k) \cdot(\partial^{|p|}/\partial {x_1}^{p_1} \cdots \partial {x_m}^{p_m})$ and in the right hand side of the first equation of (7.10)*, k and p satisfy*

$$k < n_j \quad \textit{and} \quad k + |p| \leq n_j \quad (j = 1, \ldots, N).$$

Assume that each F_i and φ_i^k are real analytic functions. Then, there exists a real analytic solution u_i $(i = 1, \ldots, N)$ of (7.10) *and it is unique in the class of real analytic functions.*

Then, for each $(B_x, B_y) \in \Xi$, there exists a real analytic solution (A_x, A_y) of the Cauchy problem (7.8) and (7.9) with the initial condition:

$$\begin{cases} \left(\dfrac{\partial A_x}{\partial x}\right)(x_0, y) = f_1(y), \quad A_x(x_0, y) = f_0(y), \\ \left(\dfrac{\partial A_y}{\partial x}\right)(x_0, y) = g_1(y), \quad A_y(x_0, y) = g_0(y), \end{cases} \tag{7.11}$$

and the real analytic solution (A_x, A_y) is unique for real analytic functions f_i and g_i $(i = 0, 1)$. By taking this process at each point (x_0, y_0) in Ω, we have a real analytic solution (A_x, A_y) of (7.8) and (7.9) in an open neighborhood of (x_0, y_0). Then, by the uniqueness theorem of the continuation of a real analytic function on a simply connected domain Ω, we have a solution (A_x, A_y) of (7.8) and (7.9) on Ω. We have (1).

For (2), we have to see $\Phi_1(A_x, A_y) = \Phi_2(A_x, A_y)$ for every $(A_x, A_y) \in \Lambda_0$, which follows from that

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\mu^{-2}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right)\right) \\ &\quad = \mu^{-2}\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y}\right) + \frac{\partial \mu^{-2}}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right) \\ &\quad = \mu^{-2}\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial}{\partial y}[A_x, A_y]\right) + \frac{\partial \mu^{-2}}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right), \end{aligned} \tag{7.12}$$

because of (5.13) and it is a similar for $(\partial/\partial y)(\mu^{-2}(\partial A_x/\partial x+\partial A_y/\partial y))$, so that we have (2).

For (3), due to (2), we only have to see ${\Phi_1}^{-1}(\Xi_1)=\Lambda_2$ which is equivalent to that:

for all $(B_x,B_y)\in\Xi$, *exists a unique* $(A_x,A_y)\in\Lambda_2$ *such that* $\Phi_1(A_x,A_y)=(B_x,B_y)$, *and vice versa.*

But, that $(B_x,B_y)=(B_z,B_{\overline{z}})\in\Xi_1$ means that it satisfies the harmonic map equation (7.1). On the other hand, $\Phi_1(A_x,A_y)=(B_x,B_y)$ means that $\Phi_1(A_z,A_{\overline{z}})=(B_z,B_{\overline{z}})$ which is equivalent to that the first two equations of (7.4) hold by definition of Φ_1, and notice here that $\Phi_1(A_x,A_y)=(B_x,B_y)$ is equivalent to the two following equations

$$\frac{\partial}{\partial x}\Big(-\mu^{-2}\Big(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}\Big)\Big)-\Big[A_x,-\mu^{-2}\Big(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}\Big)\Big]=B_x, \tag{7.13}$$

$$\frac{\partial}{\partial y}\Big(-\mu^{-2}\Big(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}\Big)\Big)-\Big[A_y,-\mu^{-2}\Big(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}\Big)\Big]=B_y, \tag{7.14}$$

which are also equivalent to

$$\frac{\partial}{\partial z}\Big(-2\mu^{-2}\Big(\frac{\partial A_z}{\partial \overline{z}}+\frac{\partial A_{\overline{z}}}{\partial z}\Big)\Big)-\Big[A_z,-2\mu^{-2}\Big(\frac{\partial A_z}{\partial \overline{z}}+\frac{\partial A_{\overline{z}}}{\partial z}\Big)\Big]=B_z, \tag{7.15}$$

$$\frac{\partial}{\partial \overline{z}}\Big(-2\mu^{-2}\Big(\frac{\partial A_z}{\partial \overline{z}}+\frac{\partial A_{\overline{z}}}{\partial z}\Big)\Big)-\Big[A_{\overline{z}},-2\mu^{-2}\Big(\frac{\partial A_z}{\partial \overline{z}}+\frac{\partial A_{\overline{z}}}{\partial z}\Big)\Big]=B_{\overline{z}}. \tag{7.16}$$

