On 4-dimensional Riemannian manifolds satisfying a certain condition on the curvature tensor

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

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§ 0. Introduction

If a Riemannian manifold M is locally symmetric, then its curvature tensor R satisfies

$$(*)\qquad R(X, Y)R=0 \qquad \text{for any tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X, Y)$ operates on R as a derivation of tensor algebra at each point of M. As a converse problem, there is a following conjecture by K. Nomizu ([1]).

CONJECTURE. *Let M be a complete, irreducible Riemannian manifold with dim $M \geqq 3$. If M satisfies the condition $(*)$, then M is a locally symmetric space.*

For this conjecture, K. Nomizu gave an affirmative answer in case that M is complete hypersuface in a Euclidean space ([2]). Furthermore there are some results in this direction ([3], [4], [5]).

In this paper, we shall consider this conjecture on 4-dimensional Rimannian manifold and prove the following theorem

THEOREM. *Let M be 4-dimensional connected Riemannian manifold with $R_{ijij}(p) \neq 0$, $p \in M$. If M satisfies the condition $(*)$, then M is a Einstein space, where $R_{ijij}(p)$ is the sectional curvature at p of M with respect to two directions e_i, e_j in the base vectors of the tangent space of M.*

COROLLARY 1. *Under the same assumption, M is locally symmetric. Proof of Corollary* 1 *is given by the following Theorem* (*cf.* [5])

THEOREM. *Let M be 4-dimensional connected Einstein space. If M satisfies the condition $(*)$, then M is locally symmetric.*

COROLLARY 2. *Let M be 4-dimensional connected Riemannian manifold satisfying the condition $(*)$. If the sectional curvature of M does not vanish, then M is a locally symmetric space.*

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§ 1. Preliminaries

We can see that there is an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ at each tangent space of M such that

$$(A) \quad R_{1213} = R_{1214} = R_{1223} = R_{1224} = R_{1314} = R_{1323} = 0\,,$$

where $R_{ijkl} = g(R(e_i, e_j)e_l, e_k)$, $1 \leqq i, j, k, l \leqq 4$, g being the Riemannian metric of M. Let us denote by $X^\wedge Y$ the endomorphism which maps Z upon $g(Z, Y)X - g(Z, X)Y$. Then we have

$$R(e_i, e_j) = \sum_{a<b} R_{ijab} e_a{}^\wedge e_b \qquad 1 \leqq a, b \leqq 4\,.$$

From the above relation we have

$$(1.1) \qquad \Big(R(e_i, e_j)R\Big)(e_k, e_l) = [\sum_{a<b} R_{ijab} e_a{}^\wedge e_b, \quad \sum_{c<d} R_{klcd} e_c{}^\wedge e_d]$$
$$- R\Big((\sum_{a<b} R_{ijab} e_a{}^\wedge e_b) e_k, e_l\Big)$$
$$- R\Big(e_k, (\sum_{a<b} R_{ijab} e_a{}^\wedge e_b) e_l\Big)$$
$$= \sum_{a<b} \sum_{c<d} R_{ijab} \Big\{R_{klcd}(\delta_{bc} e_a{}^\wedge e_d - \delta_{ac} e^\wedge e_b$$
$$+ \delta_{bd} e_d{}^\wedge e_c - \delta_{da} e_c{}^\wedge e_b + (\delta_{ak} R_{blcd}$$
$$- \delta_{bk} R_{alcd} - \delta_{bl} R_{kacd} + \delta_{al} R_{kbcd}) e_c{}^\wedge e_d\Big\}\,.$$

In 4-dimensional Riemannian manifold, only the following three cases are possible ($1 \leqq i, j, k, l \leqq 4$, $i, j, k, l \neq$)

I. $R_{ijkl} \neq 0$,

II. one of R_{ijkl} is equal to zero,

III. $R_{ijkl} = 0$,

where the symbol "$i, j, k, l \neq$" means that i, j, k, l are mutually distinct. In the following sections, we always assume that M satisfies the condition (*) and $R_{ijij} \neq 0$

§ 2. Case of $\mathbf{R}_{ijkl} \neq 0$ ($i, j, k, l \neq$)

LEMMA 2.1. $R_{ijil} = 0$.

