On Inner ρ-Derivations

Dedicated to Professor Yoshie Katsurada on the occasion of her 60th birthday

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Throughout the present paper, A will represent a ring with 1, C the center of A, and B a subring of A containing 1. We consider a (ring) endomorphism ρ of A sending 1 to 1, and set $J(\rho)=\{x\in A|\rho(x)=x\}$. Further, we use the following convention: a, a' will mean arbitrary elements of A, and b, b' elements of B.

Given $x\in A$, the map $\delta_{\rho,x}$: $A\to A$ defined by $\delta_{\rho,x}(a)=ax-x\rho(a)$ is called the *inner ρ-derivation* effected by x. In fact, $\delta_{\rho,x}$ is a $(\rho, 1)$-derivation of A: $\delta_{\rho,x}(aa')=\delta_{\rho,x}(a)\rho(a')+a\delta_{\rho,x}(a')$. Recently, in his paper [3], P. V. Praag proved the following proposition: *If A is a division ring with $[A:C]>4$ and B is a division subring properly contained in A which is invariant relative to all the inner ρ-derivations effected by elements of A, then B is a field contained in $J(\rho)$.*

In what follows, we shall prove a proposition which essentially contains Praag's, and extend [3; Corollaire] to two-sided simple rings and (right) primitive rings with non-zero socles.

LEMMA 1. *Assume that B is invariant relative to all the inner ρ-derivations effected by elements of A.*

(a) *The restriction $\rho|B$ of ρ to B is an endomorphism of B.*

(b) *If B is commutative then the additive group $[B, A]$ generated by $\{[b, a]=ab-ba|a\in A, b\in B\}$ is a B-B-submodule of A and $bd=d\rho(b)$ for every $d\in[B, A]$.*

PROOF. (a) is evident by $b-\rho(b)=b\cdot 1-1\cdot\rho(b)\in B$. If B is commutative, it is easy to see that $[B, A]$ is B-B-admissible. Further, $[b, b'a-a\rho(b')]=0$ implies $b'[b, a]=[b, a]\rho(b')$, proving (b).

LEMMA 2. *Let B be a field. Assume that B is invariant relative to all the inner ρ-derivations effected by elements of A and $B\not\subseteq J(\rho)$.*

(a) *B coincides with the centerализer $V_A(B)$ of B in A.*

(b) *$A=[B, A]+B$ and $[B, A]^2\subseteq B$.*

PROOF. (a) It suffices to prove $V_A(B)\subseteq B$. Now, let b^* be an arbitrary element of B not contained in $J(\rho)$. If v is in $V_A(B)$ then $v(b^*-\rho(b^*))=$

$b^*v - v\rho(b^*) \in B$ and $b^* - \rho(b^*)$ is a non-zero element of B (Lemma 1 (a)). Hence, v is in B.

(b) There holds $a = (\rho(b^*) - b^*)^{-1}([\rho(b^*), a] - (b^*a - a\rho(b^*))) \in [B, A] + B$ by Lemma 1 (b). Further, if d, d' are in $[B, A]$ then $(dd')(\rho^2(b^*) - \rho(b^*)) = b^*(dd') - (dd')\rho(b^*) \in B$ (Lemma 1 (b)). Since $\rho^2(b^*) - \rho(b^*) \neq 0$, we obtain $dd' \in B$.

PROPOSITION 1. *Assume that B is a proper subfield of A which is invariant relative to all the inner ρ-derivations effected by elements of A. If $B \not\subseteq J(\rho)$ and A is a prime ring then A is a simple ring with $[A:C] = 4$.*

PROOF. Since $A = [B, A] + B$ by Lemma 2 (b), $[B, A]^2 \neq 0$. In fact, $[B, A]^2 = 0$ implies a contradition that $[B, A]$ is a non-zero nlipotent ideal of A. Now, let u, u' be elements of $[B, A]$ with $uu' \neq 0$. Then, uu' is a unit of B by Lemma 2 (b) and $u'u$ is also in B and non-zero by $(uu')^2 \neq 0$. We see therefore that u is a unit. Since $bu = u\rho(b)$ by Lemma 1 (b), we obtain $b([b', a]u^{-1}) = [b', a]\rho(b)u^{-1} = [b', a](u^{-1}bu)u^{-1} = ([b', a]u^{-1})b$. Hence, $[b', a]u^{-1} \in V_A(B) = B$ (Lemma 2 (a)), namely, $[B, A] = Bu$. We have seen therefore $[A:B]_L = 2$. Combining this with $B = V_A(B)$, we readily obtain $[A:C] = 4$. (Cf. for instance [5; Proposition 7.1].)

