Discrimination of the space-time V. I.

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§ 1. Introduction.

The present paper is a continuation of [1][1], [2] and [3], and deals with the problem of the discrimination of the space-time V. In other words, we are going to establish a theory by which we can determine whether a four-dimensional Riemannian space-time defined by g_{ij} arbitrarily given in any coordinate system is a V or not. Mathematically speaking, this problem is "to determine the necessary and sufficient condition that the given g_{ij} be reducible to the form

$$ds^2 = -dx^2 - Bdy^2 - Cdz^2 + Ddt^2, \tag{1.1}$$

where B, C and D are positive valued functions of x alone".

As is easily understood, the problem of "determining whether a given space-time is V or not" is not only interesting from the standpoint of tensor analysis but also its solution is of importance when we consider the physical meanings of the given space-time. If the answer of this problem is given by some tensor equations to be satisfied by the curvature tensor K_{ijmn} made from g_{ij}, especially when the equations contain no tensor other than g_{ij}, η_{ijmn} $(=\sqrt{-g}\,\epsilon_{ijmn})$ and K_{ijmn}, we may say that the problem is solved in the most desirable form. Unfortunately, however, we have not succeeded in finding such equations at the present stage of the investigations. In the present paper and the forthcoming one [4], we shall give another way of discriminating V using the theory of characteristic system (abbreviated to c.s.) developed in [1], [2] and [3].

If we see the results of [1], it is true that if we can determine whether or not there exists a c.s. satisfying (F_1), (F_2) and (F_3) below, the purpose of the discrimination may be attained. But in order to carry out this plan, we need some devices and techniques. Now let g_{ij} be an arbitrary fundamental tensor whose signature is of type $(- - - +)$, and U be the space-time defined by this g_{ij}. Determine from g_{ij} the forms to be taken by the characteristic vectors (abbreviated to c.v.) assuming that the U is a V. If only these forms are known, we can easily determine whether the U is a V or not by substituting them into the fundamental equations (F_1), (F_2) and

1) Numbers in brackets refer to the references at the end of the paper.

(F_3), and testing whether the U satisfies these equations or not. We call such a method of determining whether the given U is a V or not a *c.v. test*. It should be noted here that in performing a c.v. test, we can ignore in a certain sense the signs of the c.v., by virtue of the freedom of ε-transformation of the c.s. elucidated in [**1**]. Thus, in the following, the sentence "$\overset{1}{u}_i$ is determined uniquely", for example, means, rigorously speaking, that "$\overset{1}{u}_i$ is determined uniquely to within its sign".

The same notations and terminologies as those in [**1**], [**2**] and [**3**] will be used, and many results of these papers will also be used without detailed elucidations. The last two sections are devoted to the appendices, in which we give some mathematical investigations which are in close connection with the contents of the text.

§ 2. Preliminaries.

A V is a four-dimensional Riemannian space-time whose metric can be brought into the form (1.1). A c.s. is composed of four unit vectors $\overset{\alpha}{u}_i$, ($\alpha, \beta, \cdots = 1, \cdots, 4$; $i, j, \cdots = 1, \cdots, 4$), called c.v., and scalars λ_a, μ_a, ($a, b = 2, 3, 4$). They satisfy

$$(\mathrm{F}_1) \qquad -\overset{1}{u}{}^i \overset{1}{u}_i = -\overset{2}{u}{}^i \overset{2}{u}_i = -\overset{3}{u}{}^i \overset{3}{u}_i = \overset{4}{u}{}^i \overset{4}{u}_i = 1, \quad \overset{\alpha}{u}{}^i \overset{\beta}{u}_i = 0, \qquad (\alpha \neq \beta);$$

$$(\mathrm{F}_{2a}) \qquad \nabla_i \overset{1}{u}_j = -\lambda_2 \overset{2}{u}_i \overset{2}{u}_j - \lambda_3 \overset{3}{u}_i \overset{3}{u}_j + \lambda_4 \overset{4}{u}_i \overset{4}{u}_j;$$

$$(\mathrm{F}_{2b}) \qquad \nabla_i \overset{a}{u}_j = \lambda_a \overset{a}{u}_i \overset{1}{u}_j, \qquad \text{(not summed for } a);$$

$$(\mathrm{F}_3) \qquad \nabla_i \lambda_a = \mu_a \overset{1}{u}_i .$$

Sometimes we deal with the scalars λ_{1a} $(=\lambda_{a1})$ and λ_{ab} $(=\lambda_{ba},\ a \neq b)$, as the members of a c.s. Here $\lambda_{\alpha\beta}$'s are the six eigenvalues of $\boldsymbol{K}_A^{\cdot B}$ $(\equiv K_{ij}^{\cdot\cdot mn}$; $A, B, \cdots = 1, 2, \cdots, 6)$, and are connected with λ_a's and μ_a's by

$$(2.1) \qquad \lambda_{1a} = (\lambda_a)^2 - \mu_a, \qquad \lambda_{ab} = \lambda_a \lambda_b, \qquad (a \neq b).$$

(See § **12** below.)

It is proved in [**1**] that U is a V when and only when it admits an orthonormal ennuple and a set of scalars satisfying (F_1), (F_2) and (F_3). They are nothing but the members of a c.s. Some equations satisfied by them are also obtained. The most important one that gives the starting point of the present research is the equation

$$(2.2) \qquad K_i^{\cdot j} \overset{\alpha}{u}{}^i = \nu_\alpha \overset{\alpha}{u}{}^j, \qquad \text{(not summed for } \alpha),$$

where $K_i^{\cdot j}$ is the Ricci tensor and the ν_α's are its eigenvalues or principal values. It is further proved in [1] that when $[K]=\{\overset{\alpha}{u}_i, \lambda_a, \mu_a, \lambda_{\alpha\beta}\}$ is a c.s. of a V, we have a coordinate system in which the metric is given by (1.1) and $\overset{\alpha}{u}_i$'s, λ_a's and μ_a's are given by

$$\overset{1}{u}_i=\delta_i^1, \quad \overset{2}{u}_i=\sqrt{B}\,\delta_i^2, \quad \overset{3}{u}_i=\sqrt{C}\,\delta_i^3, \quad \overset{4}{u}_i=\sqrt{D}\,\delta_i^4; \tag{2.3}$$

$$\lambda_2=-\beta/2, \ \lambda_3=-\gamma/2, \ \lambda_4=-\delta/2, \quad (\beta=B'/B, \ \gamma=C'/C, \ \delta=D'/D); \tag{2.4}$$

$$\mu_2=-\beta'/2, \quad \mu_3=-\gamma'/2, \quad \mu_4=-\delta'/2, \tag{2.5}$$

where a prime means the derivative with respect to x. The coordinate system is called *standard for the* $[K]$. ν_α's and $\lambda_{\alpha\beta}$'s satisfy

$$\begin{aligned} \nu_1&=-(\lambda_{12}+\lambda_{13}+\lambda_{14}), \quad & \nu_2&=-(\lambda_{12}+\lambda_{23}+\lambda_{24}),\\ \nu_3&=-(\lambda_{13}+\lambda_{23}+\lambda_{34}), \quad & \nu_4&=-(\lambda_{14}+\lambda_{24}+\lambda_{34}). \end{aligned} \tag{2.6}$$

In the above coordinate system, we have $\nu_\alpha=K_{\alpha}^{\cdot\alpha}$, (not summed for α).

§ 3. Classification of U's in terms of $\{\nu\}$, and preparatory theorems.

As is stated in the last section, the relation (2.2) is most important for our present purpose. We classify all U's in the following five types in terms of the set of the principal values $\{\nu\}$:

U_{I} : $\{\nu\}=\{a, b, c, d\}$, $(a, b, c, d\neq)$.
U_{II} : " $=\{a, a, b, c\}$, $(a, b, c\neq)$.
U_{III}: " $=\{a, a, a, b\}$, $(a\neq b)$.
U_{IV} : " $=\{a, a, b, b\}$, $(a\neq b)$.
U_{V} : " $=\{a, a, a, a\}$.

This classification corresponds to that of V's given in §2 of [2], and $U_{\mathrm{I}}, \cdots, U_{\mathrm{V}}$ correspond to $V_{\mathrm{I}}, \cdots, V_{\mathrm{V}}$ respectively. Here it should be noted that we use the same notation ν_α to denote a principal value of both U and V for brevity's sake.

The above classification will be used throughout the remainder of the present paper. As is stated in §1, we can perform a c.v. test only when $\overset{\alpha}{u}_i$'s are known. It will easily be understood, however, that if some or all of λ_a's are known further, the test will become much simpler. Further, in some special cases, we may have some methods of discrimination which, compared with that of c.v. test, are much simpler. Examples of these cases will be seen in the following.

Next we prove a theorem which will be of use in determining $\overset{1}{u}_i$ from

g_{ij} when U is given. We introduced the concept of V_0 in [2]. A V_0 is a V in which all of the six eigenvalues of $\boldsymbol{K}_A^{\cdot B}$, i.e. $\lambda_{\alpha\beta}$'s, are constants. Thus when we speak of a V which is not V_0, it means a V in which at least one of $\lambda_{\alpha\beta}$'s is not constant. Corresponding to V_0, we shall denote by U_0 a U whose six eigenvalues, which we denote by the same notations $\lambda_{\alpha\beta}$'s, are all constants.

