Normal subgroups of quadruply transitive permutation groups*

To Yoshie Katsurada on her Sixtieth Birthday

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Introduction. Let Ω be the set of symbols $1, \cdots, n$. Let G be a permutation group on Ω. Wagner [6] proved the following theorem:

If G is triply transitive and if n is odd and greater tnan 3, then every normal subgroup $(\neq 1)$ of G is also triply transitive.

In this note we prove the following theorem:

THEOREM. *If G is quadruply transitive and if n is prime to 3 and greater than 5, then every normal subgroup $(\neq 1)$ of G is also quadruply transitive.*

The outline of the proof is as follows. First of all, by the above theorem of Wagner we may assume that n is odd. Let H $(\neq 1)$ be a normal subgroup of G which is not quadruply transitive. Then without the restriction that n is prime to 3 we obtain some permutation-character theoretical results on H which are slightly more than needed in the proof. At the final point, with the restriction that n is prime to 3 we utilize results obtained above to get a contradiction.

Definitions and Notation. Let $x \in G$. Then $\alpha(x)$, $\beta(x)$, $\gamma(x)$ and $\delta(x)$ denote the numbers of 1-, 2-, 3- and 4-cycles in the permutation structure of x respectively. Let $X \subseteq G$. Then $\alpha(X)$ denotes the set of symbols of Ω each of which is fixed by X. Let X be a subgroup of G. Let φ and ϕ be class functions on X. Then $(\varphi, \phi)_X = \frac{1}{|X|} \sum_{x \in X} \varphi(x)\overline{\phi(x)}$ and $N_X(\varphi) = (\varphi, \varphi)_X$. $X_{(\Delta)}$ and X_Δ denote the global and pointwise stabilizers of Δ in X respectively. $X^{\Delta}_{(\Delta)}$ denotes the restriction of $X_{(\Delta)}$ to Δ. If $\Delta = \{1\}$, $\{1, 2\}$ or $\{1, 2, 3\}$, we also write X_1, $X_{1,2}$ or $X_{1,2,3}$ instead of X_Δ. Let Y be a subgroup of X. Then $Ns_X Y$ denotes the normalizer of Y in X. $LF(2, q)$ denotes the linear fractional group over the field of q elements.

PROOF. (a) The following permutation-character theoretical formulae for quadruply (and triply) transitive permutation groups are well-known ([4], p. 597; [7], (9.9)).

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(1) $$\sum_G \alpha = |G|\ ; \quad \sum_H \alpha = |H|\,.$$

(2) $$\sum_G \alpha^2 = 2|G|, \quad \sum_G \beta = \frac{1}{2}|G|\ ; \quad \sum_H \alpha^2 = 2|H|, \quad \sum_H \beta = \frac{1}{2}|H|.$$

(3) $$\sum_G \alpha^3 = 5|G|, \quad \sum_G \alpha\beta = \frac{1}{2}|G|, \quad \sum_G \gamma = \frac{1}{3}|G|\ ; \quad \sum_H \alpha^3 = 5|H|,$$
$$\sum_H \alpha\beta = \frac{1}{2}|H|, \quad \sum_H \gamma = \frac{1}{3}|H|.$$

(4) $$\sum_G \alpha^4 = 15|G|, \quad \sum_G \alpha^2\beta = |G|, \quad \sum_G \alpha\gamma = \frac{1}{3}|G|, \quad \sum_G \beta^2 = \frac{3}{4}|G|,$$
$$\sum_G \delta = \frac{1}{4}|G|.$$

Put $X_0 = \alpha - 1$, $X_{\underset{0}{0}} = \frac{1}{2}(\alpha-1)(\alpha-2) - \beta$, and $X_{00} = \frac{1}{2}\alpha(\alpha-3) + \beta$, .

These are all irreducible characters of G. X_0 remains irreducible in H. Furthermore,

(5) $$(X_0\ X_{\underset{0}{0}})_H = (X_0, X_{00})_H = 0\,.$$

(b) LEMMA. For all $\Delta \subseteq \Omega$ with $|\Delta| = 4$ the group $H^{\Delta}_{(\Delta)}$ is the symmetric group of degree 4.

PROOF. It suffices to show that $H^{\Delta}_{(\Delta)}$ contains an odd permutation, since $H^{\Delta}_{(\Delta)}$ is normal in $G^{\Delta}_{(\Delta)}$ which is the symmetric group of degree 4. Then it suffices to show that $H_{1,2}$ has even order, because G is quadruply transitive. Now assume that $H_{1,2}$ has odd order. Then by a theorem of Bender [1] H_1 contains a normal subgroup isomorphic to $LF(2, q)$ with q odd. But then H_1 has no transitive extension ([5], (5.2)). This is a contradiction.

