On a pseudo umbilical submanifold in a Riemannian manifold of constant curvature

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Introduction.

H. Liebmann [8] has proved that an ovaloid with constant mean curvature in a 3-dimensional Euclidean space is a sphere. The above problem for a closed hypersurface in a Riemannian manifold has been generalized by Y. Katsurada [3], [4] and K. Yano [17]. Y. Katsurada [5], [6], H. Kôjyô [5], T. Nagai [6], [12] and K. Yano [18] have given conditions for a submanifold of codimension greater than 1 in a Riemannian manifold to be pseudo umbilical by making use of integral formulas.

On the other hand M. Okumura [13] has proved that a submanifold of codimension 2 in an odd dimensional sphere is totally umbilical under certain coditions. To prove the above result, M. Okumura made use of the fact that the structure tensor of the natural normal contact structure on the odd dimensional sphere is a conformal Killing tensor of order 2 which has been defined by S. Tachibana [15].

In the previous papers [9], [10], the present author proved for a submanifold of codimension p in a sphere and a Riemannian manifold of constant curvature respectively that the submanifold is totally umbilical under certain conditions by making use of integral formulas. However, in the papers, it has been assumed that the connection of the normal bundle is trivial.

In this paper, the present author studies on a submanifold of codimension p in a Riemannian manifold of constant curvature without the condition that the connection of the normal bundle is trivial and proves that the submanifold is pseudo umbilical.

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§ 1. Conformal Killing tensors.

Recently S. Tachibana [15] and T. Kashiwada [2] have introduced the notion of conformal Killing tensor field in a Riemannian manifold. They discussed such the tensor and obtained some results.

Let $\widetilde{M}^{n+p}$ be a $(n+p)$-dimensional Riemannian manifold with the metric tensor $G_{\lambda\mu}$. We call a skew symmetric tensor field $F_{\lambda_1\cdots\lambda_p}$ a conformal Killing tensor field of order p if there exists a skew symmetric tensor field $f_{\lambda_1\cdots\lambda_{p-1}}$ such that

$$\begin{aligned}(1.1)\qquad \nabla_\lambda F_{\lambda_1\lambda_2\cdots\lambda_p}+\nabla_{\lambda_1}F_{\lambda\lambda_2\cdots\lambda_p} &= 2f_{\lambda_2\cdots\lambda_p}G_{\lambda\lambda_1} \\ &-\sum_{a=2}^{p}(-1)^a(f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p}G_{\lambda\lambda_a}+f_{\lambda\lambda_2\cdots\hat{\lambda}_a\cdots\lambda_p}G_{\lambda_1\lambda_a}),\end{aligned}$$

where $\hat{\lambda}_a$ means that λ_a is omitted and ∇_λ denotes the covariant derivative. This $f_{\lambda_1\cdots\lambda_{p-1}}$ is called the associated tensor field of $F_{\lambda_1\cdots\lambda_p}$.

§2. Submanifolds in a Riemannian manifold of constant curvature.

Let $\widetilde{M}^{n+p}$ be a $(n+p)$-dimensional Riemannian manifold of constant curvature with the metric tensor $G_{\lambda\mu}$. Then the curvature tensor $\widetilde{R}_{\lambda\mu\nu\kappa}$ of $\widetilde{M}^{n+p}$ has the form

$$(2.1)\qquad \widetilde{R}_{\lambda\mu\nu\kappa}=k(G_{\lambda\kappa}G_{\mu\nu}-G_{\lambda\nu}G_{\mu\kappa}), \qquad k=\text{const.}.$$

Let M^n be an orientable submanifold of codimension p in $\widetilde{M}^{n+p}$. We denote by $\{X^\lambda\}$, $\lambda=1,2,\cdots,n+p$, the local coordinates of $\widetilde{M}^{n+p}$ and by $\{x^i\}$, $i=1,\cdots,n$, the local coordinates of M^n. Then the submanifold M^n is locally expressed by the equation

$$(2.2)\qquad X^\lambda=X^\lambda(x^i), \qquad \begin{array}{l}\lambda=1,2,\cdots,n+p,\\ i=1,\cdots\cdots,n.\end{array}$$

