Riemannian manifolds admitting more than $n-1$ linearly independent solutions of $\nabla^2\rho+c^2\rho g=0$

Dedicated to Prof. Yoshie Katsurada on the occassion of her sixtieth birthday

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Let M be a connected C^∞-Riemannian manifold of $n(\geqq 2)$ dimensions with Riemannian metric g. Let us consider the system of partial differential equations

$$\nabla_j\nabla_i\rho+c^2\rho g_{ji}=0 \qquad (c>0) \tag{1}$$

on M, where ∇_i denote the local components of the covariant derivative with respect to the Riemannian connection associated to g $(i, j, k, \cdots=1, 2, \cdots, n)$.

In a complete Riemannian manifold the existence of a non-trivial solution of (1) uniquely determines the Riemannian manifold structure up to an isometry. In fact, the following theorem is well known.

THEOREM A (Obata [1], [2], [3]). *Let M be complete. In order for M to admit a non-trivial solution of* (1), *it is necessary and sufficient that M be isometric to a sphere $S^n\left(\frac{1}{c}\right)$ of radius $\frac{1}{c}$ in the* $(n+1)$*-dimensional Euclidean space E^{n+1}.*

In the present paper we shall deal with Riemannian manifolds, admitting more than $n-1$ linearly independent solutions of (1), instead of the assumption of completeness, and prove the following three theorems.

THEOREM B. *In order for M to admit $n+1$ solutions of* (1), *linearly independent over the real number field R, it is necessary and sufficient that M be isometrically immersed in $S^n\left(\frac{1}{c}\right)$ in E^{n+1}.*

THEOREM C. *Let M be simply connected. In order for M to admit n solutions of* (1), *linearly independent over R, it is necessary and sufficient that M be isometrically immersed in $S^n\left(\frac{1}{c}\right)$* in E^{n+1}.

THEOREM D. *If M admit $n-1$ solutions of* (1), *linearly independent over R, M is of constant curvature c^2.*

The rest of the present paper is devoted itself to the proofs of these three theorems.

Let F be the vector space over R, consisting of all the solutions of (1). We can define a positive definite bilinear form $\langle\,,\rangle$ on F, defined by

$$\langle\rho, \tau\rangle = c^2\rho\tau + g^{ij}\nabla_i\rho\nabla_j\tau \qquad \text{for} \quad \rho, \tau \in F.$$

In fact, this is constant on M, since

$$\begin{aligned}\nabla_k\langle\rho, \tau\rangle &= c^2(\tau\nabla_k\rho + \rho\nabla_k\tau) + g^{ij}(\nabla_i\rho\nabla_k\nabla_j\tau + \nabla_k\nabla_i\rho\nabla_j\tau)\\ &= 0,\end{aligned}$$

on account of the equations (1).

Let $\rho \in F$. We have

$$Z_{kji}{}^h\nabla_h\rho = 0, \tag{2}$$

as the integrability condition for (1) and

$$\nabla_l Z_{kji}{}^h\nabla_h\rho = c^2 Z_{kjil}\rho, \tag{3}$$

differentiating (2) covariantly, where Z is the concircular curvature tensor with components

$$Z_{kji}{}^h = R_{kji}{}^h - c^2(g_{ji}\delta_k^h - g_{ki}\delta_j^h),$$

and $Z_{kjil} = Z_{kji}{}^h g_{hl}$.

Proof of Theorem B. The sufficiency follows from the fact that on $S^n\left(\frac{1}{c}\right)$ in E^{n+1} defined by $\sum_{\alpha=0}^{n}(y^\alpha)^2 = \frac{1}{c^2}$ every coordinate function y^α restricted on $S^n\left(\frac{1}{c}\right)$ satisfies (1) on $S^n\left(\frac{1}{c}\right)$ and y^α's induce $n+1$ linearly independent solutions of (1) on M for an isometric immersion $i: M \to S^n\left(\frac{1}{c}\right)$ $(\alpha, \beta = 0, 1, 2, \cdots, n)$.

