Integral formulas for closed submanifolds in a Riemannian manifold

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Introduction.

In the previous paper [9][1] we have given certain generalization of integral formulas of Minkowski type and obtained some properties of a closed orientable hypersurface in a Riemannian manifold. For a submanifold in a Riemannian manifold Y. Katsurada, T. Nagai and H. Kôjyô [7], [8] obtained the following

THEOREM A (*Y. Katsurada and T. Nagai*) *Let R^n be a Riemannian manifold which admits a vector field ξ^i generating a continuous one-parameter group G of homothetic transformations in R^n and V^m a closed orientable submanifold in R^n such that*

(i) *its first mean curvature $H_1=const.$,*

(ii) *the inner product $\underset{E}{n}_i\xi^i$ has fixed sign on V^m,*

(iii) *the generating vector ξ^i is contained in the vector space spanned by m independent tangent vectors and Euler-Schouten unit vector $\underset{E}{n}^i$ at each point on V^m,*

(iv) $R_{ijhk}\underset{E}{n}^i\underset{E}{n}^h g^{\alpha\beta}B^j_\alpha B^k_\beta \geqq 0$ *at each point on V^m.*

Then every point of V^m is umbilic with respect to the vector $\underset{E}{n}^i$.[2]

THEOREM B (*Y. Katsurada and H. Kôjyô*) *Let R^n be a space of constant curvature which admits a vector field ξ^i generating a continuous one-parameter group G of conformal transformations in R^n and V^m a closed orientable submanifold in R^n such that*

(i) *its first mean curvature $H_1=const.$,*

(ii) *the inner product $\underset{E}{n}^i\xi^i$ has fixed sign on V^m,*

(iii) *the generating vector ξ^i is contained in the vector space spanned by m indepent tangent vectors and $\underset{E}{n}^i$ at each point on V^m.*

Then every point of V^m is umbilic with respect to the vector $\underset{E}{n}^i$.

THEOREM C (Y. K*atsurada and* H. K*ôjyô*) *Let R^n be a space of con-*

1) Numbers in brackets refer to the references at the end of the paper.

2) With respect to R_{ijhk}, $\underset{E}{n}^i$, $g^{\alpha\beta}$and B^i_α refer to §1 of the present paper.

stant curvature satisfying the condition of Theorem B. Suppose that V^m is a closed orientable submanifold in R^n such that

(i) *principal curvatures $\underset{E}{k_1}, \underset{E}{k_2}, \cdots, \underset{E}{k_m}$ of V^m for the normal vector $\underset{E}{n^i}$ are positive on V^m and the ν-th mean curvature $\underset{E}{H_\nu}$ $(1<\nu\leqq m-1)$ of V^m for the vector $\underset{E}{n^i}$ equals constant for any ν,*

(ii) *the inner product $\underset{E}{n_i}\xi^i$ has fixed sign on V^m,*

(iii) *the generating vector ξ^i is contained in the vector space spanned by m independent tangent vectors and $\underset{E}{n^i}$ at each point on V^m.*

Then every point of V^m is umbilic with respect to Euler-Schouten unit vector $\underset{E}{n^i}$.

The same problem for a submanifold in a Riemannian manifold has been researched by B. Y. Chen [1], [18], M. Okumura [11], [12], [19], K. Yano [15], [16], [17], [18], [19] and others. It is the aim of the present author to give certain generalization of integral formula of Minkowski type and to obtain some properties of a closed orientable submanifold in a Riemannian manifold.

Notations and general formulas on a submanifold are given in §1. In §2, we derive generalized integral formulas of Minkowski type. As a special case of §2, the later section §3 and §4 are devoted to establish several integral formulas. In §5, we give some properties of a closed orientable submanifold in a Riemannian manifold.

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§1. Notations and general formulas on a submanifold.

Let R^n be an n-dimensional orientable Riemannian manifold of class C^r $(r\geqq 3)$, and x^i, g_{ij}, " $;i$", $R^h{}_{ijk}$, $R_{ij}=R^h{}_{ijh}$ and R be local coordinates, the metric tensor, the operator of covariant differentiation with respect to the Christoffel symbols $\left\{ {h \atop ij} \right\}$ formed with the metric tensor g_{ij}, the curvature tensor, the Ricci tensor, and the curvature scalar of R^n respectively.

We now consider a closed orientable submanifold V^m of class C^3 imbedded in a Riemannian manifold R^n whose local parametric expression is

$$x^i = x^i(u^\alpha),$$

where u^α are local coordinates in V^m. Throughout this paper we will agree on the following ranges of indices unless otherwise stated:

$$1 \leqq h, i, j, \cdots \leqq n,$$
$$1 \leqq \alpha, \beta, \gamma, \cdots \leqq m,$$
$$0 \leqq \lambda, \mu, \nu, \cdots \leqq m-1$$
$$m+1 \leqq P, Q, R, \cdots \leqq n.$$

We use the convention that repeated indices imply summation.

If we put

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha},$$

then $B^i_1, B^i_2, \cdots, B^i_m$ are m linearly independent vectors tangent to V^m. The first fundamental tensor $g_{\alpha\beta}$ of V^m is given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta \tag{1.1}$$

and $g^{\alpha\beta}$ is defined by $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$, where δ^α_γ means the Kronecker deltas. We assume that m vectors $B^i_1, B^i_2, \cdots, B^i_m$ give the positive orientation on V^m and we denote by $\underset{P}{n}{}^i$ unit normal vectors of V^m such that $B^i_1, B^i_2, \cdots, B^i_m, \underset{m+1}{n}{}^i, \cdots, \underset{n}{n}{}^i$ give the positive orientation in R^n. Denoting by ";α" the operation of D-symbol due to van der Waerden-Bortolotti ([13], p. 254), we have

$$B^i_{\alpha;\beta} = H_{\alpha\beta}{}^i, \tag{1.2}$$

where $H_{\alpha\beta}{}^i$ means the Euler-Schouten curvature tensor ([13], p. 256). Then putting $H_{\alpha\beta}{}^i \underset{P}{n}_i = \underset{P}{b}_{\alpha\beta}$, we have

$$H_{\alpha\beta}{}^i = \sum_{P=m+1}^{n} \underset{P}{b}_{\alpha\beta} \underset{P}{n}{}^i, \tag{1.3}$$

$$\underset{P}{n}{}^i{}_{;\alpha} = -\underset{P}{b}{}^\gamma_\alpha B^i_\gamma, \tag{1.4}$$

where $\underset{P}{b}{}^\gamma_\alpha = g^{\beta\gamma} \underset{P}{b}_{\alpha\beta}$.

