On some hypersurfaces satisfying $R(X, Y) \cdot R_1 = 0$

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

By Kouei SEKIGAWA

1. Introduction.

The Riemannian curvature tensor R of a locally symmetric Riemannian manifold (M, g) satisfies

(*) $\qquad R(X, Y)\cdot R=0 \qquad$ for all tangent vectors X and Y,

where the endomorphism $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M. Conversely, does this algebraic condition (*) on the curvature tensor field R imply that $\nabla R=0$? K. Nomizu conjectured that the answer is positive in the case where (M, g) is complete, irreducible and dim $M \geqq 3$. But, recently, H. Takagi [5] gave an example of 3-dimensional complete, irreducible Riemannian manifold (M, g) satisfying (*) and $\nabla R \neq 0$. Moreover, the present author proved that, in an $(m+1)$-dimensional Euclidean space E^{m+1} $(m \geqq 3)$, there exist some complete, irreducible hypersurfaces which satisfy the condition (*) and $\nabla R \neq 0$. For example,

$$(1.1) \qquad M;\quad x_{m+1}=(x_1-x_2)^2 x_2+(x_1-x_2) x_3 + \sum_{a=1}^{m-3} x_{a+3} e^{a(x_1-x_2)} \qquad m \geqq 4\,,$$

$$(1.2) \qquad M;\quad x_4=(x_1-x_2)^2 x_2+(x_1-x_2) x_3\,, \qquad \text{(See [3])},$$

$$(1.3) \qquad M;\quad x_4=\frac{x_1^2 x_3-x_2^2 x_3-2 x_1 x_2}{2(1+x_3^2)}\,, \qquad \text{(See [5])},$$

where $(x_1, x_2, \cdots, x_{m+1})$ denotes a canonical coordinate system on E^{m+1}.

By these examples, we see that K. Nomizu's conjecture is negative. For theses examples, we see that the type number $k(x)$ is at most 2 for each point $x \in M$ and actually 2 at some point of M. In [2], K. Nomizu proved

THEOREM A. *Let (M, g) be an m-dimensional complete Riemannian manifold which is isometrically immersed in E^{m+1} so that the type number $k(x) \geqq 3$ at least at one point $x \in M$. If (M, g) satisfies the condition (*), then it is of the form $S^k \times E^{m-k}$, where S^k is a hypersphere in a Euclidean subspace E^{k+1} of E^{m+1} and E^{m-k} is a Euclidean subspace orthogonal to E^{k+1}.*

Now, let R_1 be the Ricci tensor field of (M, g) and R^1 be the symmetric endomorphism satisfying $R_1(X, Y)=g(R^1X, Y)$. Then, the condition (*) implies in particular

$$(**) \qquad R(X, Y)\cdot R_1=0 \qquad \text{for all tangent vectors } X \text{ and } Y.$$

In [4], the present author proved

THEOREM *B. Let (M, g) be an m-dimensional complete Riemannian manifold which is isometrically immersed in E^{m+1} so that the type number $k(x)\geqq 3$ and odd at least at one point $x\in M$. If (M, g) satisfies the condition (**), then it is of the form $S^k\times E^{m-k}$.*

In the present paper, we shall prove the followings:

THEOREM *C. Let (M, g) be an m-dimensional complete Riemannian manifold which is isometrically immersed in E^{m+1} so that the type number $k(x)\geqq 3$ and odd, or $k(x)>2m/3$ at least at one point $x\in M$. If (M, g) satisfies the condition (**), then it is of the form $S^k\times E^{m-k}$,*

THEOREM *D. Let (M, g) be an m-dimensional irreducible Riemannian manifold which is isometrically immersed in E^{m+1}. If (M, g) satisfies the condition (**) and*

$$(1.4) \qquad R(X, Y)\cdot \nabla_Z R_1=0 \qquad \textit{for all tangent vectors } X, Y \textit{ and } Z,$$

then it is a space of positive constant curvature.

