Sufficient conditions for an almost-Hermitian manifold to be Kählerian

Dedicated to Professor Y. Katsurada on her 60th birthday

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§ 0. Introduction

If an almost-Hermitian manifold M is a Kählerian manifold, then its curvature tensor R satisfies

$$(*) \qquad R(X, Y)\cdot F=0 \qquad \text{for all tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X, Y)$ operates on the almost-complex structure tensor F as a derivation at each point on M.

Conversely, does this algebraic condition (*) on the almost-complex structure tensor field F imply that M is a Kählerian manifold? For an almost-Kählerian manifold or a K-space, Kotō and the present author (Sawaki and Kotō [3]) already showed that the answer is affirmative, that is,

THEOREM A. *If an almost-Kählerian manifold or a K-space M satisfies* $S=S^*$, *then M is Kählerian, where S is the scalar curvature and* $S^*=\frac{1}{2}F^{ab}R_{abt}{}^{c}F_{c}^{t}$.

In this theorem, the condition $S=S^*$ is weaker than $R(X, Y)\cdot F=0$, in fact, $R(X, Y)\cdot F=0$ implies $S=S^*$.

This problem for an almost-Kählerian manifold has also been studied recently by Goldberg [1] and under some additional conditions the present author [4] has proved the following

THEOREM B. *If an almost-Hermitian manifold M satisfies*

(i) $R(X, Y)\cdot F=0$, $R(X, Y)\cdot \nabla F=0$ *for all tangent vectors X and Y,*

(ii) $\nabla_{[j}S_{i]h}=0$ (*or equivalently* $\nabla_t R^t{}_{jih}=0$),

(iii) *the Ricci form is definite,*

then M is Kählerian.

THEOREM C. *If a compact Hermitian manifold M satisfies*

(i) $R(X, Y)\cdot F=0$, $R(X, Y)\cdot \Omega=0$ *for all tangent vectors X and Y,*[1)]

1) In the sequel, we omit "for all tangent vectors X and Y".

(ii) *the Ricci form is definite,*

then M is Kählerian, where Ω is the exterior derivative of the 2-form $F_{ji}dx^j \wedge dx^i$.

The purpose of the present paper is to obtain other conditions for an almost-Hermitian manifold to be Kählerian. In §1 we shall give some definitions and a proposition about the purity and hybridity of tensors. In §2 we shall prepare some lemmas for later use. The main results in an almost-Hermitian manifold and an $*O$-space will be stated in §3 and §4 respectively. Particularly, in §3 we shall give an affirmative answer to the above question in the case where M is a locally symmetric and irreducible almost-Hermitian manifold (Corollary 1).

§1. Prelimiuaries

Let M be a $2n$-dim. almost-Hermitian manifold with local coordinates $\{x^i\}$ and the structure (F_j^i, g_{ji}). Then by definition we have

$$F_j^t F_t^i = -\delta_j^i, \tag{1.1}$$

$$g_{ab} F_j^a F_i^b = g_{ji}, \tag{1.2}$$

$$F_{ji} = -F_{ij} \tag{1.3}$$

where $F_{ji} = F_j^t g_{ti}$.

If an almost-Hermitian manifold M satisfies

$$\Omega_{jih} = \nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0, \tag{1.4}$$

where ∇_j denotes the operator of covariant derivative with respect to the Riemannian connection, then it is called an almost-Kählerian manifold and if it satisfies

$$\nabla_j F_{ih} + \nabla_i F_{jh} = 0, \tag{1.5}$$

then it is called K-space (or Tachibana space or nearly Kähler manifold). Moreover, an almost-Hermitian manifold is called a Hermitian manifold if it satisfies $N_{ji}{}^h = 0$ where $N_{ji}{}^h$ is the Nijenhuis tensor, that is,

$$N_{ji}{}^h = F_j^t(\partial_t F_i^h - \partial_i F_t^h) - F_i^t(\partial_t F_j^h - \partial_j F_t^h)$$

and an almost-Kählerian manifold satisfying $N_{ji}{}^h = 0$ is called a Kählerian manifold.

