A note on homotopy spheres

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§ 0. Introduction

All manifolds will mean compact oriented smooth manifolds without any further notices. In [5] J. Milnor and M. Kervaire has determined bP_m, the group of homotopy spheres which bound parallelizable manifolds. If $m=4k$ $(k\neq 1)$, then bP_{4k} is the cyclic group of order $\sigma_k/8$. In this paper we will consider the group of homotopy spheres which bound manifolds of dim m, whose Spivk normal fiber spaces are trivial. We denote it by bF_m. We show that there exists an analogy of the above fact for bF_m. We define bF_m^0 to be the group of homotopy spheres which bound manifolds of dim m whose Spivak normal fiber spaces are trivial and whose indexes are zero. Then bF_m^0 is a subgroup of bF_m. Let f_k be $1/8$ min $\{n\in \boldsymbol{Z} \mid n$ is the index of a closed manifold of dim $4k$ whose Spivak normal fiber space is trivial. $n>0\}$. Then we have

THEOREM 0. 1 i) *If* $m\geq 6$, *then* $bF_m=bP_m$. ii) *If* $m=4k$, *then the group* bF_m/bF_m^0 *is isomorphic to a cyclic group of order* f_k.

THEOREM 0. 2 *Let* d_{2n} *be the greatest common divisor of* $2^{4n-2}(2^{4n-1}-1)$ numerator $(B_m/4m)$ *and* $2\left\{2^{2n-1}\cdot(2^{2n-1}-1)\cdot a_n\cdot \text{numerator}\left(\dfrac{B_n}{4n}\right)\right\}^2$. *Then* $f_{2n}\leqq d_{2n}$. *Especially if* $k=1$, *then* $f_2=4$ *which is equivalent to* $bF_8^0\cong \boldsymbol{Z}_7$.

§ 1 is devoted to preliminaries. Theorem 0.1 will be proved in § 2. In § 3 we will give some computations and a proof of Theorem 0.2.

§ 1. Preliminaries

We quote some results due to D. Sullivan [8]. Let $F/0$ be the fiber of the map, $BSO\rightarrow BSF$. Let W be a simply connected manifold with a boundary $\partial W\neq\phi$. Let $hS(W)$ denote the concordance classes of h-smoothings h: $(W', \partial W')\rightarrow(W, \partial W)$ of W.

(1. 1) If dim $W\geqq 6$, then there is a bijection η, $hS(W)\rightarrow[W, F/0]$. Moreover if a h-smoothing, h: $(W', \partial W')\rightarrow(W, \partial W)$ corresponds to f: $W\rightarrow F/0$ by η, then the stable tangent bundle $\tau_{w'}$ of W' is equivalent to $h^*\tau_w\oplus h^*f^*(\gamma)$, where γ is a universal $F/0$-bundle [8, 9].

If ∂W is a homotopy sphere of dim $\partial W=m-1$, let a map $\bar{d}$; $hS(W)\rightarrow\theta_{m-1}$ be defined as follows. θ_{m-1} denotes the group of homotopy spheres of dim$=m-1$. Let h be as above and α the class of h. Then we put $\bar{d}(\alpha)=\{\partial W-\partial W'\}$. Let $W=W\underset{\partial W}{\cup}C(\partial W)$, where C is cone and $\bar{d}\circ\eta^{-1}=d$. D. Sullivan has defined a surgery obstruction $\mathscr{S}$: $[\hat{W}, F/0]\to \boldsymbol{Z}$ if $m=4k$. Let γ: $[W, F/0]\to[W, F/0]$ be a restriction. Then we have the following commutative diagram which is implicit in [8].

$$(1.\ 2)\quad \begin{array}{ccccc} [W, F/0] & \xrightarrow{\mathscr{S}} & \boldsymbol{Z} & \longrightarrow & \boldsymbol{Z}\sigma_k/8 \\ \downarrow & & & & \downarrow \\ [W, F/0] & \longrightarrow & \theta_{4k-1} & \longrightarrow & bP_{4k} \end{array}$$

We call a manifold W *F-parallelizable* if its Spivak normal fiber space is trivial. As an immediate consequence of (1. 1) we have the following

COROLLARY 1. 2. *Let W be a simply connected F-paralleable manifold with $\partial W\neq\phi$ and* dim $W\geqq 6$. *Then there exists a h-smoothing, h: $(W', \partial W')\to(W, \partial W)$ os that W' is a parallelizable manifold.*

(PROOF) Since W is F-parallelizable, we can choose a spherical trivialization t of the stable normal bundle ν_w of W. Then (ν_w, t) is an $F/0$-bundle [8, 9] and there is a map f: $W\to F/0$ so that $f^*(\gamma)=\nu_w$. Then h: $(W', \partial W')\to(W, \partial W)$ corresponding to f is what we want.

We use (1. 2) in §3.

§2. bF_m and bF_m^0

It follows from [6, Lemma 1] that an almost parallelizable closed manifold is F-parallelizable. Let W be an F-parallelizable manifold with ∂W a homotopy sphere. $\hat{W}=W\underset{\partial W}{\cup}C(\partial W)$ is a *p. l.* manifold. Then we have

LEMMA 2. 1 *$\hat{W}$ is an F-parallelizable p. l. manifold.*

(PROOF) The proof is the same as that of [6, Lemma 1] by considering in piecewise linear category.