COROLLARY *D. Under the same hypothesis as theorem D, furthermore, if (M, g) is complete, then it is of the form S^m, that is, a hypersphere in E^{m+1}.*

2. Reduction of the condition (**).

Let (M, g) be an m-dimensional Riemannian manifold which is isometrically immersed in an $(m+1)$-dimensional Euclidean space $E^{m+1}(m\geqq 3)$, g being the Riemannian metric induced from E^{m+1}. Let U be a neighborhood of a point $x\in M$ on which we can choose a unit vector field N normal to M. For local vector fields X and Y on U tangent to M, we have the formulas of Gauss and Weingarten:

$$(2.1) \qquad D_XY=\nabla_XY+H(X, Y)N,$$

$$(2.2) \qquad D_XN=-AX,$$

where D_X and ∇_X denote the covariant differentiations for the Euclidean connection on E^{m+1} and the Riemannian connection on M, respectively. H is the second fundamental form and A is a symmetric endomorphism satisfying $H(X, Y)=g(AX, Y)$. Then the equation of Gauss is

$$R(X, Y)=AX\wedge AY. \tag{2.3}$$

The type number $k(x)$ at a point $x\in M$ is, by definition, the rank of A at X. From (2.3), the Ricci tensor R_1 of (M, g) is given by

$$R_1(X, Y)=(\text{trace } A)\ g(AX, Y)-g(A^2X, Y). \tag{2.4}$$

For each point $x\in M$, we may take an orthonormal basis $\{e_i\}$ of the tangent space $T_x(M)$ such that $Ae_i=\lambda_i e_i$ $1\leqq i, j, h, k, \cdots\leqq m$. Then the equation (2.3) implies

$$R(e_i, e_j)=\lambda_i\lambda_j e_i\wedge e_j, \tag{2.5}$$

and (2.4) implies

$$R_1(e_i, e_i)=\lambda_i\sum_{h=1}^{m}\lambda_h-\lambda_i^2, \qquad \text{and otherwise being zero.} \tag{2.6}$$

From (2.5) and (2.6), we see that the condition (**) is equivalent to

$$\lambda_i\lambda_j(\lambda_i-\lambda_j)\Big(\sum_{h=1}^{m}\lambda_h-\lambda_i-\lambda_j\Big)=0 \qquad \text{for } i\neq j. \tag{2.7}$$

From (2.7), at each point $x\in M$, we see that essentially only the following cases are possible:

(I) $\lambda_1=\cdots=\lambda_k=\lambda,\quad \lambda_{k+1}=\cdots=\lambda_m=0,$

(II) $\lambda_1=\cdots=\lambda_t=\lambda,\quad \lambda_{t+1}=\cdots=\lambda_{t+t'}=\mu,$

$\lambda_{t+t'+1}=\cdots=\lambda_m=0,$

where $k=k(x)$, and for (II), $\lambda\neq\mu$, $t=t(x)\geqq 1$, $t'=t'(x)\geqq 1$, $k=t+t'$, $(t-1)\lambda+(t'-1)\mu=0$.

If (M, g) satisfies the condition (*), then we see that (II) can not be valid on M. From (II), if $k(x)=3$, then we see that (II) can not be valid at x.

3. Lemmas.

First, we assume that the type number $k(z)>3$ at some point $z\in M$ and (II) is valid at z. Then, by the continuity argument for the characteristic polynomial of A, we see that (II) is also valid and, furthermore, t and t' are constant near z and hence, let $W=\{x\in M;\ k(x)>3$ and (II) is valid at $x\}$, which is an open set of M. For each point $x_0\in M$, let W_0 be the connected component of x_0 in W. Then, non-zero eigenvalues of A, λ and μ are certain differentiable functions on W_0 and we can take three differentiable distributions, T_λ, T_μ and T_0 corresponding to λ, μ and 0, respectively on W_0. Let $T_1(x)=T_\lambda(x)+T_\mu(x)$ (direct sum), for each point $x\in W_0$. Then, T_1 is differentiable and, from (2. 6) and (II), we have

$$(3.1)\qquad R^1X=KX,\ \text{ for }\ X\in T_1\ \text{ and }\ R^1X=0,\ \text{ for }\ X\in T_0,$$

where $K=\lambda\mu$.
Then, by [4],

LEMMA 3. 1. *T_λ and T_μ are involutive.*

For each point $x\in W_0$, let $M_\lambda(x)$ and $M_\mu(x)$ be the maximal integral submanifolds through x of T_λ and T_μ, respectively. Then we have

LEMMA 3. 2. *λ and μ are constant on each $M_\lambda(x)$ ($M_\mu(x)$, resp.)*

Now, if $k(x)=m$ at some point $x\in M$, then

PROPOSITION 3. 3. *Let (M, g) be an m-dimensional Riemannian manifold which is isometrically immersed in E^{m+1} so that the type number $k(x)=m$ at some point $x\in M$. If (M, g) satisfies the condition (**), then it is a space of positive constant curvature.*

COROLLARY 3. 3. *Under the same hypothesis as proposition 3. 3, furthermore, if (M, g) is complete, then it is a hypersphere S^m.*