Let $\underset{E}{n}{}^i$ be Euler-Schouten unit normal vector, that is, the unit vector of the same direction to the vector $g^{\alpha\beta} H_{\alpha\beta}{}^i$,

$$\underset{E}{n}{}^i = \frac{g^{\alpha\beta} H_{\alpha\beta}{}^i}{\| g^{\alpha\beta} H_{\alpha\beta}{}^i \|}$$

([7], p. 93, [8], p. 81).

We also have the equations of Gauss and Codazzi:

$$R_{hijk} B^h_\alpha B^i_\beta B^j_\gamma B^k_\delta = R_{\alpha\beta\gamma\delta} - \sum_{P=m+1}^{n} (\underset{P}{b}_{\alpha\gamma} \underset{P}{b}_{\beta\delta} - \underset{P}{b}_{\beta\gamma} \underset{P}{q}_{\alpha\delta}), \tag{1.5}$$

$$(1.6)\qquad R_{hijk}\underset{P}{n}{}^h B^i_\alpha B^j_\beta B^k_\gamma = -(\underset{P}{b}{}_{\alpha\beta;\gamma} - \underset{P}{b}{}_{\alpha\gamma;\beta})$$
$$= -2\underset{P}{b}{}_{\alpha[\beta;\gamma]}\,, \qquad ([13],\ \text{p. } 266)$$

where $R_{\alpha\beta\gamma\delta}=g_{\alpha\epsilon}R^{\epsilon}{}_{\beta\gamma\delta}$ is the curvature tensor of the submanifold V^m, and the symbol [ ] means alternating in 2 ([13], p. 14).

If we denote by $\underset{P}{k}_1, \underset{P}{k}_2, \cdots, \underset{P}{k}_m$ the principal curvatures of V^m for the normal vector $\underset{P}{n}_i$, that is the roots of the characteristic equation

$$(1.7)\qquad |\underset{P}{b}{}_{\alpha\beta} - \underset{P}{k}g_{\alpha\beta}| = 0\,,$$

then the ν-th mean curvature $\underset{P}{H}_\nu$ is given by

$$(1.8)\qquad \binom{m}{\nu}\underset{P}{H}_\nu = \sum_{\alpha_1<\cdots<\alpha_\nu} \underset{P}{k}_{\alpha_1}\cdots\underset{P}{k}_{\alpha_\nu} = \sum_{\alpha_1,\cdots,\alpha_\nu} \underset{P}{b}{}^{\alpha_1}_{[\alpha_1}\cdots\underset{P}{b}{}^{\alpha_\nu}_{\alpha_\nu]}\,,$$

and $H_0=1$. From equation (1.7) and (1.8) it follows immediately

$$(1.9)\qquad m\underset{P}{H}_1 = \underset{P}{b}{}^\alpha_\alpha\,, \qquad \underset{P}{H}_m = \frac{\underset{P}{b}}{g'}\,,$$

where $\underset{P}{b}$ and g' are determinants of $\underset{P}{b}{}_{\alpha\beta}$ and $g_{\alpha\beta}$ respectively. Moreover we have

$$(1.10)\qquad \underset{P}{H}_1\underset{P}{H}_\nu - \underset{P}{H}_{\nu+1} = \frac{\nu\,!\,(m-\nu-1)\,!}{m\,m\,!}\sum_{\alpha_1<\cdots<\alpha_{\nu+1}} \underset{P}{k}_{\alpha_1}\cdots\underset{P}{k}_{\alpha_{\nu-1}}(\underset{P}{k}_{\alpha_\nu} - \underset{P}{k}_{\alpha_{\nu+1}})^2$$

(cf. [3], p. 292).

We note here that

$$(1.11)\qquad \underset{P}{H}{}_1^2 - \underset{P}{H}_2 = \frac{1}{(m-1)}\Big(\underset{P}{b}{}^\alpha_\beta\underset{P}{b}{}^\beta_\alpha - \frac{1}{m}\,\underset{P}{b}{}^\alpha_\alpha\underset{P}{b}{}^\beta_\beta\Big) = \frac{1}{m^2(m-1)}\sum_{\beta<\alpha}(\underset{P}{k}_\beta - \underset{P}{k}_\alpha)^2 \geqq 0$$

and consequently, if

$$\underset{P}{H}{}_1^2 - \underset{P}{H}_2 = 0\,,$$

then

$$\underset{P}{k}_1 = \underset{P}{k}_2\cdots\underset{P}{k}_m = k\,,$$

that is

$$\underset{P}{b}{}_{\alpha\beta} = \underset{P}{k}g_{\alpha\beta}\,.$$

A point of a submanifold V^m at which all principal curvatures $\underset{P}{k}_1, \underset{P}{k}_2, \cdots, \underset{P}{k}_m$

are equal, is called an umcilical point for the normal vector $\underset{P}{n^i}$.