Now we consider a V which is not V_0, and let λ be one of its non-constant $\lambda_{\alpha\beta}$'s. Since, as is easily seen, the six $\lambda_{\alpha\beta}$'s are functions of x in the coordinate system of (1. 1), the unit vector proportional to the gradient $\nabla_i\lambda$ is space-like and identical with $\overset{1}{u}_i$ to within its sign, which is of no importance in the following discussions as is elucidated in §1. Further it is also evident that any non-constant $\lambda_{\alpha\beta}$ gives the same $\overset{1}{u}_i$. Thus we have

PROPOSITION 3. 1. *Consider a U which is not U_0. Let λ be one of its non-constant $\lambda_{\alpha\beta}$'s, and $\overset{1}{u}_i$ be the unit vector proportional to $\nabla_i\lambda$. If all $\lambda_{\alpha\beta}$'s do not determine the same $\overset{1}{u}_i$, which is gradient and space-like, the U is not V. If they determine the same $\overset{1}{u}_i$ satisfying these conditions, the U has a possibility of being a V.*

Here it should be noted that it is easy to discriminate whether a vector is a gradient or not, since the condition that a vector be a gradient is given by that its rotation be 0. For example, $\overset{1}{u}_i$ is a gradient from (F_{21}).

We add another preparatory theorem. Consider a V. Then $\overset{1}{u}_i$ is a gradient. On the other hand, $\overset{a}{u}_i$ is not necessarily gradient. When $\overset{2}{u}_i$ is a gradient, for example, we have in the standard coordinate system for the c.s. $B=$const. Similar considerations lead us to

PROPOSITION 3. 2. *When $\overset{a}{u}_i$ is a gradient for some a, we have $\lambda_{1a}=\lambda_{ab}=\nu_a=0$ for all b $(\neq a)$.*

§ 4. Discrimination of V_{I}.

First we consider the problem of discriminating whether a U_{I} is a V_{I} or not. (Note that the given U_{I} may or may not be a U_0.) We can easily obtain

PROPOSITION 4. 1. *In a V_{I}, if $\overset{2}{u}_i$ or $\overset{3}{u}_i$ is a gradient, the corresponding ν_a (i.e. ν_2 or ν_3 respectively) is 0, and we cannot have the case in which both $\overset{2}{u}_i$ and $\overset{3}{u}_i$ are gradients.*

The proof of the last part is evident from the fact that if both are gradients, we have $\nu_2=\nu_3=0$, which cannot be the case.

Now we consider a U_{I}. It is evident that if at least one of the eigenvectors of $\mathrm{K}_i^{\cdot j}$ corresponding to the eigenvalues a, b, c, d is null, the U_{I} cannot be V_{I}. Next we consider the case in which all eigenvectors are non-null, and denote the unit eigenvectors by $u_{a|i}, u_{b|i}, u_{c|i}, u_{d|i}$ respectively. When the U_{I} is a V_{I}, these vectors must be identical with $\overset{\alpha}{u}_i$'s (to within their signs) by virtue of (2.2). Therefore one of them (say, $u_{d|i}$) must be time-like and the remaining three space-like, and we assume this in the following. Then, in principle, we can discriminate the U_{I} by putting

$$(4.1) \qquad u_{p|i}=\overset{1}{u}_i\,, \quad u_{q|i}=\overset{2}{u}_i\,, \quad u_{r|i}=\overset{3}{u}_i\,, \quad u_{d|i}=\overset{4}{u}_i\,,$$

where (p, q, r) is any permutation of (a, b, c), and by trying c.v. test. If the U_{I} passes the test for some permutation, it is a V_{I}, while it is not when it fails for all permutations. It will be laborious, however, to carry out this plan. So we shall give some devices in the following.

It is evident that when all of the space-like eigenvectors $u_{a|i}, u_{b|i}, u_{c|i}$ are not gradient, the U_{I} is not V_{I}. Next when only one of these vectors (say, $u_{a|i}$) is a gradient, put

$$(4.2) \qquad u_{a|i}=\overset{1}{u}_i\,, \quad u_{b|i}=\overset{2}{u}_i\,, \quad u_{c|i}=\overset{3}{u}_i\,, \quad u_{d|i}=\overset{4}{u}_i\,,$$

and try c.v. test. Then the U_{I} is a V_{I} when and only when it passes the test. (Note the freedom of i-transformation. Cf. §**4** of [**1**].) When two of the three vectors (say, $u_{a|i}$ and $u_{b|i}$) are gradients, one of a and b (say, b) must be 0 and the remaining one (i.e. a) be non-zero. Try c.v. test by putting (4.2). If the U_{I} passes the test, it is a V_{I}. When it fails anywhere, it is not V_{I}, and accordingly not V.

We find from Proposition 4.1 that one cannot have the case in which all of the three vectors are gradients. Further it is evident that if the given U_{I} is not U_0, we can determine $\overset{1}{u}_i$ by the method stated in Proposition 3.1, and, in general, this procedure simplifies the discrimination process. Thus we have completed the discussions concerning the case of V_{I}.

§5. Discrimination of V_{II}.

We proceed to U_{II}'s, which are defined by $\{\nu\}=\{a, a, b, c\}$, $(a, b, c\neq)$. They are classified into two sub-classes $U_{\mathrm{II}1}$ and $U_{\mathrm{II}2}$ corresponding to $V_{\mathrm{II}1}$ and $V_{\mathrm{II}2}$ respectively. (See §**2** of [**2**].) In $U_{\mathrm{II}1}$, all eigenvectors corresponding to the double eigenvalue a are space-like, while they are space-like or null or time-like in $U_{\mathrm{II}2}$.

First we consider the case of $U_{\text{II}1}$. If the $U_{\text{II}1}$ is a $V_{\text{II}1}$, one of the eigenvectors corresponding to the simple eigenvalues b and c, say $u_{b|i}$, is space-like and the other, say $u_{c|i}$, is time-like. Mutually orthogonal unit eigenvectors corresponding to a are given by

$$(5.1)\qquad v^*_i = v_i \cos\omega - w_i \sin\omega, \qquad w^*_i = v_i \sin\omega + w_i \cos\omega,$$

where v_i and w_i are any pair of mutually orthogonal unit space-like eigenvectors and ω is an arbitrary scalar. If $(\overset{\alpha}{u}_i)=(u_{b|i}, v^*_i, w^*_i, u_{c|i})$ or $(\overset{\alpha}{u}_i)=(v^*_i, w^*_i, u_{b|i}, u_{c|i})$ passes the test, the $U_{\text{II}1}$ is a $V_{\text{II}a}$ or $V_{\text{II}b}$ respectively. (Cf. §**4** of [**1**] and §**2** of [**2**].) If both fail in the test, the given $U_{\text{II}1}$ is not V.

Again it is evident that, when the $U_{\text{II}1}$ is not U_0, we can determine $\overset{1}{u}_i$ by the method of Proposition 3.1, and the determination of $\overset{2}{u}_i$ and $\overset{3}{u}_i$, and hence the process of the c.v. test, becomes much simpler.

Next we consider the case of $U_{\text{II}2}$. In this case, both $u_{b|i}$ and $u_{c|i}$ are space-like. Mutually orthogonal unit eigenvectors corresponding to a are given by

$$(5.2)\qquad v^*_i = v_i \cosh\sigma + w_i \sinh\sigma, \qquad w^*_i = v_i \sinh\sigma + w_i \cosh\sigma,$$

where v_i and w_i are any pair of mutually orthogonal space-like and time-like unit eigenvectors respectively and σ is an arbitrary scalar. The c.v. test by putting $\{(\overset{\alpha}{u}_i)=(u_{b|i}, u_{c|i}, v^*_i, w^*_i)$ or $(u_{c|i}, u_{b|i}, v^*_i, w^*_i)\}$ or $(\overset{\alpha}{u}_i)=(v^*_i, u_{b|i}, u_{c|i}, w^*_i)$ determines whether the $U_{\text{II}2}$ is $V_{\text{II}c}$ or $V_{\text{II}d}$ respectively. (See again §**4** of [**1**] and §**2** of [**2**].) Again, when the $U_{\text{II}2}$ is not U_0, the test becomes much simpler by using Proposition 3.1.

Here it should be noted that the meaning of the c.v. test in the above is somewhat broader than the one used in the last section, since an arbitrary function ω or σ is contained in $\overset{\alpha}{u}_i$'s, which is to be determined in the process of the test, in addition to λ_a's. As will be seen later in [**4**], however, when the U_{II} is a U_0, we can determine λ_a's to some extent, and by virtue of these circumstances, it will not be so laborious to carry out such a *generalized c.v. test.*

§6. Discriminations of $V_{\text{III}a}$ and $V_{\text{III}b}$.

Now we proceed to the discrimination of V_{III}, assuming that the given U is a U_{III} and satisfies $\{\nu\}=\{a, a, a, b\}$, $(a\neq b)$. The V_{III}'s are classified into two sub-classes $V_{\text{III}1}$ and $V_{\text{III}2}$, as is seen in §**2** of [**2**]. Similarly, we classify U_{III}'s into $U_{\text{III}1}$ and $U_{\text{III}2}$. Both are characterized by that the eigenvectors of $K_i^{\cdot j}$ corresponding to the simple eigenvalue b are time-like and space-like respectively. First we consider the case of U_{III} which is not U_0. Naturally we can assume that $\overset{1}{u}_i$ is known. The discrimination of V_{III} belonging to V_0 will be discussed later in [4].

First we deal with the case of $U_{\mathrm{III}1}$. $\overset{1}{u}_i$ must be an eigenvector corresponding to a. Take any pair of space-like unit vectors v_i and w_i belonging to the eigenspace corresponding to a, which are mutually orthogonal and orthogonal to the $\overset{1}{u}_i$. Then we can discriminate the $U_{\mathrm{III}1}$ by performing the generalized c.v. test in which $(\overset{a}{u}_i)=(\overset{1}{u}_i, v^*_i, w^*_i, u_{b|i})$, where v^*_i and w^*_i are given by (5.1).

