Positively curved complex submanifolds immersed in a complex projective space II

Dedicated to Professor Y. Katsurada on her 60th birthday

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1. Introduction

Let $P_m(\boldsymbol{C})$ be a complex projective space of complex dimension m with the Fubini-Study metric of constant holomorphic sectional curvature 1. Recently S. Tanno [6] has proved the following result.

PROPOSITION A. *Let M be an n-dimensional complete complex submanifold immersed in $P_{n+p}(\boldsymbol{C})$. If every holomorphic sectional curvature of M with respect to the induced metric is greater than $1-\dfrac{n+2}{6n^2}$, then M is complex analytically isometric to a linear subspace $P_n(\boldsymbol{C})$.*

In this paper we shall prove the following theorems.

THEOREM 1. *Let M be an n-dimensional complete complex submanifold immersed in $P_{n+p}(\boldsymbol{C})$. If every Ricci curvature of M with respect to the induced metric is greater than $n/2$, then M is complex analytically isometric to a linear subspace $P_n(\boldsymbol{C})$.*

Theorem 1 is the best possible in this direction.

THEOREM 2. *Let M be an n-dimensional complete submanifold immersed in $P_{n+p}(\boldsymbol{C})$. If every holomorphic sectional curvature of M with respect to the induced metric is greater than δ, then M is complex analytically isometric to a linear subspace $P_n(\boldsymbol{C})$, where*

$$\delta=\begin{cases}\dfrac{3n-1}{3n+1} & (n\leq 5)\\[2ex] \dfrac{2n-3}{2n-2} & (n>5).\end{cases}$$

Theorem 2 is an improvement of Proposition A.

THEOREM 3. *Let M be an n-dimensional complete complex submanifold immersed in $P_{n+p}(\boldsymbol{C})$. If $n\geq 2$ and if every sectional curvature of M with respect to the induced metric is greater than δ, then M is complex analytically*

Work done under partial support by the Sakko-kai Foundation.

isometric to a linear subspace $P_n(\boldsymbol{C})$, *where*

$$\delta=\begin{cases}\dfrac{5}{23} & (n=5)\\[2mm] \dfrac{5n-2-\sqrt{9n^2+60n+4}}{8(n-5)} & (n\neq 5).\end{cases}$$

2. Preliminaries

Let J (resp. $\tilde{J}$) be the complex structure of M (resp. $P_{n+p}(\boldsymbol{C})$) and g (resp. $\tilde{g}$) be the Kaehler metric of M (resp. $P_{n+p}(\boldsymbol{C})$). We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation with respect to g (resp. $\tilde{g}$). Then the second fundamental form σ of the immersion is given by

$$\sigma(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y.$$

Let R be the curvature tensor field of M. Then the equation of Gauss is

$$\begin{aligned} g\big(R(X, Y)Z, W\big) &= \tilde{g}\big(\sigma(X, W), \sigma(Y, Z)\big)-\tilde{g}\big(\sigma(X, Z), \sigma(Y, W)\big)\\ &+\frac{1}{4}\{g(X, W)g(Y, Z)-g(X, Z)g(Y, W)\\ &+g(JX, W)g(JY, Z)-g(JX, Z)g(JY, W)+2g(X, JY)g(JZ, W)\}. \end{aligned}$$

Let $\xi_1, \cdots, \xi_p, \tilde{J}\xi_1, \cdots, \tilde{J}\xi_p$ be local fields of orthonormal vectors normal to M. If we set, for $i=1, \cdots, p$,

$$\begin{aligned} g(A_iX, Y) &= \tilde{g}\big(\sigma(X, Y), \xi_i\big)\\ g(A_{i*}X, Y) &= \tilde{g}\big(\sigma(X, Y), \tilde{J}\xi_i\big), \end{aligned}$$

then $A_1, \cdots, A_p, A_{1*}, \cdots, A_{p*}$ are local fields of symmetric linear transformations. We can easily see that $A_{i*}=JA_i$ and $JA_i=-A_iJ$ so that, in particular, $\mathrm{tr}A_i=\mathrm{tr}A_{i*}=0$. The equation of Gauss can be written in terms of A_i's as

$$\begin{aligned} g\big(R(X, Y)Z, W\big) &= \sum\{g(A_iX, W)g(A_iY, Z)-g(A_iX, Z)g(A_iY, W)\\ &\qquad +g(JA_iX, W)g(JA_iY, Z)-g(JA_iX, Z)g(JA_iY, W)\}\\ &+\frac{1}{4}\{g(X, W)g(Y, Z)-g(X, Z)g(Y, W)\\ &+g(JX, W)g(JY, Z)-g(JX, Z)g(JY, W)+2g(X, JY)g(JZ, W)\}. \end{aligned}$$

Let S be the Ricci tensor of M. Then we have

$$(1) \qquad S(X, Y)=\frac{n+1}{2}g(X, Y)-2g(\sum A_i^2X, Y).$$

Let $\|\sigma\|$ be the length of the second fundamental form of the immersion so that $\|\sigma\|^2=2\sum \mathrm{tr}A_i^2$.