LEMMA 3. *Assume that B is a proper subring of A which is invariant relative to all the inner ρ-derivations effected by elements of A.*

(a) *If B is two-sided simple then B is a field.*

(b) *If A is a (right) primitive ring with non-zero socle S, B a completely primitive ring (with non-zero socle T), and ρ a monomorphism, then B is a field.*

PROOF. (a) Since $b(b'a - a\rho(b)) \in B$ and $b'(ba) - (ba)\rho(b') \in B$, we readily obtain $[b, b']a \in B$. If $[B, B] \neq 0$ then $B[B, B]B = B$, and hence $A = BA = (B[B, B]B)A \subseteq B$. This contradiction shows that B is a field.

(b) By [1; Theorem IV. 15.1], T can not be equal to S. As in (a), we obtain $B \supseteq [B, B]A \cup A\rho([B, B])$. If $[B, B] \neq 0$ then $\rho([B, B] \neq 0$. Noting here that the non-zero ideals $B[B, B]B$ and $B\rho([B, B])B$ contains T, we readily obtain $B \supseteq TA$ and $B \supseteq AT$. Accordingly, $T \subseteq TA = TTA \subseteq T$ and $T \subseteq AT \subseteq ATT \subseteq T$, namely, $AT = T = TA$. Hence, the ideal T of A contains S and ST is a non-zero ideal of B, which implies a contradiction $S = T$.

LEMMA 4. *If A is a simple ring with $[A:C] = 4$ and B is a field with $B = V_A(B)$, then there exists a unique automorphism ρ_B of B such that $ba - a\rho_B(b) \in B$, and then there exists an (inner) automorphism ρ' of A such that B is invariant relative to all the inner ρ'-derivations effected by elements of A.*

PROOF. Let x be an arbitrary element of A not contained in B. Then, x, 1 form a right free B-basis of A, and we obtain a map $\rho_B: B\to B$ such that ($bx-x\rho_B(b)\in B$ and hence) $ba-a\rho_B(b)\in B$. To be easily seen, ρ_B is a C-automorphism of B and uniquely determined. Now, it is well-known that ρ_B can be extended to an inner automorphism ρ' of A. (Cf. for instance [5; Theorem 7.2].)

THEOREM 1. *Let A and B be two-sided simple. Then, B is invariant relative to all the inner ρ-derivations effected by elements of A if and only if one of the following conditions is satisfied*:

(1) $B=A$.

(2) $B\subseteq J(\rho)\cap C$.

(3) $[A:C]=4$, $B=V_A(B)$, *and* $\rho|B=\rho_B$ (*see Lemma* 4).

THEOREM 2. *Let A be a primitive ring with non-zero socle, B a completely primitive ring, and ρ a monomorphism. Then, B is invariant relative to all the inner ρ-derivations effected by elements of A if and only if one of the following conditions is satisfied*:

(1) $B=A$.

(2) $B\subseteq J(\rho)\cap C$.

(3) *A is a simple ring with* $[A:C]=4$, $B=V_A(B)$, *and* $\rho|B=\rho_B$.

PROOF OF THEOREMS 1 and 2. By the validity of Lemma 4, it remains only to prove the only if part. We may assume here that B is a proper subring of A. In any rate, B is a field by Lemma 3. If $B\not\subseteq J(\rho)$ then it is the case (3) by Proposition 1 and Lemma 4. On the other hand, if $B\subseteq J(\rho)$ then B is invariant relative to all the inner derivations effected by elements of A. In case B is not of characteristic 2, $B\subseteq C$ by [5; Proposition 8.10 (b)] and [2; Corollary 1 (1′)]. In what follows, we may assume therefore that B is of characteristic 2 and $B\not\subseteq C$. Then, $[b, [b, a]]=0$ implies $b^2\in C$, in particular, $[b, a]^2\in C$. Hence, A is a simple ring with $[A:C]=4$ by [4; Theorem 2]. Choosing here $b_0\in B$ and $a_0\in A$ with $[b_0, a_0]\neq 0$, there holds $1=[b_0, a_0]^{-1}[b_0, a_0]=[b_0, [b_0, a_0]^{-1}a_0]$. Accordingly, for every $c\in C$ we have $c=[b_0, c[b_0, a_0]^{-1}a_0]\in B$, which means $C\subsetneqq B$. Consequently, $B=V_A(B)$ and it is the case (3).

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