Next we consider the case of $U_{\mathrm{III}2}$, in which $u_{b|i}$ is space-like. When $u_{b|i}$ is different from $\overset{1}{u}_i$, such a $U_{\mathrm{III}2}$ corresponds to $V_{\mathrm{III}b}$ and is denoted by $U_{\mathrm{III}b}$. Let v_i and w_i be any pair of space-like and time-like unit eigenvectors corresponding to a, which are mutually orthogonal and orthogonal to the $\overset{1}{u}_i$. Then by the generalized c.v. test in which $(\overset{a}{u}_i)=(\overset{1}{u}_i, v^*_i, u_{b|i}, w^*_i)$, where v^*_i and w^*_i are given by (5.2), we can discriminate whether the given $U_{\mathrm{III}b}$ is a $V_{\mathrm{III}b}$ or not.

Lastly, we consider a $U_{\mathrm{III}2}$ in which $\overset{1}{u}_i=u_{b|i}$. Such a $U_{\mathrm{III}2}$ is denoted by $U_{\mathrm{III}c}$ and corresponds to $V_{\mathrm{III}c}$. We must choose $\overset{2}{u}_i$, $\overset{3}{u}_i$ and $\overset{4}{u}_i$ in the eigenspace corresponding to a. This eigenspace is three-dimensional, and these vectors are given by

$$\begin{aligned} &\overset{2}{u}_i = l_2\overset{2}{v}_i + l_3\overset{3}{v}_i + l_4\overset{4}{v}_i\,, \qquad \overset{3}{u}_i = m_2\overset{2}{v}_i + m_3\overset{3}{v}_i + m_4\overset{4}{v}_i\,, \\ &\overset{4}{u}_i = n_2\overset{2}{v}_i + n_3\overset{3}{v}_i + n_4\overset{4}{v}_i\,, \end{aligned} \tag{6.1}$$

where $\overset{2}{v}_i$, $\overset{3}{v}_i$, $\overset{4}{v}_i$ are mutually orthogonal unit eigenvectors, the first two of which are space-like and the third is time-like, and $l_2, l_3, \cdots, n_4$ are coefficients of a pseudo-orthogonal transformation (not necessarily constants) which keeps (F_1) invariant.

Thus, in principle, we can discriminate the $U_{\mathrm{III}c}$ by making a generalized c.v. test. This work is very laborious, however, since we must deal with many unknown functions l_a's, m_a's and n_a's in addition to λ_a's, and it is very desirable to reduce the number of the unknown functions as far as possible. As has sometimes been stated, if λ_a's and $\overset{a}{u}_i$'s are known from g_{ij} to some extent, the work of the generalized c.v. test will become much simpler. The greater part of the remaining pages is devoted to such investigations.

§ 7. Discrimination of $V_{\mathrm{III}\,c}$, 1. Class (A).

In order to deal with the problem of discriminating $V_{\mathrm{III}\,c}$ which is not V_0, we investigate its properties more in detail by classifying all such $V_{\mathrm{III}\,c}$'s into six classes (A), (B), $\cdots$, (F). (A) is composed of $V_{\mathrm{III}\,c}$'s, none of which is V_0 but admits a c.s. satisfying $\lambda_2=\lambda_3=\lambda_4$. The meanings of the remaining classes will be elucidated in the following sections respectively. In this section, we consider the properties of $V_{\mathrm{III}\,c}$'s belonging to (A).

Then in the standard coordinate system for the c.s., we have $\beta=\gamma=\delta$, and hence from the formulas in § 2, we have

(7. 1) $$\lambda_{34}=\lambda_{42}=\lambda_{23}=(\lambda_2)^2\ (=a_1)\,,\qquad \lambda_{12}=\lambda_{13}=\lambda_{14}\ (=a_2)\,,$$

(7. 2) $$\begin{gathered}\nu_1=-3\lambda_{12}\,,\qquad \nu_2=\nu_3=\nu_4=-(\lambda_{12}+2\lambda_{34})\,,\\ (\lambda_{12}\neq\lambda_{34}\,,\qquad \nu_1\neq\nu_2)\,.\end{gathered}$$

Thus we find that $\{\lambda\}$, i.e. the set of $\lambda_{\alpha\beta}$'s, is of type

(7. 3) $$(8)\qquad \{a_1,\ a_1,\ a_1,\ a_2,\ a_2,\ a_2\},\qquad (a_1\neq a_2),$$

where the type number (8) is that used in § 13 below, and that the sequence of signs $\{\lambda\}_s$, which is composed of the signs of the magnitudes of the six-dimensional eigenvectors corresponding to the respective eigenvalues, is given by

(7. 4) $$\{\lambda\}_s=\{+\ -\ -\,,\quad +\ +\ -\}.$$

(See § 12 below.) On the other hand, as the result of § 7 of [3], the $V_{\mathrm{III}\,c}$ admits the freedom of the generalized ω-transformation of c.s., from which we find that any set of mutually orthogonal unit vectors $\overset{2}{u}_i$, $\overset{3}{u}_i$ (both space-like) and $\overset{4}{u}_i$ (time-like) in the three-dimensional eigenspace corresponding to ν_2 $(=\nu_3=\nu_4)$ can be c.v. together with $\overset{1}{u}_i$.

Conversely, from Proposition 11. 1 below, we know that if the eigenvalues and eigenvectors of $\boldsymbol{K}_A^{\cdot B}$ of a $V_{\mathrm{III}\,c}$ have the properties stated above, it must admit a c.s. satisfying $\lambda_2=\lambda_3=\lambda_4$. Thus, considering the freedom of ϵ_1-transformation, we have

Proposition 7. 1. *Let a $U_{\mathrm{III}\,c}$, which is not U_0, be given, its $\{\lambda\}$ be of type* (8), *and further its $\{\lambda\}_s$ be given by* (7. 4), *by interchanging a_1 and a_2 when necessary. Try c.v. test by using $\overset{1}{u}_i$ and any set of orthonormal vectors $\overset{2}{u}_i$, $\overset{3}{u}_i$, $\overset{4}{u}_i$ (the first two are space-like and the last one time-like) in* *the eigenspace corresponding to the triple eigenvalue of* $K_i^{\cdot j}$ *and* λ_a*'s determined from* $\lambda_2=\lambda_3=\lambda_4=\sqrt{a_1}$ *or* $-\sqrt{a_1}$. *If the test succeeds, the* $U_{\mathrm{III}\,c}$ *is a* $V_{\mathrm{III}\,c}$, *and if the procedure fails anywhere, it is not* V.

REMARK. The relation between the types of $\{\lambda\}$ and those of $\{\nu\}$ will be studied in detail in §13 below, and the results obtained there will be of use in considering the present problem, especially in checking the calculations.

§8. Discrimination of $V_{\mathrm{III}\,c}$, 2. Class (B).

Now we proceed to the discrimination of $V_{\mathrm{III}\,c}$ which is not V_0 and admits a c.s. whose two λ_a's are equal. The class (B) is composed of such $V_{\mathrm{III}\,c}$'s.

(B_4) First we consider the case of $\lambda_2=\lambda_3\neq\lambda_4$, and study the properties of $\{\lambda\}$ and $\{\nu\}$ somewhat in detail. In the standard coordinate system for the c.s., we have $\beta=\gamma\neq\delta$ and

$$4\lambda_{12}=4\lambda_{13}=2\beta'+\beta^2\,,\qquad 4\lambda_{14}=2\delta'+\delta^2\,, \tag{8.1}$$
$$4\lambda_{34}=4\lambda_{24}=\beta\delta\,,\qquad 4\lambda_{23}=\beta^2\,.$$

ν_a's are given by (2.6) with (8.1). The condition that the V be $V_{\mathrm{III}\,c}$ is given by, in terms of β and δ,

$$4\beta'+2\beta^2+2\delta'+\delta^2\neq 2\beta'+2\beta^2+\beta\delta=2\delta'+\delta^2+2\beta\delta\,. \tag{8.2}$$

Using these relations we can prove that only the following cases are possible:

(i) The most general case, i.e. the case in which $(\lambda_{12},\ \lambda_{24},\ \lambda_{14},\ \lambda_{23}\neq)$. In this case we have

$$\{\lambda\}=\Big\{a_1(12),\ a_1(13),\ a_2(24),\ a_2(34),\ a_3(14),\ a_4(23)\Big\}, \tag{8.3}$$

$$-\{\nu\}=\{2a_1+a_3,\ 2a_2+a_3,\ \text{''}\ ,\ \text{''}\ \},\qquad (a_1,\ a_2,\ a_3,\ a_4\neq), \tag{8.4}$$

where $a_1=\lambda_{12}$, $a_2=\lambda_{24}$, $a_3=\lambda_{14}$, $a_4=\lambda_{23}$, and a's must satisfy

$$a_1-a_2-a_3+a_4=0\,. \tag{8.5}$$

Here and throughout the remainder of the paper, $a_1(12)$, $a_1(13)$, $\cdots$ in $\{\lambda\}$ mean $a_1=\lambda_{12}$, $a_1=\lambda_{13}$, $\cdots$ respectively, and the members of $\{\nu\}$ are arranged in their natural order $\nu_1,\ \nu_2,\ \nu_3,\ \nu_4$. Thus we have from (8.3), $\{\lambda\}_s=\{+\ +,\ -\ -,\ -,\ +\}$.