We know that the second fundamental form σ satisfies the following differential equation ([4]).

$$\frac{1}{2}\Delta\|\sigma\|^2=\|\tilde{\nabla}\sigma\|^2+\sum \mathrm{tr}(A_\lambda A_\mu-A_\mu A_\lambda)^2-\sum(\mathrm{tr}A_\lambda A_\mu)^2+\frac{n+2}{2}\|\sigma\|^2,$$

where Δ denotes the Laplacian and $\lambda, \mu=1,\cdots,p, 1^*,\cdots,p^*$. For a suitable choice of $\xi_1,\cdots,\xi_p, J\xi_1,\cdots, J\xi_p$, the above differential equation can be written as follows ([5, 6]).

$$(2) \qquad \frac{1}{2}\Delta\|\sigma\|^2=\|\tilde{\nabla}\sigma\|^2-8\,\mathrm{tr}(\sum A_i^2)^2-2\sum(\mathrm{tr}A_i^2)^2+\frac{n+2}{2}\|\sigma\|^2.$$

3. Proof of Theorm 1

First we note that, by a theorem of Myers ([3]), M is compact.

Since $S-\frac{n}{2}g$ is positive definite, we can see from (1) that $I-4\sum A_i^2$ is positive definite, where I denotes the identity transformation. This implies

$$(3) \qquad \|\sigma\|^2<n.$$

Moreover, since A_i's are symmetric linear transformations, $\sum A_i^2$ is positive semi-definite. Since $\sum A_i^2$ and $I-4\sum A_i^2$ can be transformed simultaneously by an orthogonal matrix into diagonal forms at each point of M, $(\sum A_i^2)(I-4\sum A_i^2)$ is positive semi-definite. Hence we have

$$(4) \qquad 8\,\mathrm{tr}(\sum A_i^2)^2\leq\|\sigma\|^2.$$

On the other hand, we can see

$$(5) \qquad \sum(\mathrm{tr}\,A_i^2)^2\leq(\sum \mathrm{tr}\,A_i^2)^2=\frac{1}{4}\|\sigma\|^4.$$

From (2), (3), (4) and (5), we have

$$(6) \qquad \frac{1}{2}\Delta\|\sigma\|^2\geq\frac{1}{2}\|\sigma\|^2(n-\|\sigma\|^2)\geq 0.$$

Hence, by a well-known theorem of E. Hopf, $\|\sigma\|^2$ is a constant. This, together with (3) and (6), implies $\|\sigma\|=0$. Therefore M is a totally geodesic submanifold.

4. Proof of Theorem 2 and Theorem 3

To prove Theorem 2, we need the following Proposition due to Bishop and Goldberg (Theorem 8.1 in [2]).

PROPOSITION 1. *If every holomorphic sectional curvature of M is greater than δ, then every Ricci curvature of M is greater than μ, where*

$$\mu=\begin{cases}\dfrac{(3n+1)\delta-(n-1)}{4} & (n\leq 5)\\[2ex] (n-1)\delta-\dfrac{n-3}{2} & (n>5).\end{cases}$$

We can see that if

$$\delta=\begin{cases}\dfrac{3n-1}{3n+1} & (n\leq 5)\\[2ex] \dfrac{2n-3}{2n-2} & (n>5),\end{cases}$$

then $\mu=\dfrac{n}{2}$.

This, combined with Theorem 1, implies Theorem 2.

To prove Theorem 3, we need the following Proposition due to Berger ([1]).

PROPOSITION 2. *If $n\geq 2$ and if the sectional curvature K of M satisfies $\delta<K\leq 1$, then every holomorphic sectional curvature of M is greater than $\dfrac{\delta(8\delta+1)}{1-\delta}$.*

Let x be an arbitrary point of M and X be an arbitrary unit vector in $T_x(M)$. If $e_1=X, e_2, \cdots, e_n, Je_1, \cdots, Je_n$ is an orthonormal basis of $T_x(M)$, then

$$S(X,X)=H(X)+\sum_{i=2}^{n}\left\{K(X,e_i)+K(X,Je_i)\right\},$$

where $H(X)$ is the holomorphic sectional curvature of M determined by X and $K(X,Y)$ is the sectional curvature of M determined by X and Y. Hence, by Proposition 2, $K>\delta$ implies

$$S(X,X)>\frac{\delta(8\delta+1)}{1-\delta}+2(n-1)\delta.$$

We can see that if

$$\delta = \begin{cases} \dfrac{5}{23} & (n=5) \\ \dfrac{5n-2-\sqrt{9n^2+60n+4}}{8(n-5)} & (n \neq 5), \end{cases}$$

then $S(X, X) > \dfrac{n}{2}$.

This, combined with Theorem 1, implies Theorem 3.

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(Received September 13, 1971)