Now, in an almost-Hermitian manifold, we define the following operators [6],

$$O_{ih}^{mt} = \frac{1}{2}(\delta_i^m \delta_h^t - F_i^m F_h^t), \qquad *O_{ih}^{mt} = \frac{1}{2}(\delta_i^m \delta_h^t + F_i^m F_h^t).$$

A general tensor $T_{ji}{}^{h}$(resp. $T_{jh}{}^{i}$), for example, is said to be *pure* in j, i, if it satisfies $*O_{ji}^{ab}T_{ab}{}^{h}=0$ (resp. $*O_{jb}^{ai}T_{ah}{}^{b}=0$) and $T_{ji}{}^{h}$(resp. $T_{jh}{}^{i}$) is said to be *hybrid* in j, i, if it satisfies $O_{ji}^{ab}T_{ab}{}^{h}=0$ (resp. $O_{jb}^{ai}T_{ah}{}^{b}=0$).

Then, an almost-Hermitian manifold satisfying

$$*O_{ji}^{ab}\nabla_a F_{bh}=0 \tag{1.6}$$

is called an $*O$-space and it is well known that almost-Kählerian manifold and K-space are both $*O$-spaces.[2)]

We can easily verify the following

PROPOSITION

(1) *If $T_{jh}{}^{i}$ is pure (resp. hybrid) in j, i, then*

$$F_t{}^i T_{jh}{}^t = F_j{}^t T_{th}{}^i \qquad (\textit{resp.}\quad F_t{}^i T_{jh}{}^t = -F_j{}^t T_{th}{}^i).$$

If $T_{ji}{}^{h}$ is pure (resp. hybrid) in j, i then

$$F_j{}^t T_{ti}{}^h = F_i{}^t T_{jt}{}^h \qquad (\textit{resp.}\quad F_j{}^t T_{ti}{}^h = -F_i{}^t T_{jt}{}^h).$$

(2) *Let $T_{ji}{}^{h}$ be pure in j, i. If $S_{jk}{}^{i}$ is pure (resp. hybrid) in j, i, then $T_{jr}{}^{h}S_{ik}{}^{r}$ is pure (resp. hybrid) in j,i.*

(3) *If $T_{ji}{}^{h}$ is pure in j, i and $S_k{}^{ji}$ is hybrid in j, i, then*

$$T_{ji}{}^{h}S_k{}^{ji}=0.$$

(4) *In an almost-Hermitian manifold, $\nabla_j F_{ih}$ or $\nabla_j F^{ih}$ is pure in i, h.*

(5) *In an almost-Hermitian manifold, $R(X, Y)\cdot F=0$ means that $R_{abj}{}^{i}$ is pure in j, i and hybrid in a, b.*

§ 2. Lemmas

In this section, we shall prove some lemmas for later use. The following lemma is well known.

LEMMA 2.1. (*Yano and Mogi* [5]) *In order that an almost-Hermitian manifold be Kählerian, it is necessary and sufficient that $\nabla_j F_{ih}=0$.*

LEMMA 2.2. *In an almost-Hermitian manifold, $*O_{ji}^{ab}\nabla_a F_{bh}=0$ is equivalent to $*O_{ji}^{ab}\Omega_{abh}=0$.*

PROOF. If $*O_{ji}^{ab}\nabla_a F_{bh}=0$, then by Proposition (4), we have $*O_{ji}^{ab}\Omega_{abh}=0$. Conversely, when

$$\nabla_j F_{ih}+\nabla_i F_{hj}+\nabla_h F_{ji}+F_j{}^a F_i{}^b(\nabla_a F_{bh}+\nabla_b F_{ha}+\nabla_h F_{ab})=0$$

or by Proposition (4),

2) For example, see S. Sawaki [2].

$$\nabla_j F_{ih} + F_j^a F_i^b \nabla_a F_{bh} = \nabla_i F_{jh} + F_j^a F_i^b \nabla_b F_{ah}, \tag{2.1}$$

if we put

$$T_{jih} = \nabla_j F_{ih} + F_j^a F_i^b \nabla_a F_{bh},$$

then (2.1) shows that T_{jih} is symmetric in j, i. But, since T_{jih} is skew-symmetric in i, h, we obtain $T_{jih}=0$, that is, $*O_{ji}^{ab} \nabla_a F_{bh}=0$.

