On the geodesic projective transformation in Riemannian spaces

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

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Introduction. In a recent paper [2], the author introduced the notion of the geodesic conformal transformation and discussed Riemannian spaces which admit such transformations at each point. The geodesic conformal transformation at a point O is, roughly speaking, a local conformal transformation which leaves invariant any geodesic through O. The purpose of this paper is to give the projective analogy. Though the results are not satisfactory comparing to the conformal case, the geodesic projective transfomation seems to be interesting. Because, when we seek after the projective analogy of locally symmetric spaces, it must play a basic role.

1. Normal coordinates. Let M^n be an n dimensional analytic Riemannian space with positive definite metric g_{ij}[1]. Consider a normal coordinate $\{x^i\}$ of origin O in a normal neighbourhood U, then

$$(1.1) \qquad \begin{Bmatrix} h \\ ij \end{Bmatrix} x^i x^j = 0$$

hold good in U. Any geodesic γ through O is given by $x^i = \xi^j s$, where ξ^i is the unit tangent vector of γ at O and s means the arc length along γ. Throughout the paper we shall only use such a coodinate, and consider s to be positive, unless otherwise stated. f_i, f_{ij}, $\cdots$ mean the successive derivatives of f with respect to x^i, x^j, $\cdots$, and f', f'', $\cdots$ the ones with respect to s.

The following identities in U are well known.

$$(1.2) \qquad (g_{ij})_0 x^i x^j = g_{ij} x^i x^j = s^2,$$

$$(1.3) \qquad s_i x^i = s,$$

$$(1.4) \qquad s_{ij} x^j = 0.$$

If we put $s^i = g^{ij} s_j$ and $x_i = (g_{ij})_0 x^j$, then

$$(1.5) \qquad g_{ij} x^j = x_i,$$

$$(1.6) \qquad g_{ij}(\rho x) x^i x^j = x_i x^i = s^2 \qquad \text{for any small } \rho,$$

(1. 7) $s_i s^i = 1$

hold good, too.

2. The geodesic transformation at O. In a normal neighbourhood U of origin O in M^n, we consider a local transformation f defined by

$$f: \quad x^i = \xi^i s \in U \longrightarrow y^i = \xi^i t \in U$$

which leaves invariant each geodesic through O. We assume that t is of the form

(2. 1) $t = \rho(s)s ,$

and ρ is an analytic function of s in a domain $0 < s < \varepsilon$ for some ε. Such an f will be called a geodesic transformation at O.

As we have

(2. 2) $\partial_i y^h = \rho' s_i x^h + \rho \delta_i{}^h , \qquad \partial_i = \partial/\partial x^i ,$

it follows that

(2. 3) $x^i \partial_i y^h = (\rho' s + \rho) x^h = t' x^h ,$

(2. 4) $x^i x^j \partial_{ij} y^h = s(\rho'' s + 2\rho') x^h = s t'' x^h .$

Putting

$$\varDelta = \det(\partial_i y^h) ,$$

we can obtain the following equation :

(2. 5) $\varDelta = \rho^{n-1} t' .$

3. The geodesic projective transformation at O. Let f be a geodesic transformation at O in M^n. If we denote

$$\bar{g}_{ij} = g_{ij}(y) , \qquad g^*_{ij} = f^* \bar{g}_{ij} ,$$

then the condition for f to be projective is

(3. 1) $\left\{ \begin{matrix} h \\ ij \end{matrix} \right\}^* = \left\{ \begin{matrix} h \\ ij \end{matrix} \right\} + \phi_i \delta_j{}^h + \phi_j \delta_i{}^h ,$

where $\left\{ \begin{matrix} h \\ ij \end{matrix} \right\}^*$ are the Christoffel symbols formed by g^*_{ij} and $\phi_i = \partial_i \phi$ is a gradient vector.

When ϕ is an analytic function of s defined on a domain $0 < s < \varepsilon_1$ for some ε_1, we shall call f a geodesic projective transformation at O.

In the following, f always means such a transformation.