For any ν, if we put

$$\underset{P}{H}{}^{\alpha\beta}_{(\nu)}=\frac{1}{(m-1)!}\varepsilon^{\alpha}_{\alpha_1\cdots\alpha_\nu\beta_{\nu+1}\cdots\beta_{m-1}}\varepsilon^{\beta\beta_1\cdots\beta_{m-1}}\underset{P}{b}{}^{\alpha_1}_{\beta_1}\cdots\underset{P}{b}{}^{\alpha_\nu}_{\beta_\nu}, \tag{1.12}$$

$$\begin{aligned}\underset{P}{H}_{(\nu)\beta}&=\frac{1}{m!}\varepsilon^{\alpha_1\cdots\alpha_{\nu+1}\gamma_{\nu+2}\cdots\gamma_m}\varepsilon_{\beta\beta_2\cdots\beta_{\nu+1}\gamma_{\nu+2}\cdots\gamma_m}\underset{P}{b}{}^{\beta_2}_{\alpha_1;\alpha_2}\underset{P}{b}{}^{\beta_3}_{\alpha_3}\cdots\underset{P}{b}{}^{\beta_{\nu+1}}_{\alpha_{\nu+1}}\\&=\frac{1}{\binom{m}{\nu+1}}\underset{P}{b}{}^{\alpha_1}_{[\beta;\alpha_1}\underset{P}{b}{}^{\alpha_2}_{\alpha_2}\cdots\underset{P}{b}{}^{\alpha_\nu}_{\alpha_\nu]},\end{aligned} \tag{1.13}$$

then we have the following relations

$$g_{\alpha\beta}\underset{P}{H}{}^{\alpha\beta}_{(\nu)}=m\underset{P}{H}_\nu,\qquad b_{\alpha\beta}\underset{P}{H}{}^{\alpha\beta}_{(\nu)}=m\underset{P}{H}_{\nu+1}, \tag{1.14}$$

and

$$\underset{P}{H}{}^{\alpha\beta}_{(\nu);\alpha}=-\nu m\underset{P}{H}_{(\nu)\alpha}g^{\alpha\beta}, \tag{1.15}$$

where $\varepsilon_{\alpha_1\cdots\alpha_m}$ denotes the ε-symbol of V^m and the symbol [ ] means alternating in $\nu+1$. In particular we have

$$H^{\alpha\beta}_{(0)}=g^{\alpha\beta},\qquad H_{(0)\nu}=0, \tag{1.16}$$

$$\underset{P}{H}_{(1)\alpha}=\frac{1}{\binom{m}{2}}\underset{P}{b}{}^{\beta}_{[\alpha;\beta]}. \tag{1.17}$$

§ 2. Generalized Minkowski formulas for a closed submanifold.

We suppose that R^n admits a one-parameter continuous group G of transformations generated by an infinitesimal transformation

$$\bar{x}^i=x^i+\xi^i\delta\tau, \tag{2.1}$$

where ξ^i are the components of a contravariant vector and $\delta\tau$ is an infinitesimal. In R^n, we consider a domain U. If the domain U is simply covered by the orbits of transformations generated by ξ^i, and ξ^i is everywhere of class C^3 and $\neq 0$ in U, then we call U a regular domain with respect to the vector field (cf. [4], p. 448). If ξ^i is a Killing vector, a homothetic Killing vector, a conformal Killing vector, then the group G is called isometric, homothetic and conformal respectvely.

The vector field ξ^i is said to be conformal, homothetic, or Killing when it satisfies

$$\underset{\xi}{\mathcal{L}} g^{ij} = \xi_{i;j} + \xi_{j;i} = 2\phi(x) g_{ij}\,, \quad \underset{\xi}{\mathcal{L}} g_{ij} = 2c g_{ij}\,, \quad \underset{\xi}{\mathcal{L}} g_{ij} = 0 \tag{2.2}$$

respectively, where $\underset{\xi}{\mathcal{L}} g_{ij}$ denotes the Lie derivative of g_{ij} with respect to thə infinitesimal transformation (2.1), $\phi(x)$ is a scalar function, c is a constant and $\xi_i = g_{ij}\xi^j$ (cf. [14]). When the generating vector ξ^i is a conformal Killing vector, it satisfies

$$\begin{aligned} \underset{\xi}{\mathcal{L}} \left\{ {h \atop ij} \right\} &= \xi^h{}_{;i;j} + R^h{}_{ijk}\xi^k \\ &= \delta^h_i \phi_j + \delta^h_j \phi_i - \phi^h g_{ij}\,, \end{aligned} \tag{2.3}$$

where $\phi_i = \phi_{;i}$, $\phi^h = \phi_i g^{ih}$.

Now, we shall consider $\underset{E}{n^i}$ as one of the unit normal vectors of V^m, that is $\underset{m+1}{n^i} = \underset{E}{n^i}$ and assume that at each point on V^m the generating vector ξ^i is contained in the vector space $\mathscr{V}(B^i_1, B^i_2, \cdots, B^i_m, \underset{E}{n^i})$ spanned by $m+1$ independent vectors $B^i_1, B^i_2, \cdots, B^i_m$ and $\underset{E}{n^i}$. This assumption is always satisfies for the case $m = n-1$, that is, V^m is a hypersurface in R^n ([7], p. 94, [8], p. 83). Then we may put

$$\xi^i = \varphi^\tau B^i_\tau + p\underset{E}{n^i}\,, \tag{2.4}$$

where $p = \underset{E}{n_i}\xi^i$.

Hereafter we denote by V^m an m-dimensional closed orientable submanifold of class C^3 imbedded in a regular domain U with respect to the vector ξ^i. We assume that at any point P on V^m, the vector ξ^i is not on its tangent space.