Lemma 2.3. *In an almost-Hermitian manifold M satisfying $R(X, Y)\cdot F=0$, $\nabla_Z R(X, Y)\cdot F=0$ is equivalent to $R(X, Y)\cdot \nabla_Z F=0$.*

Proof. Since (1, 1) type tensor $R(X, Y)$ or F operates as a derivation on the tensor algebra of the tangent space $T_x(M)$ at each point $x \in M$, $R(X, Y)\cdot F=0$ is equivalent to $[R(X, Y), F]=0$ where $[A, B]$ means $AB-BA$ for (1.1) type tensors A and B.

Thus, lemma follows from the identity:

$$\nabla_Z\Big[R(X, Y), F\Big] = \Big[\nabla_Z R(X, Y), F\Big] + \Big[R(\nabla_Z X, Y), F\Big] + \Big[R(X, \nabla_Z Y), F\Big] + \Big[R(X, Y), \nabla_Z F\Big].$$

Lemma 2.4. *In an almost-Hermitian manifold M, $R(X, Y)\cdot \nabla_W \nabla_Z F=0$ implies $R(X, Y)\cdot F=0$.*

Proof. From $R(X, Y)\cdot \nabla_W \nabla_Z F=0$, we have

$$R_{abj}{}^s \nabla_c \nabla_d F_{si} + R_{abi}{}^s \nabla_c \nabla_d F_{js} = 0. \tag{2.2}$$

Transvecting (2.2) with F^{ji}, we have

$$2R_{abj}{}^s F^{ji} \nabla_c \nabla_d F_{si} = 0$$

or

again transvecting with $g^{ac} g^{bd}$, then this equation turns out to be

$$F^{ji} R^{ab}{}_j{}^s (\nabla_a \nabla_b F_{si} - \nabla_b \nabla_a F_{si}) = 0,$$

where $$F^{ji} = g^{aj} g^{bi} F_{ab}, \qquad R^{ab}{}_j{}^s = g^{ac} g^{bd} R_{cdj}{}^s \quad \text{etc.}.$$

Thus, by Ricci identity, we have

$$F^{ji} R^{ab}{}_j{}^s (R_{abs}{}^t F_{ti} + R_{abi}{}^t F_{st}) = 0$$

or

$$R^{abj}{}_s (R_{abjs} + F_j^i F_t^s R_{abi}{}^t) = 0$$

or $$\frac{1}{2}(R_{abj}{}^s + F_j^i F_t^s R_{abi}{}^t)(R^{abj}{}_s + F_c^j F_s^d R^{abc}{}_d) = 0$$

from which it follows that

$$R_{abj}{}^{s}+F_{j}^{i}F_{t}^{s}R_{abi}{}^{t}=0\,,$$

that is, $R(X, Y)\cdot F=0$, by virtue of Proposition (5).

§ 3. Theorems in an almost-Hermitian manifold

THEOREM 3. 1. *If an almost-Hermitian manifold M satisfies*

(i) $\qquad R(X, Y)\cdot F=0\,, \qquad \nabla_Z R(X, Y)\cdot F=0\,,$

(ii) $\qquad$ *the rank of the Ricci form is maximum*,

then M is Kählerian.

PROOF. By Lemma 2.3, from (i) we have $R(X, Y)\cdot\nabla_Z F=0$, that is,

(3. 1)
$$R_{abj}{}^{s}\nabla_m F_{si}+R_{abi}{}^{s}\nabla_m F_{js}=0$$

or transvecting (3.1) with g^{bj}, we have

(3. 2)
$$S^{as}\nabla_m F_{si}+R^{ab}{}_{i}{}^{s}\nabla_m F_{bs}=0\,.$$

On the other hand, making use of the first Bianchi identity, we have

$$R^{ab}{}_{i}{}^{s}\nabla_m F_{bs}=\frac{1}{2}(R^{ab}{}_{i}{}^{s}-R^{as}{}_{i}{}^{b})\nabla_m F_{bs}$$

$$=-\frac{1}{2}R^{a}{}_{i}{}^{sb}\nabla_m F_{bs}\,.$$

But, by the assumption $R(X, Y)\cdot F=0$, $R^{a}{}_{i}{}^{sb}$ is hybrid in s, b and by Proposition (4), $\nabla_m F_{bs}$ is pure in b, s and therefore the last term vanishes, by virtue of Proposition (3).

