Pontrjagin classes of Riemannian manifolds of class 2.

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

By Yasurô TOMONAGA.

The purpose of this paper is to make clear the relations between the Pontrjagin classes and the second fundamental tensors in the case of Riemannian manifolds of class 2. Let X_n be a compact orientable Riemannian manifold of dimension $n\,(\geqq 3)$. We assume that X_n is of class C^r and r is large enough. Suppose that a domain D of X_n be immersed in an $(n+2)$-dimensional euclidean space E_{n+2}. We denote the local coordinates of D or the orthogonal coordinates of E_{n+2} by $\{x^i\}$ $i=1, \cdots, n$ or $\{X^\lambda\}$ $\lambda=\dot{1}, \cdots, \dot{n}+\dot{2}$ respectively. Hereafter the latin indices range over $1, \cdots, n$ and the Greek indices range over $\dot{1}, \cdots, \dot{n}+\dot{2}$. Throughout this paper we use the following notations and formulas:

g_{ij} ·············· the metric tensor of X_n,
$B_i^\lambda = \partial X^\lambda / \partial x^i$ ··· the tangent vectors of X_n,
$N_A^\lambda\,(A=1, 2)$ ··· the unit normal vectors to X_n, i.e.
$\sum\limits_\lambda N_A^\lambda B_i^\lambda = 0 (A=1, 2, i=1, \cdots, n)$, $\qquad \sum\limits_\lambda N_A^\lambda N_B^\lambda = \delta_{AB}$,

$$(1)\quad \delta B_i^\lambda / \partial x^j \equiv \frac{\partial B_i^\lambda}{\partial x^j} - \left\{ {k \atop ij} \right\} B_k^\lambda = \sum_{\lambda=1}^{2} N_A^\lambda H_{A_{ij}}, \qquad H_{A_{ij}} = H_{A_{ij}},$$

$$(2)\quad \partial N_A^\lambda / \partial x^i = -B_j^\lambda H_{A\cdot i}^{\ j} + \sum_{B=1}^{2} T_{BA_i} N_B^\lambda, \qquad T_{AB_i} = -T_{BA_i},$$

$$(3)\quad R_{ijkl} = \sum_{A=1}^{2} (H_{A_{jk}} H_{A_{il}} - H_{A_{jl}} H_{A_{ik}}) \cdots \text{ the equations of Gauss,}$$

$$(4)\quad H_{A_{ij};k} - H_{A_{ik};j} + \sum_{B=1}^{2} (T_{BA_j} H_{B_{ik}} - T_{BA_k} H_{ijB}) = 0 \cdots \text{ the equations of Coddazi,}$$

$$(5)\quad T_{BA_i;j} - T_{BA_j;i} - H_{B_{kj}} H_{A\cdot i}^{\ k} + H_{B_{ki}} H_{A\cdot j}^{\ k} = 0 \cdots \text{ the equations of Ricci, where}$$

the semi-colon denotes the covariant differentiation.

Let P_{4k} be the $4k$-dimensional Pontrjagin class of X_n. It is expressed by the following differential form:

$$(6)\qquad P_{4k} = \alpha_k \delta \begin{pmatrix} j_1 \cdots j_{2k} \\ i_1 \cdots i_{2k} \end{pmatrix} R^{i_1}{}_{j_1[a_1 a_2} R^{i_2}{}_{|j_2| a_3 a_4} \cdots R^{i_{2k}}{}_{|j_{2k}| a_{4k-1} a_{4k}]} \underset{1}{dx^{a_1}} \underset{2}{dx^{a_2}} \cdots \underset{4k}{dx^{a_{4k}}},$$

where α_k denotes a certain constant depending only on k. First we consider the P_4. In this case we have from (3) and (6)

$$(7)\quad R^{i}_{j[a_1a_2}R^{j}{}_{|i|a_3a_4]}=\sum_{[a_1a_2a_3a_4]}\Big\{(H_{1ja_1}H_{1\cdot a_2}^{\ i}-H_{1ja_2}H_{1\cdot a_1}^{\ i})+(H_{2ja_1}H_{2\cdot a_2}^{\ i}-H_{2ja_2}H_{2a_1}^{\ i})\Big\}$$
$$\times\Big\{(H_{1ia_3}H_{1\cdot a_4}^{\ j}-H_{1ia_4}H_{1a_3}^{\ j})+(H_{2ia_3}H_{2\cdot a_4}^{\ j}-H_{2ia_4}H_{2\cdot a_3}^{\ j})\Big\}$$
$$=8H_{1\cdot[a_2}^{\ i}H_{2|i|a_3}H_{1|j|a_1}H_{2\cdot a_4]}^{\ j}=2f_{[a_2a_3}f_{a_1a_4]}\,,$$

where we put

$$(8)\qquad f_{ij}\equiv H_{1\cdot i}^{\ k}H_{2kj}-H_{1\cdot j}^{\ k}H_{2ki}\,.$$