In the sequel, we assume that $3\leqq k(z)<m$, that is, dim $T_0\geqq 1$. In the future, we shall show that, under some additional conditions, (II) can not be valid. By [4],

LEMMA 3. 4. *T_0 is involutive.*

For each point $x\in W_0$, let $M_0(x)$ be the maximal integral submanifold through x of T_0, then

LEMMA 3. 5. *Each $M_0(x)$ is totally geodesic and furthermore, a piece of an $(m\text{-}k)$-dimensional Euclidean space E^{m-k} in E^{m+1}.*

4. Main results.

Since T_λ, T_μ and T_0 are differentiable on W_0, for each point $x\in W_0$, we may choose a differentiable orthonormal frame field $\{E_i\}$ near x in such a way that $\{E_a\}$, $\{E_p\}$ and $\{E_u\}$ are bases for T_λ, T_μ and T_0, respectively. Here $1\leqq a, b, c, \cdots\leqq t$, $t+1\leqq p, q, r, \cdots\leqq t+t=k$, $k+1\leqq u, v, w, \cdots\leqq m$. From (2. 5) and (II), with respect to the above basis $\{E_i\}$, we have

$$(4.1)\qquad \begin{aligned} R(E_a, E_b) &= \lambda^2 E_a\wedge E_b\,,\\ R(E_a, E_p) &= \lambda\mu E_a\wedge E_p\,,\\ R(E_p, E_q) &= \mu^2 E_p\wedge E_q\,, \qquad \text{and otherwise being zero.}\end{aligned}$$

On the other hand, in general, for a local differentiable orthonormal frame field $\{E_i\}$ in a Riemannian manifold (M, g), we may put

$$(4.2)\qquad \nabla_{E_i} E_j = \sum_{k=1}^{m} B_{i\,jk} E_k\,,$$

where ∇_X denotes the covariant differentiation with respect to the Riemannian connection given by g and $B_{i\,jk} = -B_{i\,kj}$, $m = \dim M$.
Then, by [4], we have the followings:

$$(4.3)\qquad B_{u\,va} = B_{u\,vp} = 0\,,$$

$$(4.4)\qquad B_{a\,ub} = 0 \quad \text{for} \quad a \neq b\,, \quad \text{and} \quad B_{a\,pb} = 0\,,$$

$$(4.5)\qquad B_{p\,uq} = 0 \quad \text{for} \quad p \neq q\,, \quad \text{and} \quad B_{p\,aq} = 0\,,$$

$$(4.6)\qquad (\lambda - \mu) B_{u\,ap} + \mu B_{a\,up} = 0\,,$$

$$(4.7)\qquad (\mu - \lambda) B_{u\,pa} + \lambda B_{p\,ua} = 0\,,$$

and from (4. 6) and (4. 7)

$$(4.8)\qquad \lambda B_{p\,ua} - \mu B_{a\,up} = 0\,.$$

By considering $R(E_a, E_u)E_v = 0$ and $R(E_p, E_u)E_v = 0$, we have

$$(4.9)\qquad (t-t')(t+t'-1)\sum_{a=1}^{t}\sum_{p=t+1}^{k}(B_{a\,up})^2 = 0\,.$$

Now, for each $a\,(1 \leqq a \leqq t)$, we have

$$\begin{aligned} R(E_a, E_p)E_a &= \nabla_{E_a}\nabla_{E_p}E_a - \nabla_{E_p}\nabla_{E_a}E_a - \nabla_{[E_a, E_p]}E_a \\ &= \sum_{i=1}^{m}\Big(E_a B_{p\,ai} - E_p B_{a\,ai} + \sum_{j=1}^{m} B_{p\,aj} B_{a\,ji} \\ &\quad - \sum_{j=1}^{m} B_{a\,aj} B_{p\,ji} - \sum_{j=1}^{m}(B_{a\,pj} - B_{p\,aj}) B_{j\,ai}\Big) E_i\,. \end{aligned}$$

Thus, by using (4. 1), (4. 4), (4. 5), (4. 6), (4. 7) and (4. 8), we have

$$(4.10)\qquad \sum_{u=k+1}^{m} B_{a\,up} B_{a\,uq} = 0\,, \qquad \text{for} \quad p \neq q\,.$$

By [4], we have

$$(4.11)\qquad B_{a\,ua} = B_{p\,up} = -E_u\lambda/\lambda = -E_u\mu/\mu\,.$$

Thus, again, by using (4. 1), (4. 4), (4. 5), (4. 6), (4. 7), (4. 8), and (4. 11), we have

$$(4.12)\qquad \sum_{u=k+1}^{m}(B_{a\,up})^2 = \lambda^2/2 + (1/2\lambda\mu)\sum_{u=k+1}^{m}(E_u\lambda)^2, \quad p = t+1, \cdots, k.$$