Thus, from (3.2), we have

$$S^{as}\nabla_m F_{si}=0\,.$$

Consequently, by the assumption (ii), we have $\nabla_m F_{si}=0$ which shows that M is Kählerian, by virture of Lemma (2.1).

Since a locally symmetric and irreducible almost-Hermitian manifold is an Einstein manifold, we have the following

COROLLARY 1. *If a locally symmetric and irreducible almost-Hermitian manifold M with $S\neq0$ satisfies $R(X, Y)\cdot F=0$, then M is Kählerian.*

COROLLARY 2. *If an almost-Hermitian manifold M satisfies*

(i) $\qquad R(X, Y)\cdot\nabla_Z F=0\,, \qquad R(X, Y)\cdot\nabla_W\nabla_Z F=0\,,$

(ii) $\qquad$ *the rank of the Ricci form is maximum*,

then M is Kählerian.

PROOF. This follows directly from Lemma 2.4 and Theorem 3.1.

§ 4. Theorems in an *O-space

THEOREM 4.1. *If an *O-space M satisfies*

(i) $R(X, Y)\cdot F=0, \quad R(X, Y)\cdot \Omega=0,$

(ii) *the rank of the Ricci form is maximum,*

then M is Kählerian.

PROOF. From $R(X, Y)\cdot\Omega=0$, we have

$$R_{abj}{}^{s}\Omega_{sih}+R_{abi}{}^{s}\Omega_{jsh}+R_{abh}{}^{s}\Omega_{jis}=0 .$$

Transvecting this equation with g^{bj}, we have

$$S^{as}\Omega_{sih}+R^{ab}{}_{i}{}^{s}\Omega_{bsh}+R^{ab}{}_{h}{}^{s}\Omega_{bis}=0 \tag{4.1}$$

Now, the second term of the left hand side of (4.1), by the first Bianchi dentity, can be written as

$$R^{ab}{}_{i}{}^{s}\Omega_{bsh}=\frac{1}{2}(R^{ab}{}_{i}{}^{s}-R^{as}{}_{i}{}^{b})\,\Omega_{bsh}$$

$$=-\frac{1}{2}R^{a}{}_{i}{}^{sb}\Omega_{bsh}$$

and this last term vanishes by virtue of Proposition (3), because by the assumption $R(X, Y)\cdot F=0$, $R^{a}{}_{i}{}^{sb}$ is hybrid in s, b, and by Lemma 2.2, Ω_{bsh} is pure in s, b. Similarly, the third term of (4.1) also vanishes.

Thus, (4.1) becomes

$$S^{as}\Omega_{sih}=0$$

from which it follows that $\Omega_{sih}=0$, by virtue of (ii), that is, M is an almost-Kählerian manifold and therefore Theorem A proves the theorem.

THEOREM 4.2. *In an *O-space M satisfies*

(i) $R(X, Y)\cdot F=0, \quad R(X, Y)\cdot N=0,$

(ii) $\nabla_{[j}S_{i]h}=0 \quad$ (*or equivalently* $\nabla^{t}R_{tjih}=0$),

(iii) *the Ricci form is definite,*

then M is Kählerian.

PROOF. First we note that in an *O-space the Nijenhuis tensor can be written as

$$N_{jih}=2F_{j}^{t}(\nabla_{t}F_{ih}-\nabla_{i}F_{th}).$$

Hence, making use of the assumption $R(X, Y)\cdot F=0$, from $R(X,Y)\cdot N=0$, we have

$$2F_j^t\left[\nabla_a\nabla_b(\nabla_t F_{ih}-\nabla_i F_{th})-\nabla_b\nabla_a(\nabla_t F_{ih}-\nabla_i F_{th})\right]=0\,.$$