We put

$$(2.3)\qquad B_i^\lambda=\partial X^\lambda/\partial x^i.$$

Then n vectors B_i^λ are linearly independent vectors tangent to M^n. The Riemannian metric tensor g_{ji} on M^n induced from $G_{\lambda\mu}$ is given by

$$(2.4)\qquad g_{ji}=G_{\lambda\mu}B_j^\lambda B_i^\mu.$$

We choose p mutually orthogonal unit normal vectors N_A^λ $(A=n+1,\cdots,n+p)$. Let H_{Aji} be the second fundamental tensor with respect to N_A^λ and L_{ABj} the third fundamental tensor. Then the Gauss and Weingarten equations are given by

$$(2.5)\qquad \begin{aligned}\nabla_j B_i^\lambda &= \sum_A H_{Aji}N_A^\lambda,\\ \nabla_j N_A^\lambda &= -H_{Aj}{}^i B_i^\lambda+\sum_B L_{ABj}N_B^\lambda,\end{aligned}$$

where ∇_j denotes the covariant derivative.

The mean curvature vector field H^{λ} of M^{n} is given by

$$H^{\lambda}=\frac{1}{n}\sum_{A}H_{At}{}^{t}N_{A}{}^{\lambda}\,, \tag{2.6}$$

and H^{λ} is independent of the choice of mutually orthogonal unit normal vectors.

Now we take a unit normal vector $N_{n+1}{}^{\lambda}$ in the direction of the mean curvature vector field H^{λ}. Then $N_{n+1}{}^{\lambda}$ is determined uniquely on M^{n}. If the second fundamental tensor H_{n+1ji} with respect to $N_{n+1}{}^{\lambda}$ is proportional to the metric tensor g_{ji}, that is, satisfying $H_{n+1ji}=\alpha g_{ji}$, where α is a scalar function on M^{n}, then we say that the submanifold M^{n} is pseudo umbilical. We take $N_{n+2}{}^{\lambda},\cdots,N_{n+p}{}^{\lambda}$ such that $N_{n+1}{}^{\lambda}$, $N_{n+2}{}^{\lambda},\cdots,N_{n+p}{}^{\lambda}$ are mutually orthogonal unit normal vectors.

Since the curvature tensor $\tilde{R}_{\lambda\mu\nu\kappa}$ of $\tilde{M}^{n+p}$ has the form (2.1), the equations of Gauss, Codazzi and Ricci are written as

$$R_{kjih}=k(g_{kh}g_{ji}-g_{ki}g_{jh})+\sum_{A}(H_{Akh}H_{Aji}-H_{Aki}H_{Ajh})\,, \tag{2.7}$$

$$\nabla_{k}H_{n+1ji}-\nabla_{j}H_{n+1ki}+\sum_{A}(H_{Aji}L_{An+1k}-H_{Aki}L_{An+1j})=0\,, \tag{2.8}$$

$$\begin{aligned}&H_{n+1k}{}^{i}H_{Aji}-H_{n+1j}{}^{i}H_{Aki}+\nabla_{k}L_{n+1Aj}-\nabla_{j}L_{n+1Ak}\\&\qquad+\sum_{B}(L_{n+1Bj}L_{BAk}-L_{n+1Bk}L_{BAj})=0\,,\end{aligned} \tag{2.9}$$

where R_{kjih} denotes the curvature tensor of M^{n}.

For a normal vector N^{λ}, if the normal part of $\nabla_{j}N^{\lambda}$ vanishes identically along M^{n}, then we call that N^{λ} is parallel with respect to the connection of the normal bundle. We assume that the mean curvature vector field H^{λ} of M^{n} is parallel with respect to the connection of the normal bundle. Then we see easily that this assumption is equivalent to

$$H_{n+1t}{}^{t}=\text{const.}\,,\qquad L_{n+1Aj}=0\,. \tag{2.10}$$

From (2.10), we have

$$\nabla_{j}H_{n+1k}{}^{j}=0\,, \tag{2.11}$$

$$H_{n+1k}{}^{i}H_{Aji}-H_{n+1j}{}^{i}H_{Aki}=0 \tag{2.12}$$

by virtue of (2.8) and (2.9).