(ii) Especially when $\lambda_{12}=\lambda_{23}$ holds, we have $\beta'=0$ and

$$\{\lambda\}=\Big\{a_1(12),\ a_1(13),\ a_1(23),\ a_2(24),\ a_2(34),\ a_3(14)\Big\}, \tag{8.6}$$

$$-\{\nu\}=\{2a_1+a_3,\ 2a_1+a_2,\ \text{''}\ ,\ \text{''}\ \},\qquad (a_1,\ a_2,\ a_3\neq), \tag{8.7}$$

where $4a_1=\beta^2$, $4a_2=\beta\delta$, $4a_3=2\delta'+\delta^2$, and β and δ must satisfy

$$2a_1=a_2+a_3\,. \tag{8.8}$$

(iii) When $\lambda_{14}=\lambda_{24}$ holds, we have

$$\{\lambda\}=\big\{a_1(14),\ a_1(24),\ a_1(34),\ a_2(12),\ a_2(13),\ a_3(23)\big\}, \tag{8.9}$$

$$-\{\nu\}=\{a_1+2a_2,\ 3a_1,\ \text{"}\ ,\ \text{"}\ \},\qquad (a_1,\ a_2,\ a_3\neq), \tag{8.10}$$

where $4a_1=2\delta'+\delta^2=\beta\delta$, $4a_2=2\beta'+\beta^2$, $4a_3=\beta^2$, $(\beta'\neq0)$, and β and δ must satisfy (8.8).

(B_3) Next we consider the case of $\lambda_2=\lambda_4\neq\lambda_3$. In this case, we have $\beta=\delta\neq\gamma$ and

$$\begin{aligned}4\lambda_{12}&=4\lambda_{14}=2\beta'+\beta^2\,, & 4\lambda_{13}&=2\gamma'+\gamma^2\,,\\ 4\lambda_{23}&=4\lambda_{34}=\beta\gamma\,, & 4\lambda_{24}&=\beta^2\,.\end{aligned} \tag{8.11}$$

Similarly to the preceding case, the following three cases are possible:

(i) When $(\lambda_{12},\ \lambda_{23},\ \lambda_{34},\ \lambda_{13},\ \lambda_{24}\neq)$, we have (8.4), (8.5) and

$$\{\lambda\}=\big\{a_1(12),\ a_1(14),\ a_2(23),\ a_2(34),\ a_3(13),\ a_4(24)\big\}. \tag{8.12}$$

(ii) When $\lambda_{12}=\lambda_{24}$ holds, we have (8.7), (8.8) and

$$\{\lambda\}=\big\{a_1(12),\ a_1(14),\ a_1(24),\ a_2(23),\ a_2(34),\ a_3(13)\big\}. \tag{8.13}$$

(iii) When $\lambda_{13}=\lambda_{23}$ holds, we have (8.10), (8.8) and

$$\{\lambda\}=\big\{a_1(13),\ a_1(23),\ a_1(34),\ a_2(12),\ a_2(14),\ a_3(24)\big\}. \tag{8.14}$$

(B_2) The case in which $\lambda_3=\lambda_4\neq\lambda_2$ holds is obtained from (B_3) by interchanging the index 2 with 3 and β with γ. Thus we have the following three cases:

(i) In the general case, we have (8.4), (8.5) and

$$\{\lambda\}=\big\{a_1(13),\ a_1(14),\ a_2(23),\ a_2(24),\ a_3(12),\ a_4(34)\big\}. \tag{8.15}$$

(ii) When $\lambda_{13}=\lambda_{34}$ holds, we have (8.7), (8.8) and

$$\{\lambda\}=\big\{a_1(13),\ a_1(14),\ a_1(34),\ a_2(23),\ a_2(24),\ a_3(12)\big\}. \tag{8.16}$$

(iii) When $\lambda_{12}=\lambda_{23}$ holds, we have (8.10), (8.8) and

$$\{\lambda\}=\big\{a_1(12),\ a_1(23),\ a_1(24),\ a_2(13),\ a_2(14),\ a_3(34)\big\}. \tag{8.17}$$

Summarizing the above, we find that when a $V_{\mathrm{III}c}$ admits a c.s. whose two λ_a's are equal, its $\{\lambda\}$ is one of the following two types:

$$(8.18) \qquad (3) \quad \{a_1, a_1, a_2, a_2, a_3, a_4\}, \quad \text{and} \quad (6) \quad \{a_1, a_1, a_1, a_2, a_2, a_3\}.$$

Here $a_\rho \neq a_\sigma$ when $\rho \neq \sigma$, and the numberings (3) and (6) are those used in §**13** below.

Conversely, if a $V_{\text{III}c}$ is of type (3) or (6), the $V_{\text{III}c}$ admits a c.s. whose two λ_a's are equal if we exclude some special cases. These circumstances will be made clear in §**11** below.

Now we come back to the $V_{\text{III}c}$ admitting a c.s. whose two λ_a's are equal. (8.3), (8.12) and (8.15) are of the same type (3). But $\{\lambda\}_s$ in these cases are $\{+\,+,\ -\,-,\ -,\ +\}$, $\{+\,-,\ +\,-,\ +,\ -\}$ and $\{+\,-,\ +\,-,\ +,\ -\}$ respectively. The last two are identical with each other, while the first is different from them. Further, $\{\nu\}$ and the additional condition (8.5) are common to the last two cases. These come from the fact that any V admits the freedom of the i-transformation of. c.s. Similar circumstances also hold for the cases (ii) and (iii) in (B_3) and (B_2). Thus we can consider that the cases (B_3) and (B_2) are identical with each other by virtue of the freedom of the i-transformation of c.s. In the following, we say that (B_3) and (B_2), for example, are (23)-*conjugate*, since each case is obtained from the other by interchanging each 2 with 3 and each 3 with 2, in the indices of λ_a's, $\lambda_{\alpha\beta}$'s, etc.

As the result of the above we find that when a $V_{\text{III}c}$ of type (3) is given and it is known to admit a c.s. whose two λ_a's are equal, we can discriminate to which of (B_4) and (B_3) (or (B_2)) the given $V_{\text{III}c}$ belongs, by studying its $\{\lambda\}_s$. Similar considerations can be made with respect to the cases (ii) and (iii) of (B_4), (B_3) and (B_2).

The $\{\lambda\}_s$'s corresponding to (8.6), (8.9); (8.13), (8.14) (or (8.16), (8.17)) are $\{+\,+\,+,\ -\,-,\ -\}$, $\{-\,-\,-,\ +\,+,\ +\}$; $\{+\,-\,-,\ +\,-,\ +\}$, $\{+\,+\,-,\ +\,-,\ -\}$ respectively. These four types are exclusive with each other. Hence by using these results, we find that when a $V_{\text{III}c}$ whose $\{\lambda\}$ is of type (6) is given and it is known to admit a c.s. whose two λ_a's are equal, we can discriminate to which case the $V_{\text{III}c}$ belongs.

In the above two cases, in which $\{\lambda\}$ is of type (3) or (6), if further the case (B_a), ($a=2, 3, 4$), to which the given $V_{\text{III}c}$ belongs, is known, we can express $\overset{a}{u}_i$'s in terms of the eigenvectors $\boldsymbol{u}_{\rho|A}$ of $\boldsymbol{K}_A^{\cdot B}$ and $\overset{1}{u}_i$, at most to within a transformation of the form (5.1) or (5.2). Further the values of λ_a's can be almost determined from the eigenvalues of $\boldsymbol{K}_A^{\cdot B}$. We shall show these circumstances by taking two examples.

a) Let $\{\lambda\}$ and $\{\lambda\}_s$ be of type (3) and $\{+\,-,\ +\,-,\ +,\ -\}$ respectively, and the corresponding eigenvalues and unit eigenvectors of $\boldsymbol{K}_A^{\cdot B}$ be a_1, a_2,

a_3, a_4 and $\boldsymbol{u}_{1|A}$, $\boldsymbol{u}_{2|A}$, $\boldsymbol{u}_{3|A}$, $\boldsymbol{u}_{4|A}$ respectively. Denote the four-dimensional expressions of these vectors by $\overset{1}{u}_{ij}$, $\overset{2}{u}_{ij}$, $\overset{3}{u}_{ij}$, $\overset{4}{u}_{ij}$ respectively. Then it is easy to see that the $V_{\mathrm{III}\,c}$ belongs to (i) of (B_3) (or (B_2)) and that we have either $\{\overset{1}{u}_{ij}\overset{1}{u}^i = a\overset{2}{u}_j + b\overset{4}{u}_j,\ \overset{2}{u}_{ij}\overset{1}{u}^i = 0\}$ or $\{\overset{1}{u}_{ij}\overset{1}{u}^i = 0,\ \overset{2}{u}_{ij}\overset{1}{u}^i = a\overset{2}{u}_j + b\overset{4}{u}_j\}$, where a and b are some scalars. When the latter holds, by interchanging the numberings of a_1 and a_2, we can assume that the former holds. Further we have $\overset{3}{u}_{ij}\overset{1}{u}^i = \epsilon\overset{3}{u}_j$ and $\overset{4}{u}_{ij}\overset{1}{u}^i = 0$. (Note that the case obtained from the above by interchanging $\overset{2}{u}_i$ and $\overset{3}{u}_i$ is also possible.) Thus $\overset{a}{u}_i$'s can be obtained from $\boldsymbol{u}_{\rho|A}$'s and $\overset{1}{u}_i$ to within the sign of $\overset{3}{u}_i$ and the transformation (5.2) between $\overset{2}{u}_i$ and $\overset{4}{u}_i$. (We can also determine the linear space $a\overset{2}{u}_i + b\overset{4}{u}_i$ from the condition that it be orthogonal to both $\overset{1}{u}_i$ and $\overset{3}{u}_i$.) Next, λ_a's satisfying $\lambda_2 = \lambda_4 \neq \lambda_3$ are obtained from $(\lambda_2)^2 = a_2$ and $\lambda_2\lambda_4 = a_4$ to within their common sign.

b) Consider a $V_{\mathrm{III}\,c}$ whose $\{\lambda\}$ is of type (6) and whose $\{\lambda\}_s$ is given by $\{+\ +\ +,\ -\ -,\ -\}$, which corresponds to the case of (8.6). Denote the eigenvalues by a_1, a_2, a_3 respectively. We can determine $\overset{4}{u}_i$ by using $\overset{3}{u}_{ij}\overset{1}{u}^i = \epsilon\overset{4}{u}_j$, and then $a\overset{2}{u}_j + b\overset{3}{u}_j$. λ_a's are determined from $\lambda_2 = \lambda_3 \neq \lambda_4$, $(\lambda_2)^2 = a_1$, $\lambda_2\lambda_4 = a_2$.