From the definition of g^*_{ij}, we have

(3. 2)
$$\left\{\begin{matrix} l \\ ij \end{matrix}\right\}^* \partial_l y^h = \left\{\begin{matrix} \overline{h} \\ kl \end{matrix}\right\} \partial_i y^k \partial_j y^l + \partial_{ij} y^h .$$

On the other hand, it follows from (2. 3), (1. 1) and (2. 4) that

$$x^i x^j \left(\left\{\begin{matrix} \overline{h} \\ kl \end{matrix}\right\} \partial_i y^k \partial_j y^l + \partial_{ij} y^h \right) = s t'' x^h .$$

From (3. 1), (1. 1) and (2. 3) we have

$$x^i x^j \left\{\begin{matrix} l \\ ij \end{matrix}\right\}^* \partial_l y^h = 2\phi' s t' x^h .$$

Thus, (3. 2) gives us

$$t'' = 2\phi' t' .$$

As t' does not vanish identically,

(3. 3)
$$2\phi' = \frac{t''}{t'}$$

follows, and by integration we have

(3. 4)
$$e^{2\phi} = C_1 t' ,$$

where C_1 is a non-zero constant.

Consider the case when t' is constant. Then ϕ being constant, ϕ_i vanishes identically and hence f is a homothety.

4. The projective curvature tensor. In this section, we shall assume that a geodesic projective transformation f at O is defined without singularity at O, that is, ρ and ϕ are defined in a domain of s containin 0. As $\rho(0)$ and $\rho'(0)$ are finite, it follows from (2. 2) that

$$(\partial_i y^h)_0 = a\delta_i^h ,$$

where we have put $a = \rho(0) \neq 0$.

f being projective, we have

(4. 1)
$$W^{*h}{}_{ijk} = W^h{}_{ijk} ,$$

where $W^h{}_{ijk}$ denotes the so-called Weyl projective curvature tensor defined by

$$W^h{}_{ijk} = R^h{}_{ijk} - \frac{1}{n-1}(R_{ij}\delta_k{}^h - R_{ik}\delta_j{}^h).$$

On the other hand,

(4. 2)
$$W^{*h}{}_{ijk}\partial_h y^l = \overline{W}^l{}_{abc}\partial_i y^a \partial_j y^b \partial_k y^c$$

are valid.

If we put $s = 0$ in (4.1) and (4.2), then it follows $(W^h_{ijk})_0 = a^2 (W^h_{ijk})_0$, and hence we get

$$(W^h_{ijk})_0 = 0, \qquad \text{if } a \neq \pm 1.$$

Similarly we can get

$$(\nabla_l W^h_{ijk})_0 = 0, \qquad \text{if } a \neq 1,$$

on taking account of $\phi_i(0) = 0$.

As well known, if W^h_{ijk} vanishes identically, the space is one of constant curvature, that is,

$$R^h_{ijk} = k(g_{ij}\delta_k{}^h - g_{ik}\delta_j{}^h)$$

holds good, where

$$k = \frac{R}{n(n-1)}$$

and R is the scalar curvature. The converse is true.

It is known [4] that $\nabla_l W^h_{ijk} = 0$ is equivalent to $\nabla_l R^h_{ijk} = 0$.

5. The relation of g_x and g_y. Let f be a geodesic projective transformation at O in a Riemannian space M^n. Denoting

$$g_x = \det(g_{ij}), \qquad g_y = \det(\bar{g}_{ij}),$$

we shall find the relation between them.

From (3.1) we have

(5.1)
$$\left\{ {i \atop ij} \right\}^* = \left\{ {i \atop ij} \right\} + (n+1)\phi_j.$$

On the other hand, it follows from (3.2) that

$$\left\{ {h \atop ij} \right\}^* = \frac{\partial x^h}{\partial y^l}\left(\left\{ \overline{l \atop kp} \right\} \partial_i y^k \partial_j y^p + \partial_{ij} y^l \right).$$

Putting $h = i$, we get

$$\left\{ {i \atop ij} \right\}^* = \left\{ \overline{k \atop kp} \right\} \partial_j y^p + \partial_j \log |\Delta|.$$