§ 3. Integral formulas.

Let $\tilde{M}^{n+p}$ be a $(n+p)$-dimensional Riemannian manifold of constant curvature which admits a conformal Killing tensor field $F_{\lambda_1\cdots\lambda_p}$ of order p with the associated tensor field $f_{\lambda_1\cdots\lambda_{p-1}}$ and M^n a compact orientable submanifold of codimension p in $\widetilde{M}^{n+p}$. In this section, we assume that the mean curvature vector field H^λ of M^n is parallel with respect to the connection of the normal bundle.

Now we put

$$r = F_{\lambda_1\lambda_2\cdots\lambda_p} N_{n+1}{}^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p}, \tag{3.1}$$

$$u_i = F_{\lambda_1\lambda_2\cdots\lambda_p} B_i^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p}. \tag{3.2}$$

Then we find that r is independent of the choice of mutually orthogonal unit normal vectors in the previous papers [9], [10]. u_i is independent of the choice of $p-1$ mutually orthogonal unit normal vectors $N_{n+2}{}^\lambda, \cdots, N_{n+p}{}^\lambda$ orthogonal to $N_{n+1}{}^\lambda$. We take another $p-1$ mutually orthogonal unit normal vectors $'N_{n+2}{}^\lambda, \cdots, 'N_{n+p}{}^\lambda$ orthogonal to $N_{n+1}{}^\lambda$. Then there exists a orthogonal matrix (T_{AB}), $A, B = n+2, \cdots, n+p$ such that $\det(T_{AB}) = 1$ and $'N_A^\lambda$ $(A = n+2, \cdots, n+p)$ can be written as

$$'N_A^\lambda = \sum_B T_{AB} N_B^\lambda. \tag{3.3}$$

Therefore we find

$$\begin{aligned}
'u_i &= F_{\lambda_1\lambda_2\cdots\lambda_p} B_i^{\lambda_1}\, 'N_{n+2}{}^{\lambda_2} \cdots 'N_{n+p}{}^{\lambda_p} \\
&= \sum_{A_2,\cdots,A_p} T_{n+2A_2} \cdots T_{n+pA_p} F_{\lambda_1\lambda_2\cdots\lambda_p} B_i^{\lambda_1} N_{A_2}{}^{\lambda_2} \cdots N_{A_p}{}^{\lambda_p} \\
&= \sum_{A_2,\cdots,A_p} \operatorname{sgn}\begin{pmatrix} n+2, \cdots, n+p \\ A_2, \cdots, A_p \end{pmatrix} T_{n+2A_2} \cdots T_{n+pA_p} F_{\lambda_1\lambda_2\cdots\lambda_p} B_i^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p} \\
&= \det(T_{AB}) u_i = u_i
\end{aligned}$$

by means of the skew symmetry of $F_{\lambda_1\cdots\lambda_p}$. This shows that u_i is independent of the choice of $p-1$ mutually orthogonal unit normal vectors orthogonal to $N_{n+1}{}^\lambda$.

Differentiating (3.2) covariantly and making use of (2.5), we have

$$\begin{aligned}
\nabla_j u_i &= B_j^\lambda \nabla_\lambda F_{\lambda_1\lambda_2\cdots\lambda_p} B_i^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p} \\
&\quad + F_{\lambda_1\lambda_2\cdots\lambda_p} \sum_A H_{Aji} N_A^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p} \\
&\quad + \sum_{a=2}^{p} F_{\lambda_1\lambda_2\cdots\lambda_a\cdots\lambda_p} B_i^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots (-H_{n+aj}{}^h B_h^{\lambda_a} + \sum_A L_{n+aAj} N_A^{\lambda_a}) \cdots N_{n+p}{}^{\lambda_p} \\
&= B_j^\lambda \nabla_\lambda F_{\lambda_1\lambda_2\cdots\lambda_p} B_i^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p} + r H_{n+1ji} \\
&\quad - \sum_{a=2}^{p} H_{n+aj}{}^h F_{\lambda_1\lambda_2\cdots\lambda_a\cdots\lambda_p} B_i^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots B_h^{\lambda_a} \cdots N_{n+p}{}^{\lambda_p} \\
&\quad + \sum_{a=2}^{p} L_{n+1n+aj} F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} N_{n+1}{}^{\lambda_1} \cdots B_i^{\lambda_a} \cdots N_{n+p}{}^{\lambda_p},
\end{aligned}$$