Now it is evident from the properties of $V_{\mathrm{III}\,c}$ studied in detail in the above that the following discrimination theorem holds.

PROPOSITION 8.1. *Let a $U_{\mathrm{III}\,c}$, which is not U_0, be given, its $\{\lambda\}$ be of type* (3) *or* (6), *and its $\{\lambda\}_s$ be one of those stated in the above. Further, we assume that it is known to admit a c.s. whose two λ_a's are equal. Determine λ_a's and $\overset{a}{u}_i$'s by using the eigenvalues and eigenvectors of $\boldsymbol{K}_A^{\cdot B}$ and $\overset{1}{u}_i$. In determining these quantities, we will have the freedom of the transformation of the form* (5.1) *or* (5.2) *for $\overset{a}{u}_i$'s and that of the sign for λ_a's. Then try c.v. test by using these λ_a's and an arbitrary orthonormal set composed of $\overset{a}{u}_i$'s and $\overset{1}{u}_i$. When the test succeeds, the $U_{\mathrm{III}\,c}$ is a $V_{\mathrm{III}\,c}$. If this procedure fails anywhere, it is not V.*

The reason for the middle part of Proposition comes from the fact that any $V_{\mathrm{III}\,c}$ admitting a c.s. whose two λ_a's are equal admits the freedom of ω-transformation.

REMARK. The relation between $\boldsymbol{u}_{\rho|A}$'s and $\overset{a}{u}_i$'s in the above can easily be understood if we consider the results obtained in §12 below.

§ 9. Discrimination of $V_{\mathrm{III}\,c}$, 3. Classes (C), ⋯, (F).

In order to establish the discrimination theorem concerning the $V_{\mathrm{III}\,c}$ which admits a c.s. satisfying $(\lambda_2, \lambda_3, \lambda_4 \neq)$, we study the properties of such space-times in more detail. The method to be used is the same as those in the previous sections, but the calculations are somewhat long and tedious. So we omit them and only state the outline of the results. In the standard coordinate system for the c. s. under consideration, which satisfies $(\lambda_2, \lambda_3, \lambda_4 \neq)$, λ_a's are given by (2. 4) with $(\beta, \gamma, \delta \neq)$, and we have

$$\begin{aligned} &4\lambda_{12} = 2\beta' + \beta^2, \quad 4\lambda_{13} = 2\gamma' + \gamma^2, \quad 4\lambda_{14} = 2\delta' + \delta^2, \\ &4\lambda_{34} = \gamma\delta, \quad 4\lambda_{24} = \beta\delta, \quad 4\lambda_{23} = \beta\gamma. \end{aligned} \tag{9.1}$$

The condition that the V be $V_{\mathrm{III}\,c}$ is given by $\nu_1 \neq \nu_2 = \nu_3 = \nu_4$, i. e.

$$\begin{aligned} &\lambda_{12} + \lambda_{13} + \lambda_{14} \neq \lambda_{12} + \lambda_{23} + \lambda_{24} \\ &= \lambda_{13} + \lambda_{23} + \lambda_{34} = \lambda_{14} + \lambda_{24} + \lambda_{34}. \end{aligned} \tag{9.2}$$

If we use (9. 1) and (9. 2), we can prove that only the following cases are possible:

(C) The most general case is the one in which all six eigenvalues $\lambda_{\alpha\beta}$'s are different from one another, i. e.

$$(1) \qquad \{\lambda\} = \{a_1, a_2, a_3, a_4, a_5, a_6\}, \qquad (a_\rho \neq a_\sigma \ \text{ when } \ \rho \neq \sigma), \tag{9.3}$$

where (a_1, a_2, a_3) and (a_4, a_5, a_6) (after a suitable renumbering of a_ρ's, when necessary) are eigenvalues corresponding to plus and minus eigenvectors respectively. Here a *plus* or *minus vector* means a six-dimensional vector whose magnitude is plus or minus respectively. (Cf. § **12** below.) Thus we have

$$\{\lambda\} = \big\{a_1(12),\ a_2(13),\ a_3(23),\ a_4(14),\ a_5(24),\ a_6(34)\big\}, \tag{9.4}$$

and $\{\lambda\}_s = \{+, +, +, -, -, -\}$. Evidently, $\overset{1}{u}_{ij}\overset{1}{u}{}^i$, $\overset{2}{u}_{ij}\overset{1}{u}{}^i, \cdots$ give $\overset{2}{u}_j$, $\overset{3}{u}_j$, 0, $\overset{4}{u}_j$, 0, 0 respectively, and $a_3 = \lambda_2\lambda_3$, $a_5 = \lambda_2\lambda_4$, $a_6 = \lambda_3\lambda_4$. From this we find that $\overset{a}{u}_i$'s and λ_a's are almost known from a_ρ's, $u_{\rho|A}$'s and $\overset{1}{u}_i$.

(D) Next we have the case in which

$$(2) \qquad \{\lambda\} = \{a_1, a_1, a_2, a_3, a_4, a_5\}, \qquad (a_\rho \neq a_\sigma \ \text{ when } \ \rho \neq \sigma). \tag{9.5}$$

In more detail, the following six subcases are possible:

$$\{\lambda\} = \big\{a_1(12),\ a_1(24),\ a_2(13),\ a_3(23),\ a_4(14),\ a_5(34)\big\} \quad \text{with} \tag{9.6}$$

$$\{\lambda\}_s = \{+\ -,\ +,\ +,\ -,\ -\}, \tag{9.7}$$

$$(9.8)\qquad \{\lambda\}=\left\{a_1(12),\ a_1(23),\ a_2(13),\ a_3(14),\ a_4(24),\ a_5(34)\right\}\quad \text{with}$$

$$(9.9)\qquad \{\lambda\}_s=\{+\ +,\ +,\ -,\ -,\ -\},$$

$$(9.10)\qquad \{\lambda\}=\left\{a_1(14),\ a_1(24),\ a_2(12),\ a_3(23),\ a_4(13),\ a_5(34)\right\}\quad \text{with}$$

$$(9.11)\qquad \{\lambda\}_s=\{-\ -,\ +,\ +,\ +,\ -\},$$

and the three cases (23)-conjugate to the above three. $\{\lambda\}_s$'s of these three cases are given by (9.7), (9.9) and (9.11) respectively. As a matter of course, we must have for (9.6), for example,

$$(9.12)\qquad a_1+a_2+a_4\neq 2a_1+a_3=a_2+a_3+a_5=a_1+a_4+a_5\,,$$

which is the condition that the V be of type $V_{\mathrm{III}c}$.

Just as in the case (C), we can almost determine $\overset{a}{u}_i$'s and λ_a's from a_ρ's, $u_{\rho|A}$'s and $\overset{1}{u}_i$. If we consider (9.8), for example, we can determine $\overset{3}{u}_i$ from the six-dimensional eigenvectors corresponding to the simple eigenvalue giving plus eigenvectors, $\overset{4}{u}_i$ from minus eigenvectors corresponding to simple eigenvalue, and $\overset{2}{u}_i$ from eigenvectors corresponding to the double eigenvalue. The method of determining λ_a's from a_ρ's is evident.

(E) The third case belongs to type (3) and is of very special type given by

$$(9.13)\qquad \{\lambda\}=\{\varphi,\ \varphi,\ -2\varphi,\ -2\varphi,\ -5\varphi,\ 4\varphi\},$$

where φ is a non-vanishing scalar and is a function of x defined by $\varphi=-2m^{-2}\neq 0$, ($m\equiv 3x+c$, c being an arbitrary constant), in the standard coordinate system for the c.s. The $V_{\mathrm{III}c}$ which gives (9.13) is one of the following six kinds:

$$(9.14)\qquad \{\lambda\}=\left\{\varphi(12),\ \varphi(24),\ -2\varphi(13),\ -2\varphi(23),\ -5\varphi(14),\ 4\varphi(34)\right\},$$

$$(9.15)\qquad \text{〃}=\left\{\varphi(14),\ \varphi(24),\ -2\varphi(13),\ -2\varphi(34),\ -5\varphi(12),\ 4\varphi(23)\right\},$$

$$(9.16)\qquad \text{〃}=\left\{\varphi(12),\ \varphi(23),\ -2\varphi(14),\ -2\varphi(24),\ -5\varphi(13),\ 4\varphi(34)\right\},$$

and the three cases (23)-conjugate to the above three. In all six cases, we have

$$(9.17)\qquad \{\nu\}=\{6\varphi,\ 0,\ 0,\ 0\}.$$

$(\beta,\ \gamma,\ \delta)$ and $(B,\ C,\ D)$, corresponding to (9.14), (9.15) and (9.16) are respectively

(9. 18) $\beta=2m^{-1},\ \gamma=8m^{-1},\ \delta=-4m^{-1};\ \ B=c_2m^2,\ C=c_3m^8,\ D=c_4m^{-4},$

(9. 19) $\beta=-4m^{-1},\ \gamma=8m^{-1},\ \delta=2m^{-1};\ \ B=c_2m^{-4},\ C=c_3m^8,\ D=c_4m^2,$

(9. 20) $\beta=2m^{-1},\ \gamma=-4m^{-1},\ \delta=8m^{-1};\ \ B=c_2m^2,\ C=c_3m^{-4},\ D=c_4m^8,$

where c_2, c_3 and c_4 are arbitrary positive constants. As a matter of course, those corresponding to the (23)-conjugate cases are given from the above expressions by interchanging (β, B) with (γ, C). $\{\lambda\}_s$'s corresponding to (9. 14), (9. 15) and (9. 16) are $\{+-,\ ++,\ -,\ -\}$, $\{--,\ +-,\ +,\ +\}$ and $\{++,\ --,\ +,\ -\}$ respectively. Therefore, if a $V_{\mathrm{III}\,c}$ of type (9. 13) with $\{\lambda\}_s=\{+-,\ ++,\ -,\ -\}$, for example, is given, we can determine $\overset{a}{u}_i$'s from (9. 14) by a method similar to those in the preceding cases, and λ_a's by ($\lambda_2=f$, $\lambda_3=4f$, $\lambda_4=-2f$; $f\equiv\sqrt{-\varphi/2}$).