Let us consider a differential form of $(m-1)$-degree at a point P of V^m, defined by

$$\begin{aligned} &((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, f\xi, \underbrace{\delta\underset{E}{n}, \cdots, \delta\underset{E}{n}}_{\nu}, \underbrace{dx, \cdots, dx}_{\mathrm{m}-\nu-1})) \\ &= \sqrt{g}\,(\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, f\xi, \delta\underset{E}{n}, \cdots, \delta\underset{E}{n}, dx, \cdots, dx) \\ &= \sqrt{g}\left(\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, f\xi, \underset{E}{n}{}_{;\alpha_1}, \cdots, \underset{E}{n}{}_{;\alpha_\nu}, \frac{\partial x}{\partial u^{\alpha_{\nu+1}}}, \cdots, \frac{\partial x}{\partial u^{\alpha_{m-1}}}\right) du^{\alpha_1} \wedge du^{\alpha_2} \wedge \\ &\quad \cdots \wedge du^{\alpha_{m-1}}\,, \end{aligned} \tag{2.5}$$

where the symbol ( ) means a determinant of order n whose columns are the components of respective vectors or vector-valued differential forms, $\wedge$ denotes the exterior multiplication, and dx^i be a displacement along V^m, i.e., $dx^i = B^i_\alpha du^\alpha$, g the determinant of the metric tensor g_{ij} of R^n and f a differentiable scalae function on V^m.

Differentiating exteriorly, we have

$$
\begin{aligned}
& d((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, f\xi, \delta\underset{E}{n}, \cdots, \delta\underset{E}{n}, dx, \cdots, dx)) \\
(2.6) \qquad & = ((\delta\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, f\xi, \delta\underset{E}{n}, \cdots, \delta\underset{E}{n}, dx, \cdots, dx)) + \\
& \sum_{Q=m+2}^{n} ((\underset{E}{n}, \underset{m+2}{n}, \cdots, \delta\underset{Q}{n}, \cdots, \underset{n}{n}, f\xi, \delta\underset{E}{n}, \cdots, \delta\underset{E}{n}, dx, \cdots, dx)) + \\
& ((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, df\xi, \delta\underset{E}{n}, \cdots, \delta\underset{E}{n}, dx, \cdots, dx)) + \\
& ((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, f\delta\xi, \delta\underset{E}{n}, \cdots, \delta\underset{E}{n}, dx, \cdots, dx)) + \\
& \nu((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, f\xi, \delta(\delta\underset{E}{n}), \delta\underset{E}{n}, \cdots, \delta\underset{E}{n}, dx, \cdots, dx)) .
\end{aligned}
$$

On substituting (1.4) into the first term of the right-hand member of (2.6), we obtain

$$
\begin{aligned}
(2.7) \qquad & ((\delta\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, f\xi, \delta\underset{E}{n}, \cdots, \delta\underset{E}{n}, dx, \cdots, dx)) \\
& = m\,!\,(-1)^{(n-1)(n-m)-\nu} \underset{E}{H_{\nu+1}} p\, dA \, ,
\end{aligned}
$$

where $\underset{E}{H_{\nu+1}}$ denotes the $(\nu+1)$-th mean curvature of V^m for the normal direction $\underset{E}{n^i}$ and dA means the volume element of V^m.

By virtue of (1.4) we can see that the vectors

$$
\underset{E}{n} \times \underset{m+2}{n} \times \cdots \times \delta\underset{Q}{n} \times \cdots \times \underset{n}{n} \times \underbrace{\delta\underset{E}{n} \times \cdots \times \delta\underset{E}{n}}_{\nu} \times \underbrace{dx \times \cdots \times dx}_{m-\nu-1}
$$

$$(Q = m+2, \cdots, n)$$

have the same direction to the covariant vector $\underset{Q}{n}$. Then we obtain

$$
((\underset{E}{n}, \underset{m+2}{n}, \cdots, \delta\underset{Q}{n}, \cdots, \underset{n}{n}, f\xi, \delta\underset{E}{n}, dx, \cdots, dx)) = 0 \, .
$$

$$(Q = m+2, \cdots, n)$$

Since the vector

$$
\underset{E}{n} \times \underset{m+2}{n} \times \cdots \times \underset{n}{n} \times \underbrace{\delta\underset{E}{n} \times \cdots \times \delta\underset{E}{n}}_{\nu} \times \underbrace{dx \times \cdots \times dx}_{m-\nu-1}
$$

is orthogonal to the vectors $\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n-1}{n}$ and $\underset{n}{n}$ and $\delta\underset{E}{n^i} = -\underset{E}{b^\beta_\alpha} B^i_\beta du^\alpha$, we have

$$(2.8)\qquad ((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, df\xi, \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx)) = (m-1)!\,(-1)^{(n-1)(n-m)-\nu}\underset{E(\nu)}{H^{\alpha\beta}}\xi_\alpha f_\beta dA\,,$$

$$(2.9)\qquad ((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, f\delta\xi, \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx)) = \frac{(m-1)!}{2}(-1)^{(n-1)(n-m)-\nu} f\underset{E(\nu)}{H^{\alpha\beta}} B^i_\alpha B^j_\beta \underset{\xi}{\mathcal{L}} g_{ij} dA\,,$$

where $f_\alpha = \dfrac{\partial f}{\partial \alpha^\alpha}$, $\xi_\alpha = B^i_\alpha \xi_i$.

Since we have

$$(2.10)\qquad \delta(\underset{E}{\delta n^i}) = \Big(\underset{E}{b^\gamma_{\alpha;\beta}} B^i_\gamma + \underset{E}{b^\gamma_\alpha} \sum_{P=m+1}^{n} \underset{P}{b_{\gamma\beta}} \underset{P}{n^i}\Big) du^\alpha \wedge du^\beta\,,$$

the last term of the-right hand member of (2.6) becomes

$$(2.11)\qquad ((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, f\xi, \delta(\underset{E}{\delta n}), \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx)) = m!\,(-1)^{(n-1)(n-m)-\nu-1} f\xi^\alpha \underset{E(\nu)\alpha}{H}\, dA\,.$$