from which we have

$$\nabla^j u_j = \frac{1}{2} B^{j\lambda}(\nabla_\lambda F_{\lambda_1\lambda_2\cdots\lambda_p} + \nabla_{\lambda_1} F_{\lambda\lambda_2\cdots\lambda_p}) B_j^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p} + rH_{n+1t}{}^t$$
$$= \frac{1}{2} B^{j\lambda} \Big\{ 2f_{\lambda_2\cdots\lambda_p} G_{\lambda\lambda_1} - \sum_{a=2}^{p} (-1)^a (f_{\lambda_1\cdots\hat{\lambda}_a\cdots\lambda_p} G_{\lambda\lambda_a}$$
$$+ f_{\lambda\cdots\hat{\lambda}_a\cdots\lambda_p} G_{\lambda_1\lambda_a}) \Big\} B_j^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p} + rH_{n+1t}{}^t$$

by virtue of (1. 1) and our assumption. Thus we have

$$\nabla^j u_j = rH_{n+1t}{}^t + nf_{\lambda_2\cdots\lambda_p} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p},$$

from which we get the following integral formula

$$\int_{M^n} (rH_{n+1t}{}^t + nf_{\lambda_2\cdots\lambda_p} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p}) dM = 0 \tag{3. 4}$$

by virtue of Green's theorem.

Next differentiating $H_{n+1j}{}^i u_i$ covariantly and making use of (1. 1), (2. 5), (2. 11) and (2. 12), we have

$$\nabla^j (H_{n+1j}{}^i u_i) = (\nabla^j H_{n+1j}{}^i) u_i + H_{n+1}{}^{ji} \nabla_j u_i$$
$$= H_{n+1}{}^{ji} (B_j^\lambda \nabla_\lambda F_{\lambda_1\lambda_2\cdots\lambda_p} B_i^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p} + rH_{n+1ji}$$
$$- \sum_{a=2}^{p} H_{n+aj}{}^h F_{\lambda_1\lambda_2\cdots\lambda_a\cdots\lambda_p} B_i^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots B_h^{\lambda_a} \cdots N_{n+p}{}^{\lambda_p}$$
$$+ \sum_{a=2}^{p} L_{n+1n+aj} F_{\lambda_1\cdots\lambda_a\cdots\lambda_p} N_{n+1}{}^{\lambda_1} \cdots B_i^{\lambda_a} \cdots N_{n+p}{}^{\lambda})$$
$$= \frac{1}{2} H_{n+1}{}^{ji} B_j^\lambda (\nabla_\lambda F_{\lambda_1\lambda_2\cdots\lambda_p} + \nabla_{\lambda_1} F_{\lambda\lambda_2\cdots\lambda_p}) B_i^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p}$$
$$+ rH_{n+1ji} H_{n+1}{}^{ji}$$
$$- \sum_{a=2}^{p} H_{n+1}{}^{ji} H_{n+aj}{}^h F_{\lambda_1\lambda_2\cdots\lambda_a\cdots\lambda_p} B_i^{\lambda_1} N_{n+2}{}^{\lambda_2} \cdots B_h^{\lambda_a} \cdots N_{n+p}{}^{\lambda_p},$$

from which we find

$$\nabla^j (H_{n+1j}{}^i u_i) = rH_{n+1ji} H_{n+1}{}^{ji} + H_{n+1t}{}^t f_{\lambda_2\cdots\lambda_p} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p}.$$

Therefore we obtain the following integral formula

$$\int_{M^n} (rH_{n+1ji} H_{n+1}{}^{ji} + H_{n+1t}{}^t f_{\lambda_2\cdots\lambda_p} N_{n+2}{}^{\lambda_2} \cdots N_{n+p}{}^{\lambda_p}) dM = 0 \tag{3. 5}$$

by virtue of Green's theorem.

