# Totally geodesic foliations with compact leaves

Dedicated to Professor Y. Katsurada on her 60th birthday

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## § 1. Introduction

Concerning totally geodesic foliations, D. Ferus [5] obtained a very interesting theorem: Let $\rho(t)$ denote the largest integer such that the fibration $V'_{t,\rho(t)}\rightarrow V'_{t,1}$ of Stiefel manifolds has a global cross section. Define $\nu_m$ to be the largest integer such that $\rho(m-\nu_m)\geq\nu_m+1$. $\nu_m$ has properties;

(i) $\nu_m = m-$[highest power of $2\leq m$] for $m\leq 24$,

(ii) $\nu_m\leq(m-1)/2$,

etc. (for more details, see [5]). Then

THEOREM A (D. Ferus) *Let $(M^m, g)$ be an m-dimensional Riemannian manifold and let $T_0$ be a ν-dimensional integrable distribution on $M^m$ with the following properties*:

(1) *the maximal integral manifolds of $T_0$ are totally geodesic and complete,*

(2) *the sectional curvature of $(M^m, g)$ has the same positive value k on all planes spanned by tangent vectors $X\in T_0$ and $Y\in T_0^{\perp}$,*

*Then $\nu>\nu_m$ implies $\nu=m$.*

By (ii), $\nu\geq m/2$ implies $\nu>\nu_m$.

A natural question is: If we replace "the same positive value $k$" in (2) by "positive", what can we say?

If we assume that maximal integral manifolds (= leaves) are compact, under the weaker condition "positive" we have the same conclusion for $m=3$, 6, 7, 14, 15, etc. Namely, we have

THEOREM B. *Let $(M^m, g)$ be an m-dimensional Riemannian manifold and let $T_0$ be a ν-dimensional integrable distribution of $M^m$ with the following properties*:

(1)′ *the maximal integral manifolds are totally geodesic and compact,*

(2)′ *the sectional curvature of $(M^m, g)$ is positive on all planes spanned by tangent vectors $X\in T_0$ and $Y\in T_0^{\perp}$.*

*Then $\nu\geq m/2$ implies $\nu=m$.*

"$\nu\geq m/2$ implies $\nu=m$" is the best possible result for $m=3$, 6 and 7.

In §2 we prove Theorem B by applying a technique of T. Frankel [6]. In §3 we give some remarks.

## §2. Proof of Theorem B

A theorem of T. Frankel [6] is as follows: Let $(M^m, g)$ be a complete Riemannian manifold with positive curvature and let $V^v$ and $W^w$ be compact totally geodesic submanifolds of $(M^m, g)$ with dimension $v$ and $w$ respectively. If $v+w\geq m$, then $V^v$ and $W^w$ have a nonempty intersection.

A brief summary of the proof is as follows: If we assume that $V^v$ and $W^w$ do not intersect, then there is a shortest geodesic $x(t)$, $0\leq t\leq l=$length of $x(t)$, from $V^v$ to $W^w$. $x(t)$ strikes $V^v$ and $W^w$ orthogonally at $p=x(0)$ and $q=x(l)$. By the assumption $v+w\geq m$, we have a unit tangent vector $X_0$ to $V^v$ at $p$ such that parallel translate $X_t$ of $X_0$ along $x(t)$ has a property that $X_l$ is tangent to $W^w$ at $q$. Using $X_t$ as a variation vector, we have the variation by curves joining $V^v$ to $W^w$. Denote by $Z_t$ the unit tangent vector to $x(t)$ at $x(t)$. He used

$$L_X''(0)=g(\nabla_X X, Z)_q-g(\nabla_X X, Z)_p-\int_0^l K(X, Z)dt \tag{2.1}$$
$$=-\int_0^l K(X, Z)dt\,,$$

where $K(X, Z)=K(X_t, Z_t)$ denotes the sectional curvature for the plane determined by $X_t$ and $Z_t$. Then $K(X, Z)>0$ gives a contradiction.

Theorem B follows from the following.

THEOREM B'. *Let $(M^m, g)$ be an $m$-dimensional Riemannian manifold and let $T_0$ be a $\nu$-dimensional integrable distribution of $M^m$ with the following properties:*

(1)' *the maximal integral manifolds are totally geodesic and compact,*

(2)'' *there is a maximal integral manifold $L$ such that sectional curvature for planes spanned by $X\in T_0$ and $Y\in T_0^{\perp}$ is positive on $L$.*

*Then $\nu\geq m/2$ implies $\nu=m$.*

PROOF. Suppose that $m/2\leq\nu<m$. Let $L$ be a maximal integral manifold (=leaf) stated in (2)''. Let $p$ be an arbitrary point of $L$. Let $Z_p$ be a unit normal vector to $L$ in $M^m$ at $p$. By $\exp tZ_p$ we define a geodesic $x(t)$, $0\leq t\leq\varepsilon$. Since $L$ is compact, such an $\varepsilon$ can be chosen so that it is independent of the choice of $p$ and $Z_p$. Let $X_p$ be a unit tangent vector to $L$ at $p$. Define parallel translate $X_t$ of $X_0=X_p$ along $x(t)$. The unit tangent vector to $x(t)$ at $x(t)$ is denoted by $Z_t$ and $Z_0=Z_p$. Then $K(X_0, Z_0)>0$ at $p$. Since $t\rightarrow K(X_t, Z_t)$ is continuous, we have either

(a) $K(X_t, Z_t)>0$ for all $t: 0\leq t\leq \varepsilon$, or

(b) there is a real number $s=s(X_p, Z_p)$, $0<s<\varepsilon$, such that $K(X_t, Z_t)>0$ for $t<s$ and $K(X_s, Z_s)=0$.