Substituting this equation into (5.1), we have

$$\left\{ {i \atop ij} \right\} + (n+1)\phi_j = \left\{ \overline{i \atop ip} \right\} \partial_j y^p + \partial_j \log |\Delta|,$$

from which

$$\partial_j \left(\log \sqrt{g_x} + (n+1)\phi \right) = \partial_j \left(\log \sqrt{g_y} + \log |\Delta| \right).$$

Hence we obtain

$$\sqrt{g_y}\ \Delta = C_2 e^{(n+1)\phi}\ \sqrt{g_x}\ ,$$

where C_2 is a non-zero constant. If we take account of (2. 5) and (3. 4), **then**

(5. 2)
$$\frac{dt}{t^2 G_y} = C_3 \frac{ds}{s^2 G_x}$$

follows, where we have put

$$G_x = g_x^{(n-1)^{-1}}, \qquad G_y = g_y^{(n-1)^{-1}}$$

and C_3 is a non-zero constant.

It should be noticed that G_x and G_y generally depend on not only s but ξ^i, the direction of the geodesic.

Let us consider the Euclidean n-space E^n. Then, since $G_x = G_y = 1$, (5. 2) reduces to

$$\frac{dt}{t^2} = C_3 \frac{ds}{s^2}$$

and we get

(5. 3)
$$t = \frac{s}{C_4 s + C_3}$$

with a constant C_4. On the other hand, it is evident that there is **no** geodesic projective transformation at O in E^n except the similarity. Hence, C_4 in (5. 3) must be 0, and we get

(5. 4)
$$t = C_5 s\ ,$$

where C_5 is a non-zero constant.

6. Spaces of constant curvature. Consider a space M^n of **constant** curvature $k \neq 0$ and a geodesic projective transformation f at a point O. As well known [3, p. 169, p. 183], if we choose a normal coordinate of origin O such that $(g_{ij})_0 = \delta_{ij}$, then g_x is given by

$$g_x = \left(\frac{\sin(\sqrt{k}\ s)}{\sqrt{k}\ s} \right)^{2(n-1)}$$

where $\sin(\sqrt{k}\ s) = i\ \sinh(\sqrt{|k|}\ s)$ for $k < 0$. Thus, we have from (5. 2)

$$\frac{dt}{\sin^2(\sqrt{k}\ t)} = C_3 \frac{ds}{\sin^2(\sqrt{k}\ s)}$$

and by integration

(6. 1) $\cot(\sqrt{k}\ t) = C_3 \cot(\sqrt{k}\ s) + C_6$,

where C_3 is non-zero and C_6 any constant.

Next, we shall show that a geodesic projective transformation certainly exists in a space of positive constant curvature.

Since such a space M^n is locally isometric with a sphere S^n of radius $1/\sqrt{k}$ in the Euclidean $(n+1)$-space E^{n+1}, we can regard M^n as S^n without loss of generality.

Denoting by O, p_0 and q_0 the center, the north pole and the south pole of S^n, consider the projection ϕ from O to the Euclidean n-space E^n which is the tangent n-plane to S^n at p_0. ϕ is a projective map from S^n-$\{q_0\}$ to E^n. Any geodesic projective transformation f at p_0 on S^n is obtained as the form $\phi^{-1} \circ f_0 \circ \phi$ with a geodesic projective transformation f_0 at O on E^n, and f_0 is a similarity which leaves O invariant. As it holds that

$$\xi^i s \xrightarrow{\phi} \xi^i \frac{\tan(\sqrt{k}\ s)}{\sqrt{k}} \xrightarrow{f_0} C_5 \xi^i \frac{\tan(\sqrt{k}\ s)}{\sqrt{k}} = \xi^i \frac{\tan(\sqrt{k}\ t)}{\sqrt{k}} \xrightarrow{\phi^{-1}} \xi^i t,$$

the following relation between s and t is valid,

(6. 2) $\tan(\sqrt{k}\ t) = C_5 \tan(\sqrt{k}\ s)$, $C_5 \neq 0$.