(c) By Lemma in (b) and Lemma 2 of [5] we have several transitive permutation representations of G each of which is divided into the equal number, say s, of H-transitive constituents:

(i) The permutation representation of G on the set of all ordered quartets with distinct members on Ω. Since $\alpha(\alpha-1)(\alpha-2)(\alpha-3)$ is the character of this permutation representation, by (a) we obtain that

(6) $$\sum_H \alpha^4 = (s+14)|H|\,.$$

(ii) The permutation representation of G on the set of all 2-element subsets of the set of all ordered pairs with distinct members on Ω. Since $\frac{1}{2}\alpha(\alpha-1)(\alpha-2)(\alpha-3)+2\beta(\beta-1)$ is the character of this permutation representation, by (a) and (6) we obtain that

$$\sum_H \beta^2 = \frac{1}{4}(s+2)|H| . \tag{7}$$

(iii) The permutation representation of G on the set of all ordered pairs with distinct members on the set of all 2-element subsets of Ω. Since $\frac{1}{4}\alpha(\alpha-1)(\alpha-2)(\alpha-3)+\alpha(\alpha-1)\beta+\beta(\beta-1)$ is the character of this permutation representation, by (a), (6) and (7) we obtain that

$$\sum_H \alpha^2\beta = \frac{1}{2}(s+1)|H| . \tag{8}$$

(iv) The permutation representation of G on the set of all 2-element subsets of Ω. Since $\frac{1}{8}\alpha(\alpha-1)(\alpha-2)(\alpha-3)+\frac{1}{2}\alpha(\alpha-1)\beta+\frac{1}{2}\beta(\beta-1)+\delta$ is the character of this permutation representation, by (a), (6)–(8) we obtain that

$$\sum_H \delta = \frac{1}{4}s|H| . \tag{9}$$

(v) The permutation representation of G on the set of all 4-element subsets of Ω. Since $\frac{1}{24}\alpha(\alpha-1)(\alpha-2)(\alpha-3)+\frac{1}{2}\alpha(\alpha-1)\beta+\frac{1}{2}\beta(\beta-1)+\alpha\gamma+\delta$ is the character of this permutation representation, by (a), (6)–(9) we obtain that

$$\sum_H \alpha\gamma = \frac{1}{3}s|H| . \tag{10}$$

Now by (a) and (6)–(10) we obtain that

$$N_H(X_{\underset{0}{0}}) = 1 \tag{11}$$

and

$$N_H(X_{00}) = s . \tag{12}$$

(11) shows that $X_{\underset{0}{0}}$ remains irreducible on H.

(d) Clearly s equals the number of $Ns_H(H_{1,2})$-orbits on the set Φ of all

2-element subsets of $\Omega-\{1, 2\}$. Since H is triply transitive on Ω ([7], (9.9)), every $Ns_H(H_{1,2})$-orbit contains a 2-element subset containing 3. Since $Ns_G(H_{1,2})$ is transitive on Φ, the lengths of all $Ns_H(H_{1,2})$-orbits on Φ are equal to $\frac{(n-2)(n-3)}{2s}$. Further $2s$ divides $n-3$.

Now put

$$X_{00}=a\sum_{i=1}^{t}\chi_i\,,$$

where $\chi_1, \cdots, \chi_t$ are G-associated irreducible characters of H ([4], p. 565). Then by (12) $s=a^2t$. Now by a theorem of Frame ([7], (30.5)) the following rational number F is an integer:

$$F=\frac{\left\{\frac{1}{2}n(n-1)\right\}^2 2(n-2)\left\{\frac{(n-2)(n-3)}{2s}\right\}^s}{(n-1)\left\{\frac{n(n-3)}{2at}\right\}a^2t}=\frac{(n-1)^{s-1}(n-2)^{s+1}}{2^{s-1}\cdot a^s}\,.$$

Since n is odd and s divides $n-3$, a must be a power of 2. Since a^2 divides $n-3$ and $n-1$, $a=1$. Hence we obtain that

$$X_{00}=\chi_1+\cdots+\chi_s\,, \tag{13}$$

where $\chi_1, \cdots, \chi_s$ are G-associated distince irreducible characters of H.