Accordingly by means of (2.7), (2.8), (2.9) and (2.11) it follows that

$$(2.12)\qquad \begin{aligned} &\frac{1}{m!} d((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, f\xi, \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx)) \\ &= (-1)^{(n-1)(n-m)-\nu}\Big\{\Big(\underset{E}{H_{\nu+1}} p + \frac{1}{2m}\underset{E}{H^{\alpha\beta}_{(\nu)}} B^i_\alpha B^j_\beta \underset{\xi}{\mathcal{L}} g_{ij} - \nu\xi^\alpha \underset{E}{H_{(\nu)\alpha}}\Big) f \\ &\quad + \frac{1}{m}\underset{E}{H^{\alpha\beta}_{(\nu)}} \xi_\alpha f_\beta\Big\} dA\,. \end{aligned}$$

Integrating both members of (2.12) over the whole submanifold V^m and applying Stokes' theorem, we have

$$(2.13)\qquad \begin{aligned} &\frac{1}{m!}\int_{\partial V^m} ((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, f\xi, \underset{E}{\delta n}, \cdots, \underset{E}{\delta n}, dx, \cdots, dx)) \\ &= (-1)^{(n-1)(n-m)-\nu}\int_{V^m}\Big\{\Big(\underset{E}{H_{\nu+1}} p\, dA + \frac{1}{2m}\underset{E}{H^{\alpha\beta}_{(\nu)}} B^i_\alpha B^j_\beta \underset{\xi}{\mathcal{L}} g_{ij} dA \\ &\quad - \nu\xi^\alpha \underset{E}{H_{(\nu)\alpha}} dA\Big) f + \frac{1}{m}\underset{E}{H^{\alpha\beta}_{(\nu)}} \xi_\alpha f_\beta dA\Big\}, \end{aligned}$$

where ∂V^m means the boundary of V^m. Since the submanifold V^m is closed, it follows that

$$\text{(I)} \qquad \int_{V^m} f \underset{E}{H_{\nu+1}} p\, dA + \frac{1}{2m} \int_{V^m} f \underset{E}{H^{\alpha\beta}_{(\nu)}} B^i_\alpha B^j_\beta \underset{\xi}{\mathcal{L}} g_{ij}\, dA - \nu \int_{V^n} f \xi^\alpha \underset{E}{H_{(\nu)\alpha}}\, dA$$
$$+ \frac{1}{m} \int_{V^m} \underset{E}{H^{\alpha\beta}_{(\nu)}} \xi_\alpha f_\beta\, dA = 0 .$$

This formula is nothing but the generalization of the formula established by Y. Katsurada and H. Kôjyô [7] p. 96.

§ 3. Minkowski formulas concerning a conformal transformation.

In this section we shall discuss the formula (I) for a conformal Killing vector ξ^i.

Let G be a group of conformal transformations, then from equations (1. 1), (1. 14) and (2. 2) we obtain

$$\underset{E}{H^{\alpha\beta}_{(\nu)}} B^i_\alpha B^j_\beta \underset{\xi}{\mathcal{L}} g_{ij} = 2\mathrm{m}\phi \underset{E}{H_\nu} .$$

Therefore (I) is rewritten in the following form:

$$\text{(3. 1)} \qquad \int_{V^m} \left\{ \left(\underset{E}{H_{\nu+1}} p + \underset{E}{H_\nu} \phi - \nu \xi^\alpha \underset{E}{H_{(\nu)\alpha}} \right) f + \frac{1}{m} \underset{E}{H^{\alpha\beta}_{(\nu)}} \xi_\alpha f_\beta \right\} dA = 0 .$$

On substituting f=const. into the formula (3. 1), we obtain

$$\text{(I)}_c \qquad \int_{V^m} (\underset{E}{H_{\nu+1}} p + \underset{E}{H_\nu} \phi - \nu \xi^\alpha \underset{E}{H_{(\nu)\alpha}})\, dA = 0 .$$

For $\nu=0$, we have

$$\text{(II)}_c \qquad \int_{V^m} (\underset{E}{H_1} p + \phi)\, dA = 0 .$$

Formula (II)$_c$ is due to Y. Katsurada, H. Kôjyô and T. Nagai ([7], p. 94 and [8], p. 82).

If our manifold R^n is a space of constant Riemann curvature, that is,

$$\text{(3. 2)} \qquad R_{hijk} = \kappa (g_{hj} g_{ik} - g_{hk} g_{ij}) ,$$

we obtain $\underset{E}{H_{(\nu)\alpha}} = 0$ from (1. 6), (1. 13) and (3. 2), and consequently from (I)$_c$ we obtain

$$\text{(3. 3)} \qquad \int_{V^m} (\underset{E}{H_{\nu+1}} p + \underset{E}{H_\nu} \phi)\, dA = 0 .$$

This formula is due to Y. Katsurada H. Kôjyô ([7], p. 96).

Now, let us consider a differential form of $(m-1)$-degree at a point of the submanifold V^m, defined by

$$((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi_{;i}\underset{E}{n}^i, \underbrace{dx, \cdots, dx}_{m-1})) \overset{\text{def}}{=} \sqrt{g}\,(\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi_{;i}\underset{E}{n}^i, dx, \cdots, dx).$$

Differentiating exteriorly, and applying the Stokes' theorem, we have

$$\frac{1}{(m-1)!}\int_{\partial V^m}((\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, \xi_{;i}\underset{E}{n}^i\, dx, \cdots, dx))$$
$$=(-1)^{(n-1)(n-m)}\int_{V^m}(R_{hijk}\underset{E}{n}^h B^i_\alpha \xi^k B^k_\beta g^{\alpha\beta}+mq)dA$$

by virtue of (2.3), where $q=\underset{E}{n}_i\phi^i$.

On making use of that the submonifold V^m is colsed, we have

$$\int_{V^m}(R_{hijk}\underset{E}{n}^h B^i_\alpha \xi^j B^k_\beta g^{\alpha\beta}+mq)dA=0. \tag{3.4}$$

Let G be the group of homothetic transformations, that is, $\phi\equiv$const., then we have

$$\int_{V^m}R_{hijk}\underset{E}{n}^h B^i_\alpha \xi^j B^k_\beta g^{\alpha\beta}dA=0. \tag{3.5}$$

Using the Green's theorem, K. Yano derived above formulas (3.4) rnd (3.5) ([16], pp. 382, 383).