(F) The fourth case belongs to type (6) and is of very special type given by

$$\{\lambda\}=\{0,\ 0,\ 0,\ \phi,\ \phi,\ -\phi\}, \tag{9. 21}$$

where ϕ is a non-constant function of x given by $\phi=(1/4)(1-2p)n^{-2}$, ($n\equiv px+q$, p and q being arbitrary constants satisfying $p\neq 0$, 1/2, 1), in the standard coordinate system for the c. s. The $V_{\mathrm{III}\,c}$ which gives (9. 21) is one of the following three kinds:

$$\{\lambda\}=\Big\{0(12),\ 0(23),\ 0(24),\ \phi(13),\ \phi(14),\ -\phi(34)\Big\}, \tag{9. 22}$$

$$\{\lambda\}=\Big\{0(14),\ 0(24),\ 0(34),\ \phi(12),\ \phi(13),\ -\phi(23)\Big\}, \tag{9. 23}$$

and the case which is (23)-conjugate to (9. 22). (Note that (9. 23) is self-(23)-conjugate.) $\{\lambda\}_s$'s for (9. 22) and (9. 23) are $\{++-,\ +-,\ -\}$ and $\{---,\ ++,\ +\}$ respectively. Throughout the three cases, we have

$$\{\nu\}=\{-2\phi,\ 0,\ 0,\ 0\}. \tag{9. 24}$$

In the standard coordinate system for the c. s., we have for (9. 22) and (9. 23)

(9. 25) $\beta=0,\ \gamma=n^{-1},\ \delta=(2p-1)n^{-1};\ \ B=c_2,\ C=c_3n^{1/p},\ D=c_4n^{2-1/p},$

(9. 26) $\beta=n^{-1},\ \gamma=(2p-1)n^{-1},\ \delta=0;\ \ B=c_2n^{1/p},\ C=c_3n^{2-1/p},\ D=c_4,$

respectively. As before we have the result: Let a $V_{\mathrm{III}\,c}$ of type (9. 21) with $\{\lambda\}_s=\{++-,\ +-,\ -\}$, for example, be given, where ϕ is a non-constant scalar whose gradient is proportional to $\overset{1}{u}_i$. Then we can determine $\overset{a}{u}_i$'s and λ_a's to some extent by using (9. 22) and $\{\lambda_2=0,\ \lambda_3\lambda_4=-\phi;\ (\lambda_2,\ \lambda_3,\ \lambda_4\neq)$ respectively.

§ 10. Discrimination of $V_{\mathrm{III}\,c}$, 4.

If we examine the results of the preceding three sections, we can find that $V_{\mathrm{III}\,c}$'s of type (3) and those of type (6) appear in § 8 and § 9. First we consider the case of (3). $\{\lambda\}$'s of these $V_{\mathrm{III}\,c}$'s are classified into the following three groups:

$$(10.1)\qquad \begin{cases} (8.3) \text{ with } (8.5). & \{\lambda\}_s = \{++,\ --,\ -,\ +\}. \\ (8.12) \quad " \quad " . & \quad " \quad = \{+-,\ +-,\ +,\ -\}. \\ (8.15) \quad " \quad " . & \quad " \quad = \{+-,\ +-,\ +,\ -\}. \end{cases}$$

$$(10.2)\qquad \begin{cases} (9.14) \text{ with } (9.18). & \{\lambda\}_s = \{+-,\ ++,\ -,\ -\}. \\ (9.15) \quad " \quad (9.19). & \quad " \quad = \{--,\ +-,\ +,\ +\}. \\ (9.16) \quad " \quad (9.20). & \quad " \quad = \{++,\ --,\ +,\ -\}. \end{cases}$$

(10. 2′) { The three cases which are (23)-conjugate to those given in (10. 2). $\{\lambda\}_s$'s are identical with those in (10. 2) respectively.

The λ_a's for the $V_{\mathrm{III}\,c}$'s of the three lines in (10. 1) satisfy the conditions $(\lambda_2=\lambda_3\neq\lambda_4)$, $(\lambda_2=\lambda_4\neq\lambda_3)$ and $(\lambda_3=\lambda_4\neq\lambda_2)$ respectively. On the other hand, those of the $V_{\mathrm{III}\,c}$'s in (10. 2) and (10. 2′) satisfy $(\lambda_2, \lambda_3, \lambda_4 \neq)$. The $V_{\mathrm{III}\,c}$'s in (10. 1) belong to (B_a)'s, and those in (10. 2) and (10. 2′) to (E).

On the other hand, it is shown in [7. 3] of [3] that any c. s. of a $V_{\mathrm{III}\,c}$ is obtained from a c. s. by at most ε-, i-, ω- and generalized ω-transformations. If we use this theorem, we can easily find that the $V_{\mathrm{III}\,c}$'s belonging to (10. 2) or (10. 2′) (these space-times are really the same) cannot be included in those belonging to (10. 1).

Now we shall show another method of arriving at the same conclusion without using [7. 3] of [3]. It is evident from the considerations of $\{\lambda\}_s$'s, which are intrinsic to the space-times, that the first and second $V_{\mathrm{III}\,c}$'s in (10. 2) and (10. 2′) cannot be those in (10. 1). Now we consider the third case in (10. 2). From the considerations of $\{\lambda\}_s$'s, we can conclude that if this $V_{\mathrm{III}\,c}$ belongs to those in (10. 1), it must be the one given in the first line. Then, if we compare (9. 16) with (8. 3), and consider that the double eigenvalue corresponding to plus eigenvectors are φ and a_1 respectively, we have $\varphi=a_1$. Similarly we have

$$(10.3)\qquad \varphi=a_1, \quad -2\varphi=a_2, \quad 4\varphi=a_3, \quad -5\varphi=a_4.$$

But a_ρ's given by (10. 3) cannot satisfy (8. 5). Therefore the $V_{\mathrm{III}\,c}$ of the third line of (10. 2) cannot be the one in (10. 1).

In the same way, we can arrive at the same conclusion with respect to (10.2′). Summarizing the above, we have

PROPOSITION 10.1. *The $V_{\mathrm{III}\,c}$'s in* (10.2) *or* (10.2′) *cannot be included in those of* (10.1).

From this Proposition we find that when a $V_{\mathrm{III}\,c}$ having the property written in any line of (10.1) and (10.2) (or (10.2′)) is given, we can proceed by assuming that the λ_a's of the $V_{\mathrm{III}\,c}$ satisfy the conditions stated after (10.2′) respectively.

Next we proceed to the case of type (6). The $V_{\mathrm{III}\,c}$'s of this type are given by

$$(10.4)\quad \begin{cases} (8.6) \text{ with } (8.8). & \{\lambda\}_s=\{+\,+\,+,\ -\,-,\ -\}. \\ (8.9) \quad " \quad " \ . & \quad " \ =\{-\,-\,-,\ +\,+,\ +\}. \\ (8.13) \quad " \quad " \ . & \quad " \ =\{+\,-\,-,\ +\,-,\ +\}. \\ (8.14) \quad " \quad " \ . & \quad " \ =\{+\,+\,-,\ +\,-,\ -\}. \end{cases}$$

$$(10.5)\quad \begin{cases} (9.22) \text{ with } (9.25). & \{\lambda\}_s=\{+\,+\,-,\ +\,-,\ -\}. \\ (9.23) \quad " \quad " \ . & \quad " \ =\{-\,-\,-,\ +\,+,\ +\}. \end{cases}$$

The cases {(8.16) with (8.8)} and {(8.17) with (8.8)}, which are (23)-conjugate to the third and fourth cases in (10.4) respectively, and the case which is (23)-conjugate to the first line of (10.5) are omitted in the list for brevity's sake. $\{\lambda\}_s$'s are the same only for (9.22) in (10.5) and (8.14) in (10.4), and for (9.23) in (10.5) and (8.9) in (10.4).

Just as in the preceding case, we can find that the $V_{\mathrm{III}\,c}$'s in (10.5) cannot be included in those of (10.4) by using the theorem [7.3] of **[3]**. Again, however, we shall prove this by another method without using this theorem. As a result, we will obtain some interesting kinds of $V_{\mathrm{III}\,c}$'s.