PROOF. From (1.1) we obtain

$$0=g\Big(\big(R(e_1, e_2)R\big)(e_2, e_3)e_2, e_3\Big)=2R_{1234}R_{2324}\,,$$

$$0=g\Big(\big(R(e_1, e_3)R\big)(e_2, e_3)e_1, e_3\Big)=R_{1423}R_{1334}\,,$$

$$0=g\Big(\big(R(e_1, e_4)R\big)(e_2, e_4)e_1, e_4\Big)=R_{1324}R_{1434}\,.$$

From these relations and $R_{ijkl}\neq 0$ $(i, j, k, l\neq)$, we get

$$R_{2324}=R_{1334}=R_{1434}=0\,. \tag{2. 1}$$

From the above results and (1. 1) we have

$$0=g\Big(\big(R(e_1, e_2)R\big)(e_2, e_3)e_1, e_4\Big)=R_{1234}R_{1424}\,,$$

$$0=g\Big(\big(R(e_1, e_3)R\big)(e_2, e_3)e_2, e_3\Big)=R_{1324}R_{2324}\,,$$

$$0=g\Big(\big(R(e_1, e_4)R\big)(e_2, e_4)e_2, e_4\Big)=2R_{1423}R_{2434}\,.$$

Thus we obtain

$$R_{1424}=R_{2334}=R_{2434}=0\,. \tag{2. 2}$$

Therefore by means of (A), (2. 1) and (2. 2) we obtain $R_{ijil}=0$. *Q.E.D.*

LEMMA 2. 2 $R_{1212}=R_{3434}$, $R_{1313}=R_{2424}$, $R_{1414}=R_{2323}$.

PROOF. *From* (1. 1) *we have*

$$0=g\Big(\big(R(e_1, e_2)R\big)(e_1, e_3)e_1, e_4\Big)=R_{1212}(R_{1423}+R_{1324})-R_{1234}(R_{1313}-R_{1414})\,, \tag{2. 3}$$

$$0=g\Big(\big(R(e_1, e_2)R\big)(e_2, e_3)e_2, e_4\Big)=R_{1212}(R_{1423}+R_{1324})+R_{1234}(R_{2323}-R_{2424})\,, \tag{2. 4}$$

$$0=g\Big(\big(R(e_1, e_3)R\big)(e_1, e_4)e_1, e_2\Big)=R_{1313}(R_{1234}-R_{1423})-R_{1324}(R_{1212}-R_{1414})\,, \tag{2. 5}$$

$$0=g\Big(\big(R(e_1, e_3)R\big)(e_2, e_3)e_3, e_4\Big)=R_{1313}(R_{1234}-R_{1423})-R_{1324}(R_{3434}-R_{2323})\,, \tag{2. 6}$$

$$0=g\Big(\big(R(e_2, e_3)R\big)(e_2, e_4)e_1, e_2\Big)=R_{2323}(R_{1234}+R_{1324})-R_{1423}(R_{2424}-R_{1212})\,, \tag{2. 7}$$

$$0=g\Big(\big(R(e_2, e_3)R\big)(e_3, e_4)e_1, e_3\Big)=R_{2323}(R_{1234}+R_{1324})-R_{1423}(R_{1313}-R_{3434})\,. \tag{2. 8}$$

From (2. 3) and (2. 4), we have that

(2. 9) $$R_{2323}-R_{2424}+R_{1313}-R_{1414}=0\,.$$

From (2. 5) and (2. 6) we obtain that

(2. 10) $$R_{3434}-R_{2323}-R_{1212}+R_{1414}=0\,.$$

From (2. 7) and (2. 8) we have that

(2. 11) $$R_{1313}-R_{3434}-R_{2424}+R_{1212}=0\,.$$

From (2. 9), (2. 10) and (2. 11), we have

$$R_{1313}=R_{2424}\,.$$

Then by means of (2. 9), (2. 10) and (2. 11) it follows that

$$R_{1414}=R_{2323}\,,\qquad R_{1212}=R_{3434}\,.$$ Q.E.D.