Conversely, suppose that there are $n+1$ solutions of (1) on M, linearly independent over R. We can assume without loss of generality that they are orthonormal with respect to the bilinear form $\langle\,,\rangle$, i.e.,

$$\langle\rho^\alpha, \rho^\beta\rangle = c^2\rho^\alpha\rho^\beta + g^{ij}\nabla_i\rho^\alpha\nabla_j\rho^\beta = \delta^{\alpha\beta}. \tag{4}$$

We claim that

$$c^2\sum_{\alpha=0}^{n}(\rho^\alpha)^2 = 1 \tag{5}$$

and

$$g = \sum_{\alpha=0}^{n}\nabla\rho^\alpha\otimes\nabla\rho^\alpha, \tag{6}$$

which imply that the map $\tilde{\rho}: M \to E^{n+1}$, $x \mapsto (\rho^\alpha(x))$, is an isometric immersion into $S^n\left(\frac{1}{c}\right)\subset E^{n+1}$. To see this, we shall calculate the lengths of the gradient $\nabla\phi$ of the function $\phi=\sum_{\alpha=0}^{n}(\rho^\alpha)^2$ and of the tensor $g-\sum_{\alpha=0}^{n}\nabla\rho^\alpha\otimes\nabla\rho^\alpha$:

$$\begin{aligned}\|\nabla\phi\|^2&=\sum_{\alpha,\beta=0}^{n}\rho^\alpha\rho^\beta g^{ij}\nabla_i\rho^\alpha\nabla_j\rho^\beta\\&=\sum_{\alpha,\beta=0}^{n}\rho^\alpha\rho^\beta(\delta^{\alpha\beta}-c^2\rho^\alpha\rho^\beta)\\&=\phi(1-c^2\phi)\geqq 0\end{aligned}$$

and

$$\begin{aligned}\left\|g-\sum_{\alpha=0}^{n}\nabla\rho^\alpha\otimes\nabla\rho^\alpha\right\|^2&=n-2\sum_{\alpha=0}^{n}g^{ij}\nabla_i\rho^\alpha\nabla_j\rho^\alpha+\sum_{\alpha,\beta=0}^{n}(g^{ij}\nabla_i\rho^\alpha\nabla_j\rho^\beta)^2\\&=n-2\sum_{\alpha=0}^{n}\left\{1-c^2(\rho^\alpha)^2\right\}+\sum_{\alpha,\beta=0}^{n}(\delta^{\alpha\beta}-c^2\rho^\alpha\rho^\beta)^2\\&=c^4\phi^2-1\geqq 0\,,\end{aligned}$$

which prove (5) and (6) at the same time.

PROOF of THEOREM D. Let us assume that M admit $n-1$ solutions ρ^λ $(\lambda=1,2,\cdots,n-1)$ of (1) on M, linearly independent over R, which may be supposed to be orthonormal with respect to the bilinear form $\langle\,,\rangle$ without loss of generality. Let x be an arbitrary point of M. If $n-1$ covectors $\nabla\rho^\lambda$ are linearly dependent at x, then there are $n-1$ constants a_λ such that x, is a stationary point of $\rho=\sum_{\lambda=1}^{n-1}a_\lambda\rho^\lambda\in F$ with $\sum_{\lambda=1}^{n-1}(a_\lambda)^2=1$, i.e., $\nabla\rho=\sum_{\lambda=1}^{n-1}a_\lambda\nabla\rho^\lambda$ vanishes at x. Then the concircular curvature tensor Z vanishes at x on account of (3) and $\langle\rho,\rho\rangle=c^2\rho(x)^2=1$. On the other hand, if $n-1$ covectors $\nabla\rho^\lambda$ are linearly independent at x, it is easy to see that the sectional curvature for every plane section at x is constant c^2, because of the identities $Z_{kji}{}^h\nabla_h\rho^\lambda=0$ at x. Thus M must be of constant curvature c^2.

PROOF of THEOREM C. The sufficiency is contained in the proof of Theorem B. Therefore, we only prove the necessity. Let ρ^a $(a=1,2,\cdots,n)$ be n solutions of (1) on M, linearly independent over R, which we may assume to be orthonormal with respect to the bilinear form $\langle\,,\rangle$. At first we note that M is of constant curvature c^2 by Theorem D. Then the integrability conditions for (1) are automatically satisfied. Hence for every point x of M the vector space F_x of germs of solutions at x of (1) is of $n+1$ dimensions over R and $F=\bigcup_{x\in M}F_x$ is a Riemannian vector bundle over M with fibre E^{n+1}, the metric of which is canonically induced from the bilinear form $\langle , \rangle$. Since ρ^a determine a field of orthonormal n-frames in F, our problem is reduced to the extendability of this field of orthonormal n-frames to a field of orthonormal $(n+1)$-frames in F. However, if M is simply connected, it is a simple consequence of the theory of fibre bundles (cf., e.g., 13.9. Corollary of Steenrod [4]).

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