From (3. 5) $-\frac{1}{n} H_{n+1t}{}^t \times$ (3. 4), we have the following integral formula

$$\int_{M^n} r\left\{H_{n+1ji}H_{n+1}{}^{ji} - \frac{1}{n}(H_{n+1t}{}^t)^2\right\} dM = 0\,. \tag{3.6}$$

THEOREM 3.1. *Let $\widetilde{M}^{n+p}$ be a $(n+p)$-dimensional Riemannian manifold of constant curvature which admits a conformal Killing tensor field $F_{\lambda_1\cdots\lambda_p}$ of order p and M^n a compact orientable submanifold of codimension p in $\widetilde{M}^{n+p}$. Suppose that the mean curvature vector field H^λ of M^n is parallel with respect to the connection of the normal bundle. If the function r has fixed sign on M^n, then the submanifold M^n is pseudo umbilical.*

PROOF. From the following relation

$$\left(H_{n+1ji} - \frac{1}{n}H_{n+1t}{}^t g_{ji}\right)\left(H_{n+1}{}^{ji} - \frac{1}{n}H_{n+1t}{}^t g^{ji}\right) \tag{3.7}$$
$$= H_{n+1ji}H_{n+1}{}^{ji} - \frac{1}{n}(H_{n+1t}{}^t)^2\,,$$

we see that $H_{n+1ji}H_{n+1}{}^{ji} - \frac{1}{n}(H_{n+1t}{}^t)^2$ is non negative. Therefore we have

$$H_{n+1ji}H_{n+1}{}^{ji} - \frac{1}{n}(H_{n+1t}{}^t)^2 = 0$$

by virtue of (3.6) and our assumption, which shows that the submanifold M^n is pseudo umbilical by means of (3.7).

In the case of $p=1$ and $p=2$, we obtain the following corollaries in the previous papers [9], [10].

COROLLARY 3.2. *Let $\widetilde{M}^{n+1}$ be a $(n+1)$-dimensional Riemannian manifold of constant curvature which admits a conformal Killing vector field F_λ and M^n a compact orientable hypersurface in $\widetilde{M}^{n+1}$. Assume that the mean curvature of M^n is constant. If $F_\lambda C^\lambda$ has fixed sign on M^n, then the hypersurface M^n is umbilical, where C^λ denotes a unit normal vector of M^n.*

The above corollary is included in the theorem of Y. Katsurada [3], [4].

COROLLARY 3.3. *Let $\widetilde{M}^{n+2}$ be a $(n+2)$-dimensional Riemannian manifold of constant curvature which admits a conformal Killing tensor field $F_{\lambda\mu}$ of order 2 and M^n a compact orientable submanifold of codimension 2 in $\widetilde{M}^{n+2}$. Assume that the mean curvature vector field H^λ of M^n is parallel with respect to the connection of the normal bundle. If $F_{\lambda\mu}C^\lambda D^\mu$ has fixed sign on M^n, then the submanifold M^n is totally umbilical, where C^λ and D^λ denote mutually orthogonal unit normal vectors of M^n.*

We assume that the connection of the normal bundle of M^n is trivial.

Then we get the following result given in the previous papers [9], [10].

COROLLARY 3. 4. *Let $\widetilde{M}^{n+p}$ be a $(n+p)$-dimensional Riemannian manifold of constant curvature which admits a conformal Killing tensor field $F_{\lambda_1\cdots\lambda_p}$ of order p and M^n a compact orientable submanifold of codimension p in $\widetilde{M}^{n+p}$. Suppose that the mean curvature vector field H^λ of M^n is parallel with respect to the connection of the normal bundle and the connection of the normal bundle is trivial. If the function r has fixed sign on M^n, then the submanifold M^n is totally umbilical.*

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