Transvecting this equation with $\frac{1}{2}F_k^j$ and using Ricci identity, we have

$$(4.2)\qquad R_{abk}{}^s(\nabla_s F_{ih}-\nabla_i F_{sh})+R_{abi}{}^s(\nabla_k F_{sh}-\nabla_s F_{kh})+R_{abh}{}^s(\nabla_k F_{is}-\nabla_i F_{ks})=0\,.$$

Multiplying (4.2) by $g^{bk}(\nabla^a F^{ih}-\nabla^i F^{ah})$, we have

$$(4.3)\qquad \begin{aligned} S^{as}(\nabla_s F_{ih}-\nabla_i F_{sh})(\nabla_a F^{ih}-\nabla^i F_a{}^h) &= -R^{ab}{}_i{}^s(\nabla_b F_{sh}-\nabla_s F_{bh})(\nabla_a F^{ih}-\nabla^i F_a{}^h)\\ &\quad +R^{ab}{}_h{}^s\nabla_i F_{bs}(\nabla_a F^{ih}-\nabla^i F_a{}^h)+R^{abhs}(\nabla_b F_{is})\nabla^i F_{ah}\\ &\quad -R^{ab}{}_h{}^s(\nabla_b F_{is})\nabla_a F^{ih}\,. \end{aligned}$$

We are now going to show that the right hand side of (4.3) vanishes. Its first term, by the first Bianchi identity, turns out to be

$$\begin{aligned} R^{ab}{}_i{}^s(\nabla_b F_{sh}-\nabla_s F_{bh})(\nabla_a F^{ih}-\nabla^i F_a{}^h) &=\frac{1}{2}(R^{ab}{}_i{}^s-R^{as}{}_i{}^b)(\nabla_b F_{sh}-\nabla_s F_{bh})(\nabla_a F^{ih}-\nabla^i F_a{}^h)\\ &=-\frac{1}{2}R^a{}_i{}^{sb}(\nabla_b F_{sh}-\nabla_s F_{bh})(\nabla_a F^{ih}-\nabla^i F_a{}^h) \end{aligned}$$

and therefore, by virtue of Proposition (3), it vanishes, because $R^a{}_i{}^{sb}$ is hybrid in s, b and $\nabla_b F_{sh}-\nabla_s F_{bh}$ is pure in s, b. By the same method, we can see that the second and third terms vanish. For the last term, by the assumption (ii), we have

$$R^{ab}{}_h{}^s(\nabla_b F_{is})\nabla_a F^{ih}=\nabla_a(R^{ab}{}_h{}^s F^{ih}\nabla_b F_{is})-R^{ab}{}_h{}^s F^{ih}\nabla_a\nabla_b F_{is}\,.$$

In the right hand side of this equation, by Proposition (3) we have $R^{ab}{}_h{}^s F^{ih}\nabla_b F_{is}=0$, because by Proposition (2) $R^{ab}{}_h{}^s F^{ih}$ is hybrid in s, i and $\nabla_b F_{is}$ is pure in i, s. For the last term, since $R^{ab}{}_h{}^s$ is skew-symmetric in a, b, and by the assumption $R(X, Y)\cdot F=0$, $\nabla_a\nabla_b F_{is}$ is symmetric in a, b, it vanishes.

Consequently, (4.3) becomes

$$S^{as}(\nabla_s F_{ih}-\nabla_i F_{sh})(\nabla_a F^{ih}-\nabla^i F_a{}^h)=0$$

from which it follows that, by the assumption (iii),

$$\nabla_s F_{ih}-\nabla_i F_{sh}=0\,.$$

But, since $\nabla_s F_{ih}$ is skew-symmetric in i, h, we have $\nabla_s F_{ih}=0$ which shows that M is Kählerian.

We conclude this section with the following theorem where the condition $R(X, Y)\cdot F=0$ has been removed.

THEOREM 4.3. *If a compact almost-Kählerian manifold M satisfies*

(i) $R(X, Y)\cdot N=0$,

(ii) $\nabla_{[j}S_{i]h}=0$ (*or equivalently* $\nabla^t R_{tjih}=0$),

(iii) *the Ricci form is semi-negative definite*,

then M is Kählerian.