First, from (4. 9), if $t \neq t'$, then we see that $B_{a\,up} = 0$. Next, we assume that $t = t'$. Then we see that $\lambda = -\mu$. Thus, from (4. 10) and (4. 12), if $m-k$

$<k/2$, that is, $k>2m/3$, then, for some $p_0(t+1\leqq p_0\leqq k)$, $B_{a u p_0}=0$, $u=k+1, \cdots, m$. Thus, form (4. 12), we see that $B_{a u p}=0$. Therefore, by [4] we have theorem C.

Next, we shall prove theorem D. From (3. 1), (4. 2) and lemma 3. 2, we have

$$(\nabla_{E_a} R^1) E_b = K \sum_{u=k+1}^{m} B_{a b u} E_u , \tag{4. 13}$$

$$(\nabla_{E_a} R^1) E_p = K \sum_{u=k+1}^{m} B_{a p u} E_u .$$

Thus, from (4. 1) and (4. 13), we have

$$\begin{aligned}(R(E_a, E_p)\cdot \nabla_{E_a} R^1) E_a &= -(\nabla_{E_a} R^1)(R(E_a, E_p) E_a) \\ &= K^2 \sum_{u=k+1}^{m} B_{a p u} E_u , \\ (R(E_a, E_b)\cdot \nabla_{E_a} R^1) E_b &= -(\nabla_{E_a} R^1)(R(E_a, E_b) E_b) \\ &= -K^2 \sum_{u=k+1}^{m} B_{a a u} E_u .\end{aligned} \tag{4. 14}$$

Thus, from (1. 3) and (4. 14), we have

$$B_{a p u} = 0 \quad \text{and hence} \quad B_{p a u} = 0 , \tag{4. 15}$$

$$B_{a u a} = 0 \quad \text{and hence} \quad B_{p u p} = 0 . \tag{4. 16}$$

Therefore, from (4. 3), (4. 4), (4. 5), (4. 6), (4. 7), (4. 15) and (4. 16), we see that T_λ, T_μ and T_0 are parallel on W_0. But, this contradicts to (4. 1). Thus, if (M, g) satisfies (**) and (1. 3), and furthermore, $k(z)\geqq 3$ at $z\in M$, then (II) can not be valid at z. Thus, (I) is valid at z. Then, let $W=\{x\in M;\ k(x)\geqq 3 \text{ at } x\}$, which is an open set of M. For each point $x_0\in W$, let W_0 be the connected component of x_0 in W. Then, from (2. 5) and (2. 6), at each point $x\in W_0$, we have

$$R(e_a, e_b) = \lambda^2\, e_a\wedge e_b , \qquad \text{and otherwise being zero,} \tag{4. 17}$$

$$R_1(e_a, e_a) = (k-1)\lambda^2 , \qquad \text{and otherwise being zero,} \tag{4. 18}$$

where $1\leqq a, b, c\cdots\leqq k$, $k+1\leqq u, v, w, \cdots\leqq m$.

Then, non-zero eigenvalue λ of A is a differentiable function on W_0 and we may take two differentiable distributions T_1 and T_0 corresponding to λ and 0, respectively on W_0. For each point $x\in W_0$, we may choose a differentiable orthonormal frame field $\{E_i\}$ near x in such a way that $\{E_a\}$ and $\{E_u\}$ are bases for T_1 and T_0, respectively. Then, by the equation of Codazzi, we have

$$(4.19)\qquad E_a\lambda=0\,,$$

$$(4.20)\qquad B_{a\,ub}=0 \quad \text{for} \quad a\neq b\,, \quad \text{and} \quad B_{u\,va}=0\,,$$

$$(4.21)\qquad B_{a\,ua}=-E_u\lambda/\lambda\,.$$

Furthermore, from (1. 3), by the similar ones as the previous arguments, we see that T_1 and T_0 are parallel on W_0. Thus, λ is constant on W_0. Since M is connected, we see that $W_0=M$. Thus we have

PROPOSITION 4. 1. *Let (M, g) be an m-dimensional Riemannian manifold which is isometrically immersed in E^{m+1} so that the type number $k(x)\geqq 3$ at least at one point $x\in M$. If (M, g) satisfies (**) and (1. 3), then (M, g) is locally of the form $M_1\times M_2$, where M_1 is a k-dimensional space of constant curvature λ^2 and M_2 is an (m-k)-dimensional locally flat space (more precisely, a piece of an (m-k)-dimensional Euclidean space E^{m-k}).*