Our first problem is to determine whether or not the space-time of (9.22) with (9.25) can be included as a special case in those belonging to (8.14) with (8.8). For convenience' sake, we consider the problem by using (8.17) with (8.8) in place of (8.14) with (8.8). (Note that both are (23)-conjugate to each other and both space-times are the same.) In (9.22), β, γ and δ are given by (9.25), and this equation is obtained by solving

$$(10.6)\qquad \beta=0\,,\qquad 2\gamma'+\gamma^2=-\gamma\delta=2\delta'+\delta^2\,,$$

under the condition $(0,\gamma,\delta\neq)$. The process is as follows: If we eliminate δ from (10.6), we have

$$(10.7)\qquad \gamma''=2\gamma'^2/\gamma\,.$$

If $\gamma'\doteq 0$, we have $\gamma=-\delta=$const., and we cannot have $\gamma=\delta$. If $\gamma'\neq 0$, we have (9.25), p and q in n being integration constants. In this solution, $\gamma=\delta$ is equivalent to $p=1$. (It should be noted here that the $V_{\text{III}c}$ of type $\gamma=-\delta=\text{const.}$ belongs to V_0, which is excluded from the present discussions.)

On the other hand, from the condition that (9.22) with (9.25) be a special case of (8.17) with (8.8), we have

$$(10.8) \qquad 0=a_1, \quad \psi=a_2, \quad -\psi=a_3, \qquad (2a_1=a_2+a_3),$$

and (β, γ, δ) satisfying $(\gamma=\delta\neq\beta)$ must be obtained from

$$(10.9) \qquad 2\beta'+\beta^2=\beta\gamma=0, \quad 2\gamma'+\gamma^2=-\gamma^2=4\psi.$$

By solving this, we have

$$(10.10) \qquad \beta=0, \quad \gamma=\delta=w^{-1}, \qquad (w\equiv x+c),$$

where c is an arbitrary constant. In this case, we have (8.17) with (10.8) and $\psi=-w^{-2}/4$, and we cannot have $\gamma'=0$. This solution is identical with the one obtained from (9.25) by putting $p=1$.

Thus the line element of the $V_{\text{III}c}$ which is defined by (8.17) with (8.8) and satisfies (9.21) is given by

$$(10.11) \qquad ds^2=-dx^2-c_2dy^2-c_3wdz^2+c_4wdt^2, \qquad (w\equiv x+c),$$

where c_a's are arbitrary positive constants, while that of the $V_{\text{III}c}$ defined by (9.22) with (9.25) is given by

$$(10.12) \quad ds^2=-dx^2-c_2dy^2-c_3n^{1/p}dz^2+c_4n^{2-1/p}dt^2, \quad (n=px+q;\ p\neq 0, 1/2, 1).$$

Then we can easily show that (10.11) cannot admit a c.s. $[K']$ satisfying $(\lambda_2'=0, \lambda_2', \lambda_3', \lambda_4'\neq)$ by using the fact that we have $\overset{1}{u}{}_i'=\overset{1}{u}_i$, $\overset{2}{u}{}_i'=\overset{2}{u}_i$ and $(\overset{3}{u}{}_i', \overset{4}{u}{}_i')$ are obtained from $(\overset{3}{u}_i, \overset{4}{u}_i)$ by a relation similar to (5.2), which comes from the consideration of the relation (8.17). Similarly we can prove that (10.12) cannot admit a c.s. $[K']$ satisfying $(\lambda_2'=0, \lambda_3'=\lambda_4'\neq 0)$. As a result, we find that the $V_{\text{III}c}$ defined by (10.11) and that by (10.12) are intrinsically different from each other. Thus we have solved the problem. The conclusion can also be obtained from (10.14) below.

Now we add some formulas which are of use in discriminating the two space-times studied in the above. We can easily find that ψ is $-\gamma^2/4=-w^{-2}$ for (10.11) and $-\gamma\delta/4=-(2p-1)n^{-2}/4$ for (10.12). Then if we define a scalar M by

$$(10.13) \qquad M=-(1/4)(-\psi)^{-3/2}\overset{1}{u}{}^i\nabla_i\psi,$$

we have

$$(10.14)\qquad \begin{aligned}&\text{(a)}\quad M=\epsilon_1 \qquad \text{for (10.11), and}\\ &\text{(b)}\quad M=\epsilon_1 h,\quad (h\equiv p(2p-1)^{-1/2}\neq 1), \qquad \text{for (10.12)},\end{aligned}$$

which characterize both space-times respectively. The reason why ϵ_1 $(=\pm 1)$ appears in (10.14) comes from (10.13) and the fact that any c.s. admits the freedom of ϵ_1-transformation, by which we can change the sign of $\overset{1}{u}_i$, while ψ is intrinsic to the space-time and is common to all c.s.

Summarizing the above, we find that when a $V_{\mathrm{III}\,c}$ of type (9.21), whose $\{\lambda\}_s$ is $\{++-, +-, -\}$, is given, it is the $V_{\mathrm{III}\,c}$ defined by (10.11) or (10.12) according as (a) or (b) respectively.

We can make similar considerations on the $V_{\mathrm{III}\,c}$ defined by (9.23) in (10.5) and that by (8.9) in (10.4). $V_{\mathrm{III}\,c}$ of type (9.21) whose $\{\lambda\}_s$ is $\{---, ++, +\}$ is one of the following two space-times:

$$(10.15)\qquad ds^2=-dx^2-c_2 w dy^2-c_3 w dz^2+c_4 dt^2,$$

$$(10.16)\qquad ds^2=-dx^2-c_2 n^{1/p} dy^2-c_3 n^{2-1/p} dz^2+c_4 dt^2.$$

The ψ's of these two $V_{\mathrm{III}\,c}$'s satisfy (10.14a) and (10.14b) respectively just as in the preceding case. Evidently these results are of use in the theory of the discrimination of $V_{\mathrm{III}\,c}$.

Lastly it should be remarked that the considerations concerning the c.s. hitherto made are consistent, as a matter of course, with the theory of the freedom of c.s. of $V_{\mathrm{III}\,c}$ developed in [3].

§ 11. Discrimination of $V_{\mathrm{III}\,c}$, 5.

As the result of §§ 7, 8, ⋯, 10, we find that all $V_{\mathrm{III}\,c}$'s, which are not V_0, are classified into the following three classes:

1) $V_{\mathrm{III}\,c}$'s, each of which admits a c.s. satisfying $(\lambda_2=\lambda_3=\lambda_4)$. In this case, $\{\lambda\}$ is of type (8).

2) $V_{\mathrm{III}\,c}$'s, each of which admits a c.s. whose two λ_a's are equal. In this case, $\{\lambda\}$ is of one of the following two subtypes: 2a)⋯(3), and 2b)⋯(6).

3) $V_{\mathrm{III}\,c}$'s, each of which admits a c.s. satisfying $(\lambda_2, \lambda_3, \lambda_4 \neq)$. In this case, $\{\lambda\}$ is of one of the following four subtypes: 3a)⋯(1), 3b)⋯(2), 3c)⋯a special type of (3), i.e. (9.13), and 3d)⋯a special type of (6), i.e. (9.21).

The properties of these classes of $V_{\mathrm{III}\,c}$'s have been studied in detail. Especially the types of $\{\lambda\}_s$'s and the methods to determine $\overset{a}{u}_i$'s and λ_a's from $\overset{1}{u}_i$, a_p's and $u_{p|A}$'s have been made clear, although they have been omitted in some cases for brevity's sake. From these considerations, we find that the type of $\{\lambda\}$ of any $V_{\mathrm{III}\,c}$, which is not V_0, is one of (1), (2), (3), (6) and (8), and that we have the following converse theorems:

PROPOSITION 11. 1. *When* $\{\lambda\}$ *of a* $V_{\mathrm{III}\,c}$, *which is not* V_0, *is of type* {(1) *or* (2)} *or* (8), *it belongs to* 3) *or* 1) *of the above list respectively.*

PROPOSITION 11. 2. *Let* $\{\lambda\}$ *of a* $V_{\mathrm{III}\,c}$, *which is not* V_0, *be of type* (6). *Especially when it is of type* (9. 21) *and satisfies* (10. 14b), *the* $V_{\mathrm{III}\,c}$ *belongs to* 3d), *otherwise to* 2b).

PROPOSITION 11. 3. *Let* $\{\lambda\}$ *of a* $V_{\mathrm{III}\,c}$, *which is not* V_0, *be of type* (3). *Especially when it is of type* (9. 13), *it belongs to* 3c), *otherwise to* 2a).

If we use the above Propositions together with the results obtained in the previous sections, the discriminations of all $V_{\mathrm{III}\,c}$'s are not difficult. The first example is given by Proposition 7. 1, which is an application of Proposition 11. 1. It is evident that the combination of Propositions 8. 1 and 11. 2 gives the complete method of discriminating $V_{\mathrm{III}\,c}$ belonging to 2). In general, we have

PROPOSITION 11. 4. *Let a* $U_{\mathrm{III}\,c}$ *which is not* U_0 *be given. Determine to which class of* 1), 2), 3) (*or to which subclass, in the case of* 2) *or* 3)) *it belongs, by calculating* a_ρ*'s and* $u_{\rho|A}$*'s, by making* $\{\lambda\}_s$ *clear, and by using the above theorems. Further, determine* $\overset{a}{u_i}$*'s and* λ_a*'s as far as possible by using the methods stated in* §§7, 8, $\cdots$ *and the present section. Try c.v. test by using these quantities. If the* $U_{\mathrm{III}\,c}$ *succeeds in the test, it is a* $V_{\mathrm{III}\,c}$, *and if the procedure fails anywhere, it is not V.*

§12. Appendix 1. Some properties of the six-dimensional space $\boldsymbol{M}$.