§ 4. Integral formulas in R^n admitting a scalar field such that $\rho_{;i;j}=h(\rho)g_{ij}$.

In this section we assume that the Riemannian manifold admits a non-constant scalar field ρ such that

$$\rho_{;i;j}=h(\rho)g_{ij}, \qquad \rho_i=\rho_{;i}, \tag{4.1}$$

where $h(\rho)$ is a differentiable function of ρ, and assume that $\rho^i=g^{ij}\rho_j$ lies in the vector space $\mathscr{V}$ $(B^i_1, \cdots, B^i_m, \underset{E}{n}^i)$ spanned by the vectors $B^i_1, \cdots, B^i_m$ and $\underset{E}{n}^i$ at each point of V^m. Then we may put

$$\rho^i=\psi^r B^i_r+\alpha\underset{E}{n}^i \tag{4.2}$$

on the submanifold V^m.

We consider a differential form of $(m-1)$-degree at a point P of the submanifold V^m defined by

$$((\underset{E}{n}, \underset{m+1}{n}, \cdots, \underset{n}{n}, f\Phi, \underbrace{\delta\underset{E}{n}, \cdots, \delta\underset{E}{n}}_{\nu}, \underbrace{dx, \cdots, dx}_{m-\nu-1}))$$
$$\overset{\text{def}}{=}\sqrt{g}\,(\underset{E}{n}, \underset{m+2}{n}, \cdots, \underset{n}{n}, f\Phi, \delta\underset{E}{n}, \cdots, \delta\underset{E}{n}, dx, \cdots, dx),$$

where $\Phi=\rho^i\frac{\partial}{\partial x^i}$. Differentiating exteriorly and making use of calculations analogous to those of § 2, we have the following integral formula:

$$\int_{V^m}\left\{(\underset{E}{H}_{\nu+1}\alpha+\underset{E}{H}_\nu h-\nu\rho^\alpha\underset{E}{H}_{(\nu)\alpha})f+\frac{1}{m}\underset{E}{H}^{\alpha\beta}_{(\nu)}\rho_\alpha f_\beta\right\}dA=0\,, \tag{4.3}$$

where $\alpha=\underset{E}{n}^i\rho_{;i}$, $\rho_\alpha=\rho_{;i}B^i_\alpha$. On substituting $f=$const. into the formula (4.3), we obtain

$$\int_{V^m}(\underset{E}{H}_{\nu+1}\alpha+\underset{E}{H}_\nu h-\nu\rho^\alpha\underset{E}{H}_{(\nu)\alpha})dA=0\,, \tag{I$'$}$$

in particular for $\nu=0$ we have

$$\int_{V^n}(\underset{E}{H}_1\alpha+h)dA=0\,. \tag{II$'$}$$

§ 5. Some properties of a closed orientable submanifold.

In this section we shall show the following seven theorems for a closed orientable submanifold V^m in a Riemannian manifold R^n.

THEOREM 5.1. *Let R^n be a Riemannian manifold which admits a continuous one-parameter group G of conformal transformations and V^m a closed orientable submanifold sucd that*

(i) $\underset{E}{H}_\nu=$*const. and* $\xi^\alpha\underset{E}{H}_{(\nu)\alpha}=0$ *for any* ν $(1\leqq\nu\leqq m-1)$,

(ii) $\underset{E}{k}_1>0,\ \underset{E}{k}_2>0,\cdots,\underset{E}{k}_m>0$ *for and* ν $(2\leqq\nu\leqq m-1)$,

(iii) $\xi^i\in\mathscr{V}\ (B^i_1,B^i_2,\cdots,B^i_m,\underset{E}{n}^i)$,

(iv) *the inner product* $\underset{E}{n}_i\xi^i$ *does not change the sign on* V^m.

Then every point of V^m is umbilic with respect to Euler-Schouten vector $\underset{E}{n}$.

PROOF. On substituting the assumption $\xi^\alpha\underset{E}{H}_{(\nu)\alpha}=0$ into the formula $(\mathrm{I})_c$ in § 3, we obtain

$$\int_{V^m}(\underset{E}{H}_{\nu+1}p+\underset{E}{H}_\nu\phi)dA=0\,. \tag{III$_c$}$$

From $(\mathrm{III})_c$ and $(\mathrm{II})_c$ in § 3, we obtain

$$\int_{V^m}(\underset{E}{H}_{\nu+1}p+\underset{E}{H}_\nu\phi)dA=0\,,$$

$$\int_{V^m}(\underset{E}{H}_1\underset{E}{H}_\nu p+\underset{E}{H}_\nu\phi)dA=0$$

because of $\underset{E}{H}_\nu=\text{constant}$. Therefore we have

$$\int_{V^m}(\underset{E}{H}_1\underset{E}{H}_\nu-\underset{E}{H}_{\nu+1})p\,dA=0\,. \tag{5.1}$$

Due to (1.10) and the assumption (ii) (iii) and (iv), the integrand on the left side of equation (5.1) keeps a constant sign; the relation is possible, only when the integrand vanishes identically, which in turn implies

$$\underset{E}{H}_1\underset{E}{H}_\nu-\underset{E}{H}_{\nu+1}=0\,,$$

that is,

$$\underset{E}{k}_1=\underset{E}{k}_2=\cdots=\underset{E}{k}_m$$

at all points of the submanifold V^m. Accordingly every point of V^m is umbilic with respect to Euler-Schouten vector $\underset{E}{n}$.

Theorem 5.1 has been obtained by T. Nagai ([10], p. 153) for $\nu=1$. In the case where R^n admits a group G of proper homothetic transformations, Theorem 5.1 has been obtained by Y. Katsurada and T. Nagai for $\nu=1$ i.e., Theorem A stated in the introduction. In the case where R^n is a space of constant curvature, Theorem 5.1 becomes Theorem B and Theorem C stated in the introduction.