But, by inserting both (7.14) and (7.15) into

$$\frac{\partial}{\partial \overline{z}}B_z+\frac{\partial}{\partial z}B_{\overline{z}}=0, \tag{7.17}$$

we obtain

$$\begin{aligned}&\frac{\partial^2}{\partial \overline{z}\partial z}\Big(-2\mu^{-2}\Big(\frac{\partial A_z}{\partial \overline{z}}+\frac{\partial A_{\overline{z}}}{\partial z}\Big)\Big)-\frac{\partial}{\partial \overline{z}}\Big[A_{\overline{z}},-2\mu^{-2}\Big(\frac{\partial A_z}{\partial \overline{z}}+\frac{\partial A_{\overline{z}}}{\partial z}\Big)\Big]\\&\quad+\frac{\partial^2}{\partial z\partial \overline{z}}\Big(-2\mu^{-2}\Big(\frac{\partial A_z}{\partial \overline{z}}+\frac{\partial A_{\overline{z}}}{\partial z}\Big)\Big)-\frac{\partial}{\partial z}\Big[A_{\overline{z}},-2\mu^{-2}\Big(\frac{\partial A_z}{\partial \overline{z}}+\frac{\partial A_{\overline{z}}}{\partial z}\Big)\Big]\\&=0,\end{aligned} \tag{7.18}$$

which is just the biharmonic map equation for $(A_z, A_{\overline{z}})$: (6.1) $\delta\widetilde{\Theta}{-}0$. By the same way, one can see also immediately (A_x, A_y) satisfies the biharmonic map equation (5.20) if (B_x, B_y) satisfies the harmonic map equation (5.15) by using Theorem 5.2, (5.6) and (5.22). Thus, we obtain ${\Phi_1}^{-1}(\Xi_1) = \Lambda_2$ and (3). □

Remark The solution (A_x, A_y) in (1) of Theorem 7.3 can be chosen in such a way that they satisfy the integrability condition (5.13) at the initial value (x_0, y),

$$\frac{\partial A_y}{\partial x}(x_0, y) - \frac{\partial A_x}{\partial y}(x_0, y) + [A_x(x_0, y), A_y(x_0, y)] = 0, \tag{7.19}$$

for each y, i.e., the initial functions f_0, f_1 and g_1 may be chosen to satisfy that

$$\frac{\partial A_x}{\partial y}(x_0, y) = g_1(y) + [f_0(y), f_1(y)]. \tag{7.20}$$

Finally, we introduce a loop group formulation for biharmonic maps.
We *first*, consider a $\mathfrak{g}^{\mathbb{C}}$-valued 1-forms

$$\beta_\nu = \frac{1}{2}(1-\nu)B_z\,dz + \frac{1}{2}(1-\nu^{-1})B_{\overline{z}}\,d\overline{z} \tag{7.21}$$

for a parameter $\nu \in S^1$, which satisfy that

$$d\beta_\nu + [\beta_\nu \wedge \beta_\nu] = 0 \qquad (\forall\ \nu \in S^1), \tag{7.22}$$

where for the definition of $[\beta_\nu \wedge \beta_\nu]$, see (3.13).

Next, we consider $\mathfrak{g}^{\mathbb{C}}$-valued 1-forms

$$\alpha_\nu = \frac{1}{2}(1-\nu)A_z\,dz + \frac{1}{2}(1-\nu^{-1})A_{\overline{z}}\,d\overline{z} \tag{7.23}$$

which satisfy that

$$\begin{cases} \dfrac{\partial}{\partial z}(\delta\,\alpha_\nu) - \left[\dfrac{1}{2}(1-\nu)A_z, \delta\,\alpha_\nu\right] = B_z, \\ \dfrac{\partial}{\partial \overline{z}}(\delta\,\alpha_\nu) - \left[\dfrac{1}{2}(1-\nu)A_{\overline{z}}, \delta\,\alpha_\nu\right] = B_{\overline{z}}, \\ \qquad d\,\alpha_\nu + [\alpha_\nu \wedge \alpha_\nu] = 0, \end{cases} \tag{7.24}$$

for each $\nu \in S^1$. Here, the co-differentiation $\delta\,\alpha_\nu$ of α_ν is given by

$$\delta\,\alpha_\nu = -2\mu^{-2}\left(\frac{1}{2}(1-\nu)\frac{\partial}{\partial \overline{z}}A_z + \frac{1}{2}(1-\nu^{-1})\frac{\partial}{\partial z}A_{\overline{z}}\right). \tag{7.25}$$

Then, the mapping $\psi_\nu : \Omega \to G$ satisfying ${\psi_\nu}^*\theta = \alpha_\nu$ is a biharmonic map of (Ω, g) into (G, h) where $g = \mu^2 g_0$ for a positive C^∞ function μ on Ω.

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