Putting

$$(9)\qquad \omega=f_{ij}\underset{1}{dx^i}\underset{2}{dx^j}\,,$$

we see from (7) that $P_4=\alpha\omega^2$, where α denotes a certain constant. In general we have the

THEOREM 1. *In the local sense $P_{4k}=\beta_k\omega^{2k}$, where β_k denotes a certain constant depending ouly on k.*

Next we consider the properties of the tensor f_{ij}. We seen from (5) and (8) that

$$(10)\qquad f_{ij}=T_{21i;j}-T_{21j;i}\,.$$

Hence we have the

THEOREM 2. *ω is a null form in the local sense.*

Let us consider the rotation of the orthogonal 2-frame $(N_1^\lambda, N_1^\lambda)$:

$$(11)\qquad \overline{N}_{\bar 1}^{\lambda}=N_1^{\lambda}\cos\theta-N_2^{\lambda}\sin\theta\,,\qquad \overline{N}_{\bar 2}^{\lambda}=N_1^{\lambda}\sin\theta+N_2^{\lambda}\cos\theta\,.$$

Meanwhile we have from (1) and (11)

$$(12)\qquad \delta B_i^\lambda/\partial x^j=\sum_{A=1}^{2}N_A^\lambda H_{Aij}=\sum_{\bar A=\bar 1}^{\bar 2}\overline{N}_{\bar A}^{\lambda}\overline{H}_{\bar Aij}=\sum_{A,\bar A}N_A^\lambda c_{\bar AA}\overline{H}_{\bar Aij}\,,$$

where we put

$$c_{\bar 11}=\cos\theta\,,\quad c_{\bar 12}=-\sin\theta\,,\quad c_{\bar 21}=\sin\theta\,,\quad c_{\bar 22}=\cos\theta\,.$$

Hence we have

$$(13)\qquad H_{Aij}=\sum_{\bar A}c_{\bar AA}\overline{H}_{\bar Aij}$$

which leads to

$$(14)\qquad f_{ij}=H_{1\cdot i}^{\ k}H_{2kj}-H_{1\cdot j}^{\ k}H_{2ki}=\sum_{\bar A,\bar B}c_{\bar A1}\overline{H}_{\bar A\cdot i}^{\ k}c_{\bar B2}\overline{H}_{\bar Bkj}-\sum_{\bar A,\bar B}c_{\bar A1}\overline{H}_{\bar A\cdot j}^{\ k}c_{\bar B2}\overline{H}_{\bar Bki}$$
$$=c_{\bar 11}\overline{H}_{\bar 1\cdot i}^{\ k}c_{\bar 22}\overline{H}_{\bar 2kj}+c_{\bar 21}\overline{H}_{\bar 2\cdot i}^{\ k}c_{\bar 12}\overline{H}_{\bar 1kj}-c_{\bar 11}\overline{H}_{\bar 1\cdot j}^{\ k}c_{\bar 22}\overline{H}_{\bar 2ki}-c_{\bar 21}\overline{H}_{\bar 2\cdot j}^{\ k}c_{\bar 12}\overline{H}_{\bar 1ki}$$
$$=(\overline{H}_{\bar 1\cdot i}^{\ k}\overline{H}_{\bar 2kj}-\overline{H}_{\bar 1\cdot j}^{\ k}\overline{H}_{\bar 2ki})\cos^2\theta+(\overline{H}_{\bar 1\cdot i}^{\ k}\overline{H}_{\bar 2kj}-\overline{H}_{\bar 1\cdot j}^{\ k}\overline{H}_{\bar 2ki})\sin^2\theta$$
$$=\overline{H}_{\bar 1\cdot i}^{\ k}\overline{H}_{\bar 2kj}-\overline{H}_{\bar 1\cdot j}^{\ k}\overline{H}_{\bar 2ki}=\bar f_{ij}\,.$$

Hence we havc the

THEOREM 3. *The skew-symmetric tensor f_{ij} is invariant under the rotation of the orthogonal frame* $(N_1^\lambda, N_2^\lambda)$.

The above argument is the local one. If X_n is immersed in an E_{n+2} toghter with the diffentiable orthogonal 2-frame $(N_1^\lambda, N_{2,}^\lambda)$ then Theorem 1 and 2 hold in the global sense. Therefore all Pontrjagin classes vanish in such a case.

Utsunomiya University, Japan.

(Received December 1, 1971)