PROOF. First we note that in an almost-Kählerian manifold, the Nijenhuis tensor can be written as

$$N_{ji}{}^h=2F_t{}^h\nabla^t F_{ji}.$$

Thus, from (i) we have

$$R_{abs}{}^h F_t{}^s\nabla^t F_{ji}-R_{abj}{}^s F_t{}^h\nabla^t F_{si}-R_{abi}{}^s F_t{}^h\nabla^t F_{js}=0$$

or transvecting this equation with δ_h^b,

$$S_{as}F_t{}^s\nabla^t F_{ji}+R_{ahj}{}^s F_t{}^h\nabla^t F_{si}+R_{ahi}{}^s F_t{}^h\nabla^t F_{js}=0. \tag{4.4}$$

Again transvecting (4.4) with $F_k{}^i\nabla^a F^{jk}$, we have

$$\begin{aligned}S_{as}F_t{}^s(\nabla^t F_{ji})\,F_k{}^i\nabla^a F^{jk}+R_{ahj}{}^s F_t{}^h(\nabla^t F_{si})\,F_k{}^i\nabla^a F^{jk}&\\+R_{ahi}{}^s F_t{}^h(\nabla^t F_{js})\,F_k{}^i\nabla^a F^{jk}&=0.\end{aligned} \tag{4.5}$$

Now, the each term of (4.5), making use of Proposition (1), turns out to be the following

$$S_{as}F_t{}^s(\nabla^t F_{ji})\,F_k{}^i\nabla^a F^{jk}=S_{as}F_{ti}(\nabla^s F_j{}^t)\,F_k{}^i\nabla^a F^{jk}=S_{as}(\nabla^s F_{jk})\,\nabla^a F^{jk},$$

$$R_{ahj}{}^s F_t{}^h(\nabla^t F_{si})\,F_k{}^i\nabla^a F^{jk}=R_{ahj}{}^s F_{ti}(\nabla^h F_s{}^t)\,F_k{}^i\nabla^a F^{jk}=R_{ahj}{}^s(\nabla^h F_{sk})\,\nabla^a F^{jk},$$

$$R_{ahi}{}^s F_t{}^h(\nabla^t F_{js})\,F_k{}^i\nabla^a F^{jk}=R_{ahi}{}^s F_{tj}(\nabla^h F^t{}_s)\,F_k{}^j\nabla^a F^{ki}=R_{ahi}{}^s(\nabla^h F_{ks})\,\nabla^a F^{ki},$$

respectively.

Hence, (4.5) reduces to

$$S_{as}(\nabla^s F_{jk})\,\nabla^a F^{jk}+2R_{ahj}{}^s(\nabla^h F_{sk})\,\nabla^a F^{jk}=0. \tag{4.6}$$

Integrating (4.6) over M and making use of Green's theorem and the assumption (ii), we have

$$\int_M S_{as}(\nabla^s F_{jk})\,\nabla^a F^{jk}\,dV-2\int_M R_{ahj}{}^s(\nabla^a\nabla^h F_{sk})\,F^{jk}\,dV=0$$

where dV denotes the volume element of M, or by Ricci identity

$$\int_M S_{as}(\nabla^s F_{jk})\nabla^a F^{jk} dV + \int_M R_{ahj}{}^s (R^{ah}{}_s{}^t F_{tk} + R^{ah}{}_k{}^t F_{st}) F^{jk} dV = 0$$

or

$$\int_M S_{as}(\nabla^s F_{jk})\nabla^a F^{jk} dV - \int_M R_{ahj}{}^s (R^{ahj}{}_s + F_k{}^j F_s{}^t R^{ahk}{}_t) dV = 0 . \tag{4.7}$$

(4.7) can be written as

$$\int_M S_{as}(\nabla^s F_{jk})\nabla^a F^{jk} dV$$
$$-\frac{1}{2}\int_M (R_{ahj}{}^s + F_j{}^k F_t{}^s R_{ahkt})(R^{ahj}{}_s + F_c{}^j F_s{}^d R^{ahc}{}_d) dV = 0 .$$

Consequently, by the assumption (iii), we have

$$R_{ahj}{}^s + F_j{}^k F_t{}^s R_{ahk}{}^t = 0 ,$$

that is, $R(X, Y)\cdot F=0$ and therefore, Theorem A proves the theorem.

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