Denote by $T^1L$ and $N^1L$ the unit tangent bundle of $L$ and the unit normal bundle of $L$ in $M^m$, respectively. We define a subspace $\Delta(T^1L, N^1L)$ of the product $T^1L\times N^1L$ as a set of elements of the form $(X_p, Z_p)$. Since $L$ is compact $\Delta(T^1L, N^1L)$ is compact. We define a function $f$ on $\Delta(T^1L, N^1L)$ by

$$f(X_p, Z_p)=\min\left\{s(X_p, Z_p), \varepsilon\right\}.$$

Then $f$ is continuous and positive-valued on $\Delta(T^1L, N^1L)$, and hence $f$ attains the mimimum $\delta<0$. Let $U$ be the $\delta$-neighborhood, i.e., $U=\{x\in M^m;$ distance $(x, L)<\delta\}$. Since leaves are compact, $U$ contains a leaf $L'$ different from $L$. $L$ and $L'$ are disjoint. Let $x(t)$, $0\leq t\leq l$, be a shortest geodesic from $L$ to $L'$. Here we have $l<\delta$. Now, our construction of $U$ leads a conradiction just as in the proof of T. Frankel's theorem. Hence, we have $\nu=m$.

## §3. Remarks

[A] For nullity, $K$-nullity, relative nullity, and their indeces, see K. Abe [1], S. S. Chern and N. H. Kuiper [2], Y. H. Clifton and R. Maltz [3], etc.

The set $G$ where the index of $K$-nullity is the mimimum value is open in $M^m$. If $(M^m, g)$ is complete, then leaves of the $K$-nullity foliation on $G$ are complete (Y. H. Clifton and R. Maltz [3], K. Abe [1]). Combinig this with Theorem A we have

*Let $(M^m, g)$ be a complete Riemannian manifold. If the index of $K$-nullity $>\nu_m$ and $K>0$, then $(M^m, g)$ is of constant curvature $K$.*

[B] The results on $K$-nullity are applied to the relative nullity of submanifolds $(M^m, g)$ of a space $(^*M^{m+p}, {}^*g)$ of constant curvature $K$. See Theorems 2 and 3 in D. Ferus [5].

[C] The complex versions are also obtained: As for holomorphic $K$-nullity, see K. Abe [1], Theorem 2. 2. 1. See also D. Ferus [5], Theorem 2.

[D] Let $(M^m, J, G)$ be a complex *m-dimensional* complex hypersurface of a complex projective space $CP^{m+1}(K)$ with constant holomorphic sectional curvature $K$. For a unit normal $\xi_1$, $J\xi_1=\xi_2$ is also a unit normal. The rank of the 2nd fundamental form $A_1$ with respect to $\xi_1$ is intrinsic (K. Nomizu and B. Smyth [11]) and is called the rank of $(M^m, J, G)$ at each point. The rank of $(M^m, J, G)$ is $=2m-2\nu$, where $\nu$ is the index of relative nullity ($=$complex dimension of the relative nullity). By Theorem B (or more generally by Theorem A) we have

*Assume that a complete Kählerian manifold* $(M^m, J, G)$ *is isometrically and holomorphically immersed in a* $CP^{m+1}(K)$. *If the rank of* $(M^m, J, G)$ *is* $\leq m$ *(more generally* $< 2m - \nu_{2m}$*) at every point, then* $(M^m, J, G)$ *is imbedded as a projective hyperplane in* $CP^{m+1}(K)$.

This is a generalization of a theorem of K. Nomizu ([10], Theorem 1). See also K. Abe [1]. A result of K. Nomizu and B. Smyth ([11], Theorem 6) is generalized to

*Assume that a complete Kählerian manifold* $(M^m, J, G)$ *is immersed isometrically and holomorphically in a* $CP^{m+1}(K)$, $m \geq 2$. *Then the rank of* $(M^m, J, G)$ *can not be identically equal to* 2.

For $m=1$, the quadrics are the only closed complex curves in $CP^2(K)$ of rank identically equal to 2, see [11].

As a natural consequence of the above proposition we have a generalization of a result of K. Nomizu ([10], Theorem 2).

*Let* $m \geq 2$. *Assume that a complete Kählerian manifold* $(M^m, J, G)$ *is immersed isometrically and holomorphically in a* $CP^{m+1}(K)$. *If the sectional curvature of* $(M^m, J, G)$ *is* $\geq 1/4$ *for every tangent plane, then* $(M^m, J, G)$ *is imbedded as a projective hyperplane.*

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