THEOREM 5.2. *Let R^n be a Riemannian manifold which admits a non-constant scalar field ρ such that $\rho_{;i;j}=h(\rho)g_{ij}$ and V^m a closed orientable submanifold such that*

(i) $\underset{E}{H}_\nu=const.$ *and* $\rho^a\underset{E}{H}_{(\nu)a}=0$ *for any* ν $(1\leqq\nu\leqq m-1)$,

(ii) $\underset{E}{k}_1>0,\ \underset{E}{k}_2>0,\cdots,\underset{E}{k}_m>0$ *for any* ν $(2\leqq\nu\leqq m-1)$,

(iii) $\rho^i\in\mathscr{V}\ (B_1^i,\ B_2^i,\cdots,B_m^i,\ \underset{E}{n}^i)$,

(iv) *the inner product* $\alpha=\underset{E}{n}^i\rho_i$ *does not change the sign on* V_m.

Then every point of V^m is umbilic with respect to Euler-Schouten vector $\underset{E}{n}$.

PROOF. On substituting the assumption (i) into the formula (I′) in §4, we have

$$\int_{V^m}(\underset{E}{H}_{\nu+1}\alpha+\underset{E}{H}_\nu h)dA=0\,. \tag{III′}$$

From (III′) and (II′) in §4, we obtain

$$\int_{V^m}(\underset{E}{H}_{\nu+1}\alpha+\underset{E}{H}_\nu h)dA=0\,,$$

$$\int_{V^m} (\underset{E}{H_1}\underset{E}{H_\nu}\alpha + \underset{E}{H_\nu}h)dA = 0$$

because of $\underset{E}{H_\nu} = \text{constant}$. Therefore we have

$$\int_{V^m} (\underset{E}{H_1}\underset{E}{H_\nu} - \underset{E}{H_{\nu+1}})\alpha dA = 0 , \tag{5.2}$$

which holds if and only if $\underset{E}{H_1}\underset{E}{H_\nu} - \underset{E}{H_{\nu+1}} = 0$. Thus we can see the conclusion.

For $\nu = 1$, this theorem reduces to a result due to K. Yano ([15], p. 505).

THEOREM 5. 3. *Let R^n be a Riemannian manifold which admits a continuous one-parameter group G of conformal transformations and V^m a closed orientable submanifold such that*

(i) $\underset{E}{H_1}p + \phi \leqq 0$ $(or \geqq 0)$ *and* $\xi^\alpha \underset{E}{H_{(\nu)\alpha}} = 0$ *for any* ν $(1 \leqq \nu \leqq m-1)$,

(ii) $\underset{E}{k_1} > 0,\ \underset{E}{k_2} > 0, \cdots, \underset{E}{k_m} > 0$ *for any* ν $(2 \leqq \nu \leqq m-1)$,

(iii) $\xi^i \in \mathscr{V}\ (B_1^i,\ B_2^i, \cdots, B_m^i,\ \underset{E}{n^i})$,

(iv) *the inner product* $p = \underset{E}{n_i}\xi^i$ *does not change the sign on* V^m.

Then every point of V^m is umbilic with respect to Euler-Schouten vector $\underset{E}{n}$.

PROOF. From our assumption (i) and $(\mathrm{II})_c$ in §3 we have the relation

$$\underset{E}{H_1}p = -\phi . \tag{5.3}$$

Substituting (5. 3) into the formula $(\mathrm{III})_c$, we obtain

$$\int_{V^n} (\underset{E}{H_1}\underset{E}{H_\nu} - \underset{E}{H_{\nu+1}})p dA = 0 ,$$

which hold if and only if

$$\underset{E}{H_1}\underset{E}{H_\nu} - \underset{E}{H_{\nu+1}} = 0 .$$

Then we obtain the conclusion.

THEOREM 5. 4. *Let R^n be a Riemannian manifold which admits a continuous one-parameter group G of conformal transformations and V^m a closed orientable submanifold such that*

(i) $\underset{E}{H_{\nu+1}}p + H_\nu\phi \leqq 0$ $(or \geqq 0)$ *and* $\xi^\alpha \underset{E}{H_{(\nu)\alpha}} = 0$ *for any* ν $(1 \leqq \nu \leqq m-1)$,

(ii) $\underset{E}{k_1} > 0,\ \underset{E}{k_2} > 0, \cdots, \underset{E}{k_m} > 0$,

(iii) $\xi^i \in \mathscr{V}\ (B_1^i,\ B_2^i, \cdots, B_m^i,\ \underset{E}{n^i})$,

(iv) *the inner product* $p = \underset{E}{n_i}\xi^i$ *does not change the sign on* V^m.

Then every point of V^m is umbilic with respect to the vector $\underset{E}{n}$.

PROOF. From our assumption (i) and $(\mathrm{III})_c$ we have the relation

$$\underset{E}{H}_{\nu+1} = -\underset{E}{H}_\nu \phi\,. \tag{5.4}$$

Substituting (5.4) into the formula $(\mathrm{II})_c$ in § 3, we obtain

$$\int_{V^m} \frac{1}{\underset{E}{H}_\nu}(\underset{E}{H}_1 \underset{E}{H}_\nu - \underset{E}{H}_{\nu+1}) p\, dA = 0\,, \tag{5.5}$$

which holds if and only if $\underset{E}{H}_1\underset{E}{H}_\nu-\underset{E}{H}_{\nu+1}=0$. Thus we can see the conclusion.