The following Lemma is well known.

LEMMA 2. 3. If $nR_{ij}R^{ij}=R^2$, then M is an Einstein space, where R_{ij} mean the components of Ricci tensor, n is the dimension of M and R the scalar curvature.

PROPOSITION 2. 4. M is an Einstein space.

PROOF. By Lemma 2. 1. and Lemma 2. 2. we obtain

$$R_{jl}=0 \qquad (j\neq l)\,,$$
$$R_{11}=R_{22}=R_{33}=R_{44}\,.$$

Thus by Lemma 2. 3., M is a Einstein space.

§ 3. Case of that one of R_{ijkl} is equal to zero $(i, j, k, l\neq)$

LEMMA 3. 1. If $R_{1234}=0$, then $R_{1212}=0$.

PROOF. From $R_{1234}=0$, we get

$$R_{1324}=R_{1423}(\neq 0)\,.$$

From 1. 1 we obtain

$$0=g\Big(\big(R(e_1, e_2)R\big)(e_1, e_3)e_1, e_4\Big)=R_{1212}(R_{1423}+R_{1324})$$
$$=2R_{1212}R_{1324}\,.$$

Therefore we have $R_{1212}=0$. Q.E.D.

LEMMA 3. 2 If $R_{1423}=0$, then $R_{2323}=0$.

PROOF. From $R_{1423}=0$, we get

$$R_{1234}=R_{1324}(\neq 0)\,.$$

From (1. 1) wehave

$$0=g\Big(\big(R(e_1,\,e_2)R\big)(e_2,\,e_3)e_2,\,e_3\Big)=2R_{1234}R_{2324}\,,$$

Thus we have $R_{2324}=0$.
From the above result and (1. 1) we have

$$\begin{aligned}0=g\Big(\big(R(e_2,\,e_3)R\big)(e_2,\,e_4)e_1,\,e_2\Big)&=R_{2323}(R_{1324}+R_{1234})\\&=2R_{1234}\mathrm{R}_{2323}\,.\end{aligned}$$

Therefore we obtain $R_{2323}=0$. Q. E. D.

LEMMA 3. 3. If $R_{1324}=0$, then $R_{1313}=0$.

PROOF. From $R_{1324}=0$, we get

$$R_{1234}=-R_{1423}(\neq 0)\,.$$

From (1. 1) we obtain

$$\begin{aligned}0=g\Big(\big(R(e_1,\,e_3)R(e_1,\,e_4)e_1,\,e_2\Big)&=R_{1313}(R_{1234}-R_{1423})\\&=2R_{1234}R_{1313}\,.\end{aligned}$$

Thus we have $R_{1313}=0$. Q. E. D.

§ 4. Case of $\mathrm{R}_{ijkl}=0$ $(i, j, k, l \neq)$.

LEMMA 4. 1. $R_{ijil}=0$.

PROOF. From (1. 1) we obtain

$$0=g\Big(\big(R(e_1,\,e_2)R\big)(e_1,\,e_3)e_2,\,e_4\Big)=R_{1212}R_{2334}\,,$$

$$0=g\Big(\big(R(e_1,\,e_2)R\big)(e_1,\,e_3)e_3,\,e_4\Big)=R_{1212}R_{2334}\,,$$

$$0=g\Big(\big(R(\,e_1,\,e_2)R\big)(e_1,\,e_4)e_1,\,e_4\Big)=2R_{1212}R_{,1424}\,,$$