(e) A double coset $(Ns_HH_{1,2})x(Ns_HH_{1,2})$ of H with respect to $Ns_HH_{1,2}$ is called real, if it coincides with $(Ns_HH_{1,2})x^{-1}(Ns_HH_{1,2})$. Let f be the number of real cosets of H with respect to $Ns_HH_{1,2}$. Then by a theorem of Frame [3] we obtain that

$$\begin{aligned}
f&=\frac{1}{|H|}\sum_{x\in H}\left\{\frac{1}{2}\alpha(\alpha-1)+\beta\right\}(x^2)\\
&=\frac{1}{|H|}\sum\left\{\frac{1}{2}(\alpha+2\beta)(\alpha+2\beta-1)+2\delta\right\}\\
&=(s+2)|H| \qquad\qquad \text{(by (a), (6)–(9))}\\
&=\frac{1}{|H|}\sum_{x\in H}(1_H+X_0+\chi_1+\cdots+\chi_s)(x^2)\\
&=2+\frac{1}{|H|}\sum_{x\in H}\chi_1(x^2)+\cdots+\chi_s(x^2)\,.
\end{aligned}$$

This implies that $\sum_{x\in H}\chi_1(x^2)=\cdots=\sum_{x\in H}\chi_s(x^2)=|H|$. Thus the representations corresponding to $\chi_1, \cdots, \chi_s$ are real ([2], (3.5)).

REMARK. The argument in (e) can be used to obtain (9), since $\chi_1, \cdots, \chi_s$ are G-associated and hence $\sum_{x\in H}\chi_1(x^2)=\cdots=\sum_{x\in H}\chi_s(x^2)$.

(f) By (b) and by Lemma 4 and Remark (iii) of [6] $H_{1,2,3}$ has at most 1 orbit of odd length and at most 2 orbits of lengths prime to 3 on Ω-{1, 2, 3}. Since $H_{1,2,3}$ has exactly s orbits of length $\frac{n-3}{s}$, we obtain that

(14) $$s=2 \quad \text{and} \quad n\equiv 3 \pmod 4$$

or

(15) $$\frac{n-3}{s}\equiv 0 \pmod 6.$$

(g) Now we assume that n is prime to 3. Then in (f) (14) holds. Hence the inertia group of χ_1 in G has index 2. By induction on $G:H$ we may assume that $G:H=2$. Then we obtain that X_{00} is the character of G induced by χ_1 of H. In particular, X_{00} vanishes outside H. Hence every element x of G with $\alpha(x)\geq 4$ belongs to H. In particular, $G_{1,2,3,4}$ is contained in H.

If $n\equiv 1 \pmod 3$, then $\frac{n-3}{2}\equiv 2 \pmod 3$. By the proof of Lemma 4 and Remark (iii) of [6] this is a contradiction. Hence we obtain that $n\equiv 2 \pmod 3$.

Now let S be a Sylow 3-subgroup of $G_{1,2,3,4}$. Since $n\equiv 2 \pmod 3$, S leaves one more point, say 5, invariant. By a theorm of Witt ([7], 9.4)) $Ns_G S$ is quadruply transitive on $\alpha(S)$. Since S is a Sylow 3-subgroup of $G_{1,2,3}$ and $H_{1,2,3}$, we have that $Ns_G S \subsetneqq H$. If $S\neq 1$, then, since $|\alpha(S)|\equiv 2 \pmod 3$, by induction on n, we obtain that $H\cap NsS$ is quadruply transitive on $\alpha(S)$ or $|\alpha(S)|=5$. If $H\cap NsS$ is quadruply transitive on $\alpha(S)$, then $H\cap(Ns_G S)_{1,2,3,4}\subsetneqq(Ns_G S)_{1,2,3,4}$. This is a contradiction. If $|\alpha(S)|=5$ and $H\cap Ns_G S$ is not quadruply transitive on $\alpha(S)$, $H\cap Ns_G S$ cannot contain a 4-cycle. This contradicts (4) and (9). Hence we obtain that $S=1$.

By (3) we have that $\sum_{G-H}\gamma=\frac{1}{3}|H|$. By (3) and (10) $\sum_{G-H}\alpha\gamma=0$. Since $S=1$, we have that $\alpha(x)\leq 2$ if $\gamma(x)>0$, where $x\in G$. Now let y be an element of $G-H$ such that $\gamma(y)>0$. Since $n\equiv 2 \pmod 3$, the cycle structure of y contains a cycle C whose length is prime to 3. Then clearly we may assume that the length of C is a power of 2, and hence is equal to 2. But then $X_{00}(y)>0$, which is a contradiction.

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