THEOREM 5.5. *Let R^n be a Riemannian manifold which admits a continuous one-parameter group G of conformal transformations and V^m a closed orientable submanifold such that*

(i) $-\dfrac{\phi}{\underset{E}{H}_1} \geqq p$ (*or* $\leqq p$) *and* $\xi^\alpha \underset{E}{H}_{(\nu)\alpha}=0$ *for any* ν $(1\leqq\nu\leqq m-1)$,

(ii) $\underset{E}{k}_1>0$, $\underset{E}{k}_1<0, \cdots, \underset{E}{k}_m>0$ *for any* ν $(2\leqq\nu\leqq m-1)$ *and* $\underset{E}{H}_1>0$ (*or* <0) *for* $\nu=1$,

(iii) $\xi^i \in V\,(B_1^i, B_2^i, \cdots, B_m^i, \underset{E}{n}^i)$,

(iv) *the inner product* $p=\underset{E}{n}_i\, \xi^i$ *does not change the sign on* V^m.

Then every point of V^m is umbilic with respect to Euler-Schouten vector $\underset{E}{n}$.

PROOF. By virtue of our assumptions and $(\mathrm{II})_c$ in § 3, we obtain the following relation

$$p = -\frac{\phi}{\underset{E}{H}_1}\,. \tag{5.6}$$

Substituting (5.6) into $(\mathrm{III})_c$, we obtain

$$\int_{V^m} (\underset{E}{H}_1 \underset{E}{H}_\nu - \underset{E}{H}_{\nu+1}) p\, dA = 0\,,$$

which holds if and only if $\underset{E}{H}_1\underset{E}{H}_\nu-\underset{E}{H}_{\nu+1}=0$. Then we obtain the conclusion.

In the case that R^n is a space of constant curvature, Theorem 5.3 ond Theorem 5.5 have been obtained by Y. Katsurada and H. Kôjyô ([7]).

THEOREM 5.6. *Let R^n be a Riemannian manifold which admits a continuous one-parameter group O of conformal transformations and V^m a closed orientable submanifold such that*

(i) $-\dfrac{\underset{E}{H}_\nu}{\underset{E}{H}_{\nu+1}}\phi \geqq p$ (*or* $\leqq p$) *and* $\xi^\alpha \underset{E}{H}_{(\nu)\alpha}=0$ *for any* ν $(1\leqq\nu\leqq m-1)$,

(ii) $\underset{E}{k}_1>0, \underset{E}{k}_2>0, \cdots, \underset{E}{k}_m>0$,

(iii) $\xi^i \in \mathscr{V}(B_1^i, B_2^i, \cdots, B_m^i, \underset{E}{n}^i)$,

(iv) *the inner product* $p=\underset{E}{n}_i\xi^i$ *does not change the sign on* V^m.

Then every point of V^m *is umbilic with respect to Euler-Schouten vector* $\underset{E}{n}$.

PROOF. The formula $(\mathrm{III})_c$ is rewritten as follows

$$\int_{V^m} \underset{E}{H}_{\nu+1}\left(p+\frac{\underset{E}{H}_\nu}{\underset{E}{H}_{\nu+1}}\phi\right)dA=0\,.$$

By virtue of our assumptions, we have the following relation

$$p=-\frac{\underset{E}{H}_\nu}{\underset{E}{H}_{\nu+1}}\phi\,. \tag{5.7}$$

Substituting (5.7) into $(\mathrm{II})_c$ in §3, we obtain

$$\int_{V^m}\frac{1}{\underset{E}{H}_\nu}(\underset{E}{H}_1\underset{E}{H}_\nu-\underset{E}{H}_{\nu+1})p\,dA=0\,,$$

which holds if and only if $\underset{E}{H}_1\underset{E}{H}_\nu-\underset{E}{H}_{\nu+1}=0$. Then we obtain the conclusion.

THEOREM 5.7. *Let* R^n *be a Riemannian manifold which admits a continuous one-parameter group* G *of conformal transformations and* V^m *a closed orientable submanifold such that*

(i) $\underset{E}{H}_\nu^{\frac{1}{\nu}}p=-\phi$ *for any* ν $(2\leqq\nu\leqq m-1)$,

(ii) $\underset{E}{H}_1>0, \underset{E}{H}_2>0, \cdots, \underset{E}{H}_\nu>0$,

(iii) $\xi^i \in \mathscr{V}(B_1^i, B_2^i, \cdots, B_m^i, \underset{E}{n}^i)$,

(iv) *the inner product* $p=\underset{E}{n}_i\xi^i$ *does not change the sign on* V^m.

Then every point of V^m *is umbilic with respect to Euler-Schouten vector* $\underset{E}{n}$.

PROOF. The following lemma is well-known.

LEMMA. *If* $\underset{E}{H}_1, \underset{E}{H}_2, \cdots, \underset{E}{H}_\nu$ $(2\leqq\nu\leqq m-1)$ *are positive, then we have*

$$\underset{E}{H}_1\geqq \underset{E}{H}_2^{\frac{1}{2}}\geqq\cdots\geqq\underset{E}{H}_\nu^{\frac{1}{\nu}}\,, \tag{5.8}$$

where the equality implies that V^m is umbilic with respect to the vector $\underset{E}{n}$, *i. e.*, $\underset{E}{k_1}=\underset{E}{k_2}=\cdots=\underset{E}{k_m}$. (cf. [2], p. 52).

On substituting the assumption (i) into the formula $(\mathrm{II})_c$, we obtain

$$\int_{V^m}(\underset{E}{H_1}-\underset{E}{H_\nu^{\frac{1}{\nu}}})p\,dA=0\,. \tag{5.9}$$

Due to the inequality (5.8) the integrand in the left side of equation (5.9) keeps a constant sign, and therefore

$$\underset{E}{H_1}-\underset{E}{H_\nu^{\frac{1}{\nu}}}=0\,,$$

which implies that V^m is umbilic with respect to the vector $\underset{E}{n}$.

REMARK. If R^n admits a special concircular scalar field ρ such that

$$\rho_{;i;j}=c\rho g_{ij}\,,\qquad c=\text{const.},$$

then we can prove that V^m in the preceding theorems is isometric to a sphere. (cf. [6], [10]).

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