We have introduced in §1 the six-dimensional symmetric tensor $\boldsymbol{K}_A^{\cdot B}$. This comes from the idea that any antisymmetric tensor in a Riemannian space-time can be considered as a vector in a six-dimensional space $\boldsymbol{M}$. Generalizing such an idea, we have established theories of m-vectors in an n-dimensional space. One of these is the one developed by the present author [5], and is of use in the present discrimination theory as has been seen frequently. The main results restricted to those which have an intimate connection with the present theory are as follows:

The fundamental tensor in $\boldsymbol{M}$ is defined by

$$(12.1)\qquad \boldsymbol{g}_{AB}=2g_{[i[m}g_{j]n]}\,,\quad \boldsymbol{g}^{AB}=2g^{[i[m}g^{j]n]}\,,\quad \boldsymbol{g}_{AB}\boldsymbol{g}^{BC}=\delta_A^C\,,$$

where $A\equiv(ij)$, $B\equiv(mn)$, $(A, B, C=1, 2, \cdots, 6)$. Here the correspondence between the index (e. g. A) of the six-dimensional expression of a quantity and the pair of indices (e. g. (ij)) of the four-dimensional expression of the same quantity is arbitrary, with a proviso that the correspondence should be kept unaltered during a research. Any antisymmetric tensor p_{ij} in the four-dimensional space-time has a one to one correspondence with a vector $\boldsymbol{p}_A$ in $\boldsymbol{M}$ by the relation $p_{ij}\ (=-p_{ji})=\boldsymbol{p}_A$. The raising and lowering of the indices in $\boldsymbol{M}$ are done by using $\boldsymbol{g}^{AB}$ and $\boldsymbol{g}_{AB}$ defined by (12.1) respectively. With such a consideration, we can put $\boldsymbol{K}_A^{\cdot B}\equiv\boldsymbol{K}_{ij}^{\cdot\cdot mn}$. Then it is evident that $\boldsymbol{K}_{AB}$ is a symmetric tensor in $\boldsymbol{M}$.

Using these results in the present case, we have

PROPOSITION 12.1. *The six eigenvalues of* $\boldsymbol{K}_A^{\cdot B}$ *in* $\boldsymbol{M}$ *are* λ_{1a} *and* λ_{ab} $(a, b=2, 3, 4,\ a\neq b;\ \lambda_{ab}=\lambda_{ba})$, *and the unit eigenvectors corresponding to these values are given by*

$$\boldsymbol{u}_{1a|A}=2\overset{1}{u}_{[i}\overset{a}{u}_{j]}\,,\qquad \boldsymbol{u}_{ab|A}=2\overset{a}{u}_{[i}\overset{b}{u}_{j]}\,. \tag{12.2}$$

PROOF. From (2.11) of [1] and the above (12.2), we have

$$\begin{aligned}\boldsymbol{K}_{AB}&=\lambda_{12}\boldsymbol{u}_{12|A}\boldsymbol{u}_{12|B}+\lambda_{13}\boldsymbol{u}_{13|A}\boldsymbol{u}_{13|B}+\cdots\\&\quad-\lambda_{14}\boldsymbol{u}_{14|A}\boldsymbol{u}_{14|B}-\lambda_{24}\boldsymbol{u}_{24|A}\boldsymbol{u}_{24|B}-\cdots\,.\end{aligned} \tag{12.3}$$

On the other hand, from the orthonormality condition (F_1), we can easily see that the six vectors $(\boldsymbol{u}_{12|A},\ \boldsymbol{u}_{13|A},\ \boldsymbol{u}_{23|A})$ and $(\boldsymbol{u}_{14|A},\ \boldsymbol{u}_{24|A},\ \boldsymbol{u}_{34|A})$ are mutually orthogonal plus and minus unit vectors respectively. In other words, they form an orthonormal ennuple in $\boldsymbol{M}$. From these, the proof is evident.

§13. Appendix 2. Relations between the types of the eigenvalues of $\boldsymbol{K}_A^{\cdot B}$ and those of $K_i^{\cdot j}$.

As is seen in §3, we classify all V's into five classes V_{I}, V_{II}, $\cdots$ and V_{V} from the standpoint of the eigenvalues of $K_i^{\cdot j}$. Now we denote the types of these V's by I, II, $\cdots$ and V for brevity's sake. On the other hand, we can classify the types of the set of the six eigenvalues of $\boldsymbol{K}_A^{\cdot B}$ into the following eleven types:

(1) $\{a_1, a_2, a_3, a_4, a_5, a_6\}$, (2) $\{a_1, a_1, a_2, a_3, a_4, a_5\}$,
(3) $\{a_1, a_1, a_2, a_2, a_3, a_4\}$, (4) $\{a_1, a_1, a_1, a_2, a_3, a_4\}$,
(5) $\{a_1, a_1, a_2, a_2, a_3, a_3\}$, (6) $\{a_1, a_1, a_1, a_2, a_2, a_3\}$,
(7) $\{a_1, a_1, a_1, a_1, a_2, a_3\}$, (8) $\{a_1, a_1, a_1, a_2, a_2, a_2\}$,
(9) $\{a_1, a_1, a_1, a_1, a_2, a_2\}$, (10) $\{a_1, a_1, a_1, a_1, a_1, a_2\}$,
(11) $\{a_1, a_1, a_1, a_1, a_1, a_1\}$.

Here $a_\rho\neq a_\sigma$ $(\rho, \sigma=1, 2, \cdots, 6)$ when $\rho\neq\sigma$. The numbes of different eigenvalues belonging to these eleven types are 6; 5; 4, 4; 3, 3, 3; 2, 2, 2; 1 respectively. This classification has been used in §§**7**, **8**, $\cdots$ and **11**.

Now we consider the relation between these two classifications. Evidently, any ν_α is obtained as a linear combination of a_ρ's. Our present purpose is to make clear, by using this relation, what kinds of I, II,$\cdots$ and V can be obtained from any of (1), (2),$\cdots$ and (11). Then, as will be seen in the following, we can make clear at the same time, from what kinds of (1), (2),$\cdots$ and (11), a given type of I, II, $\cdots$ and V can be obtained.

We deal with the problem only from algebraic point of view. We take as an example, the V's of type (8) and make clear what types of I, II, $\cdots$ and V are possible for these V's. The ν_α's and $\lambda_{\alpha\beta}$'s are connected by (2.6). Since the V's under consideration are of type (8), we have the following 10 $(={}_6C_3/2\,!)$ cases to consider:

(i) $\lambda_{12}=\lambda_{13}=\lambda_{14}=a_1\,,\quad \lambda_{34}=\lambda_{24}=\lambda_{23}=a_2\,.$

(ii) $\lambda_{12}=\lambda_{13}=\lambda_{23}=a_1\,,\quad \lambda_{14}=\lambda_{24}=\lambda_{34}=a_2\,.$ (2, 3, 4)

(iii) $\lambda_{12}=\lambda_{13}=\lambda_{24}=a_1\,,\quad \lambda_{14}=\lambda_{34}=\lambda_{23}=a_2\,.$ (2, 3, 4)

(iii′) $\lambda_{12}=\lambda_{13}=\lambda_{34}=a_1\,,\quad \lambda_{14}=\lambda_{24}=\lambda_{23}=a_2\,.$ (2, 3, 4)

Here (2, 3, 4) means the two cases obtained from the left one by the cyclic changes of the indices 2, 3 and 4 of $\lambda_{\alpha\beta}$'s, and it should be noted that, since $a_1\neq a_2$, the cases obtained from the above by interchanging a_1 with a_2 are classified into the same cathegory.

Since (2.6) is symmertic with respect to the three indices 2, 3 and 4, and (iii) and (iii′) are (23)-cojugate to each other, it is evident that, in dealing with the present problem, the three cases contained in (ii) or (iii) or (iii′) give the same results and that (iii′) can be dealt with similarly to (iii). Thus we have only to consider the three cases explicitly written in (i), (ii) and (iii). In these cases, ν_α's are given respectively by

(i) $\nu_1=-3a_1,\quad \nu_2=\nu_3=\nu_4=-(a_1+2a_2)\neq\nu_1\,,$

(ii) $\nu_1=\nu_2=\nu_3=-(2a_1+a_2)\,,\quad \nu_4=-3a_2\neq\nu_1\,,$

(iii) $\nu_1=\nu_2=-(2a_1+a_2)\,,\quad \nu_3=\nu_4=-(a_1+2a_2)\neq\nu_1\,.$

Thus we find that a V of type (8) is of type III or IV according as it is of type {(i) or (ii)} or {(iii) or (iii′)} respectively.

We can make similar considerations concerning the remaining types of V's (1), (2), $\cdots$, (7), (9), $\cdots$. In some cases, however, the circumstances are somewhat complicated. For example, in the case of type (3), we have 45 $(={}_6C_2\cdot{}_4C_2/2\,!)$ cases to consider. But we omit them for brevity's sake and only give the following rough results:

PROPOSITION 13. 1. *We have the following corresponding table, the meaning of which will easily be understood*:

(1) → I, III.	(2) → I, II, III.	(3) → I, II, III, IV.
(4) → I, II.	(5) → I, II, V.	(6) → I, II, III, IV.
(7) → I, II, IV.	(8) → III, IV.	(9) → II, V.
(10) → II.	(11) → V.	

Reversing the order, we have

I ← (1), (2), (3), (4), (5), (6), (7).
II ← (2), (3), (4), (5), (6), (7), (9), (10).
III ← (1), (2), (3), (6), (8).
IV ← (3), (6), (7), (8).
V ← (5), (9), (11).

We obtained more detailed results for all subcases like those appearing in the discussions concerning the type (8). But we omit them for brevity's sake. Further it should be noted again that the results obtained in the above are those only from algebraic point of view. At any rate, the results are compatible with those in the preceding sections.

(To be continued.)

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