$$0=g\Big(\big(R(e_{,1}\,e_2)R\big)(e_1,\,e_4)e_3,\,e_4\Big)=R_{1212}R_{2434}\,,$$

$$0=g\Big(\big(R(e_1,\,e_2)R\big)(e_2,\,e_3)e_3,\,e_4\Big)=R_{1212}R_{1334}\,,$$

$$0=g\Big(\big(R(e_1,\,e_2)R\big)(e_2,\,e_4)e_3,\,e_4\Big)=R_{1212}R_{1434}\,.$$

From these relations and $R_{1212}\neq 0$, we have

$$R_{2324}=R_{2334}=R_{1424}=R_{2434}=R_{1334}=R_{1434}=0\,.$$

Therefore by means of (A) and above results we obtain $R_{ijil}=0$. Q. E. D.

Lemma 4. 2 $R_{1313}=R_{2323}$ and $R_{1414}=R_{2424}$.

PROOF. From (1. 1) we have

$$0=g\Big(\big(R(e_1, e_2)R\big)(e_1, e_3(e_2, e_3\Big)=R_{1212}(R_{2323}-R_{1313}),$$

$$0=g\Big(\big(R(e_1, e_2)R\big)(e_1, e_4)e_2, e_4\Big)=R_{1212}(R_{2424}-R_{1414}).$$

Then from $R_{1212}\neq 0$, we have

$$R_{2323}=R_{1313}, \qquad R_{2424}=R_{1414}. \qquad \text{Q.E.D.}$$

LEMMA 4. 3. $R_{1212}=R_{1313}=R_{1414}=R_{2323}=R_{2424}=R_{3434}$.

PROOF. From (1. 1) we have

$$0=g\Big(\big(R(e_1, e_3)R\big)(e_1, e_4)e_3, e_4\Big)=R_{1313}(R_{3434}-R_{1414}),$$

$$0=g\Big(\big(R(e_1, e_3)R\big)(e_2, e_3)e_1, e_2\Big)=R_{1313}(R_{1212}-R_{2323}),$$

$$0=g\Big(\big(R(e_1, e_4)R\big)(e_2, e_4)e_1, e_2\Big)=R_{1414}(R_{1212}-R_{2424}),$$

$$0=g\Big(\big(R(e_1, e_4)R\big)(e_3, e_4)e_1, e_3\Big)=R_{1414}(R_{1313}-R_{3434}).$$

Thus, by means of $R_{1313}\neq 0$, $R_{1414}\neq 0$, and Lemma 4. 2., we have

$$R_{1212}=R_{1313}=R_{1414}=R_{2323}=R_{2424}=R_{3434}. \qquad \text{Q.E.D}$$

PROPOSITION 4. 4. *M is a space of constant curvature.*
This proposition is proved without difficulty.

§ 5. Proof of theorem

By Lemma 3. 1., 3. 2., and 3. 3., under the assumptions in the Theorem, case of that one of R_{ijkl} is equal to zero does not occur $(i, j, k, l \neq)$. Therefore Theorem is proved by proposition 2. 4. and Proposition 4. 4.

Bibliography

[1] K. NOMIZU and K. YANO: Some results related to the equivalance problem in Riemannian geometry, United States-Japan Sem. Diff. Geom., Kyoto, 1965.

[2] K. NOMIZU: On hypersurfaces satisfying a certain condition on the curvature tensor, Tohoku Math. J., 20 (1968), 46–59.

[3] K. SEKIGAWA and H. TAKAGI: On the conformally flat spaces satisfying a certain condition on the Ricci tensor, Tohoku Math. J., 23 (1971), 1–11.

[4] K. SEKIGAWA and S. TANNO: Sufficient conditions for a Riemannian manifold

to be locally symmetric, Pacific J. Math., 34 (1971) 175–162.
[5] K. SEKIGAWA: On 4-dimensional connected Einstein space satisfying the condition $R(X, Y)R=0$, Sci. Rep. Niigata Univ. Ser. 7 A (1969) 29–31.

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