Thus, any space of positive constant curvature $k \neq 0$ admits a geodesic projective transformation at any point O, and C_6 in (6. 1) must be 0. As the space is locally symmetric, C_5 in (6. 2) can be negative and hence takes any value except 0.

7. The existence of a geodesic projective transformation in a space of constant curvature. Let M^n be a space of constant curvature $k \neq 0$. For the case of $k > 0$, we have seen the existence of a geodesic projective transformation at any point. In this section, the existence will be proved for $k < 0$ and > 0 at the same time.

For this purpose we shall use the following well known facts.

a. A projective transformation is such a transformation that it maps any geodesic to a geodesic.

b. The intersection of two totally geodesic subspaces is totally geodesic.

c. In a space of constant curvature, there is an $n-1$ dimensional totally geodesic subspaces X_{n-1} which is tangent to any given $(n-1)$-plane section, [3, p. 144].

d. In a space of constant curvature $k \neq 0$, let U be a normal neighbourhood such that $(g_{ij})_0 = \delta_{ij}$. Then, the equation of any totally geodesic X_{n-1} in U is

(7. 1) $a_i \xi^i \tan(\sqrt{k}\ s) = C$,

where a_i and C are constant, and for $k<0$ we understant that $\tan(\sqrt{k}\,s)$ $=i\tanh(\sqrt{|k|}\,s)$ and C is pure imaginary, [3, p. 186].

Now, we shall show that a geodesic transformation f at O given by

$$\tan(\sqrt{k}\,t)=C_5\tan(\sqrt{k}\,s)$$

is acutally projective in M^n.

In fact, any point $(\xi^i s)$ on (7.1) is mapped to $(\xi^i t)$ on $a_i\xi^i\tan(\sqrt{k}\,t)=B$ for a constant B, and hence f maps any X_{n-1} in U to another totally geodesic Y_{n-1}. Let α be a geodesic in U and P be a point on α. Consider a frame $e_1,\cdots,e_n$ at P such that e_1 is tangent to α. For each $(n\text{-}1)$-plane section $e_1,\cdots,e_{i-1},e_{i+1},\cdots,e_n$, there is an $X_{n-1}^{(i)}$ which contains α. As f transforms $X_{n-1}^{(i)}$ to $Y_{n-1}^{(i)}$, α is mapped to another geodesic which is the intersection of $Y_{n-1}^{(i)}$, $i=2,\cdots,n$. Thus our assertion is proved.

8. Einstein spaces. We shall consider in this section a geodesic projective transformation f at O in an Einstein space, and obtain a differental equation for t.

f being projective, the Ricci tensors of g_{ij} and g_{ij}^* satisfy

(8. 1) $$R_{ij}^*=R_{ij}-(n-1)\tau_{ij},$$

where

$$\tau_{ij}=\nabla_i\phi_j-\phi_i\phi_j\,.$$

As the space is Einstein, (8. 1) becomes

(8. 2) $$kg_{ij}^*=kg_{ij}-\tau_{ij}\,.$$

Multiplying (8. 2) by $x^i x^j$ and taking account of

$$g_{ij}^* x^i x^j=\bar{g}_{kp}\partial_i y^k\partial_j y^p x^i x^j=t'^2 s^2\,,$$

we have

(8. 3) $$\tau_{ij}x^i x^j=k(1-t'^2)s^2\,.$$

On the other hand, since

$$\phi_i x^i=\phi's\,,\qquad \phi_i=\phi's_i=\frac{\phi'}{s}x_i\,,$$

$$\tau_{ij}x^i x^j=(\partial_i\phi_j-\phi_i\phi_j)x^i x^j=(\phi''-\phi'^2)s^2\,,$$

we get from (8. 3)

(8. 4) $$\phi''-\phi'^2=k(1-t'^2)\,.$$

If we eliminate ϕ from (8. 4) by making use of (3. 3), then

(8. 5) $$\{t, s\} = 2k(1 - t'^2)$$

is obtained as the differential equation for t, where

$$\{t, s\} = \frac{t'''}{t'} - \frac{3}{2}\left(\frac{t''}{t'}\right)^2$$

is the Schwarzian derivative of t with respect to s.

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