Next, we shall assume that the type number $k(x)\leqq 2$ on M. If the type number $k(x)\leqq 1$ on M, then, from (2. 5), we see that $R=0$ on M, that is, (M, g) is locally flat and hence reducible. Thus, it is sufficient to deal with the case where the type number $k(x)\leqq 2$ on M and actually 2 at some point of M. Then, let $W=\{x\in M\,;\ k(x)=2 \text{ at } x\}$, which is an open set of M. For each point $x_0\in W$, let W_0 be the connected component of x_0 in W. Then, from (2. 5) and (2. 6), at each point $x\in W_0$, we may assume that

$$(4.22)\qquad R(e_1, e_2)=K\,e_1\wedge e_2\,, \qquad \text{and otherwise being zero,}$$

$$(4.23)\qquad R_1(e_1, e_1)=R_1(e_2, e_2)=K\,, \qquad \text{and otherwise being zero,}$$

where $K=\lambda_1\lambda_2$.

Since $R=0$ on the complement of W in M, from (4. 22) and (4. 23), we see that (M, g) satisfies (*) and hence (**). Then, K is a differentiable function on W_0, since $K=\text{trace } R^1/2$, and we may take two differentiable distributions T_1 and T_0 corresponding to K and 0, respectively on W_0. For each point $x\in W_0$, we may choose a differentiable orthonormal frame field $\{E_i\}$ near x in such a way that $\{E_a\}$ and $\{E_u\}$ are bases for T_1 and T_0, respectively. Then, from (4. 22) and (4. 23), with respect to the basis $\{E_i\}$, we have

$$(4.24)\qquad R(E_1, E_2)=K\,E_1\wedge E_2\,, \qquad \text{and otherwise being zero,}$$

$$(4.25)\qquad R^1E_1=K\,E_1\,, \quad R^1E_2=K\,E_2\,, \qquad \text{and otherwise being zero.}$$

First, by the equation of Codazzi, we have

$$(4.26)\qquad B_{u\,va}=0\,.$$

From (4. 2) and (4. 25), we have

$$(4.27)\qquad (\nabla_{E_1} R^1) E_1 = (E_1 K) E_1 + K \sum_{u=3}^{m} B_{11u} E_u ,$$

$$(\nabla_{E_1} R^1) E_2 = (E_1 K) E_2 + K \sum_{u=3}^{m} B_{12u} E_u .$$

From (1. 3) and (4. 27), we have

$$(R(E_1, E_2)\cdot \nabla_{E_1} R^1) E_1 = K^2 \sum_{u=3}^{m} B_{12u} E_u = 0 ,$$

that is, $B_{12u}=0$. Similarly, by considering $(R(E_1, E_2)\cdot \nabla_{E_1} R^1) E_2=0$, $(R(E_1, E_2)\cdot \nabla_{E_2} R^1) E_1=0$ and $(R(E_1, E_2)\cdot \nabla_{E_2} R^1) E_2=0$, we have

$$(4.28)\qquad B_{abu} = 0 .$$

Thus, from (4. 26) and (4. 28), we see that T_1 and T_0 are parallel on W_0 and hence, since $R=0$ on the complement of W in M, (M, g) is reducible. Therefore, we have theorem D.

REMARK. Another examples of complete, irreducible Riemannian manifolds satisfying the condition (*) and $\nabla R \neq 0$:

$$M ; \quad x_{m+1} = (x_1 - x_2)^2 x_2 + (x_1 - x_2) x_3 + \sum_{a=1}^{m-3} x_{a+3} (x_1 - x_2)^{a+3} \quad \text{in} \quad E^{m+1}, \quad m \geqq 4 .$$

Niigata University

References

[1] S. KOBAYASHI and K. NOMIZU: Foundations of Differential Geometry, Vol. I, Interscience Publisher, New York, 1963.

[2] K. NOMIZU: On hypersurfaces satisfying a certain condition on the curvature tensor, Tōhoku Math. J., 20 (1968), 41–59.

[3] K. SEKIGAWA: On some hypersurfaces satisfying $R(X, Y)\cdot R=0$, to appear.

[4] K. SEKIGAWA: Some hypersurfaces in a Euclidean space, Science Reports of Niigata Univ. Series A, 8 (1971), 19–25.

[5] H. TAKAGI: An example of Riemannian manifolds satisfying $R(X, Y)\cdot R=0$ but not $\nabla R=0$, to appear.

(Received on November 29, 1971)