By Lemma 2. 1 we need not consider an almost F-parallelizable manifold.

PROOF OF THEOREM 0. 1 i) Let Σ bound an F-parallelizable manifold W. We embed W in D^{N+1} (N is sufficiently large) so that $\Sigma\subset S^N$. If we choose a spherical trivialization t of the normal disk bundle of W, then it follows from Lemma 2. 1 that $t|\Sigma$ is reduced to a framing. If we consider the construction of the homomorphism, $\theta_{m-1}/bP_m\to\Pi_{m-1}^S/\mathrm{Im}\,J$ [5, Theorem 4. 1], then it is easy to see $bF_m/bP_m\to\Pi_{m-1}^S/\mathrm{Im}\,J$ is a zero map, which completes the proof.

LEMMA 2. 2 *If W^{4k} is an F-parallelizable closed manifold, then the index of W ($\equiv I(W)$) is divisible by* 8.

(PROOOF) Let ν_w be a stable normal bundle of W, E its associated disk bundle and t: $E \to D^N$, a spherical trivialization of E so that t is transverse regular on zero of unit N-disk D^N. If we put $W' = t^{-1}(0)$, and π the projection: $W' \to W$, then $\tau_{w'} \oplus \varepsilon^N = \pi^* \tau_w \oplus \pi^* \nu_w$. Since
$$\begin{array}{ccc} -(\tau_w + \varepsilon^N) & \to & -\pi^*(\tau_w \oplus \nu_w) \\ \downarrow & & \downarrow \\ W' & \longrightarrow & W \end{array}$$
is a normal map, $I(W) - I(W')$ is divisible by 8. $I(W') = 0$, since W' is parallelizable, so $I(W)$ is divisible by 8.

LEMMA 2. 3 *Let Σ_i bound F-parallelizable manifold W_i of* dim $4k$ $(i=0, 1)$. *Then $\Sigma_0 - \Sigma_1 \in bF^0_{4k}$ if and only if $I(W_0) \equiv I(W_1)$* (mod $8 \cdot f_k$)

(PROOF) The proof is just the same as that of [5, Theorem 7. 5].

PROOF OF THEOREM 0. 1 ii) We can define an injective homomorphism; $bF_{4k}/bF^0_{4k} \to \boldsymbol{Z}_{f_k}$ by mapping Σ_0 into 1/8 $I(W_0)$ (mod f_k). It follows from Theorem 0. 1, i) that this is surjective.

§ 3. On the number f_k and computations

Let (W, Σ) be an F-parallelizable manifold of dim $4k$. Then the image d ([W, $F/0$]) is contained in bF^0_{4k}. In facts, if h; $(W', \partial W') \to (W, \partial W)$ is a h-smoothing, then W' becomes an F-parallelizable manifold [2, Theorem 3. 6] and $I(W \# - W') = I(W) - I(W') = 0$. Moreover we will show Lemma 3. 3. So we can give some elements of bF^0_{4k} by using (1. 2) if we choose a convenient manifold W. In the sequel let W^{4k} satisfy the following conditions (C); 1) ∂W is a homotopy sphere, 2) W is parallelizable, 3) W is $(2k-1)$ connected, 4) H_{2k} $(W; \boldsymbol{Z})$ is of rank l. Then $H_{2k}(W)$ becomes a free module of rank l. Then W has the homotopy type of $\bigvee_{i=1}^{l} S^{2k}_i$. Let α be the composition map;
$S_{4k-1} \to \partial W \subsetneq W \xrightarrow{h} \bigvee_{i=1}^{l} S^{2k}_i \xrightarrow{\beta} S^{2k}$, where $S_{4k-1} \to \partial W$ is a map of degree 1, h a homotopy equivalence, β any map.

LEMMA 3. 1 *Let W and α be as above, then the suspension of α is zero.*

(PROOF) Let $V = W \underset{\partial W}{\cup} -(W - \operatorname{Int} D^{4k})$. There is an extension $\bar{R}$ over V of α. In fact, since ∂W is a retraction of $(W - \operatorname{Int} D^{4k})$, we have a retract R; $(W - \operatorname{Int} D^{4k}) \to \Sigma$. It is clear that $R|S^{4k-1}$; $S^{4k-1} \to \Sigma$ is of degree 1. Consider the following diagram,

$$\begin{array}{ccc} \Pi_{4k-1}(S^{2k}) & \longrightarrow & \Pi^s_{2k-1} \\ & \searrow^{A_1} & \downarrow A_2 \\ & & \Omega^{fr}_{2k-1} \end{array}$$

, where both A_1 and A_2 are the homomorphism constructed by the usual transversal arguments. We fix an framing of τ_V which is induced from that of $W \underset{\Sigma}{\cup} (-W)$. Let $\bar{R}$ be transversal regular on the base point of S^{2k} and $\bar{R}^{-1}(pt)=N$. Then $A_1(\alpha)=[\partial N$ with an induced framing$]=0$. Q. E. D.

PROPOSITION 3. 2 *If* (*C*) *without* (2) *holds for* W, $k=2n$, *and* $\xi \in \widetilde{KO}(\hat{W})$, *then* $\langle ph(\xi), [\hat{W}]\rangle \equiv 0$ (mod 1).

(PROOF) Let k be the inverse map of h. Since $\widetilde{KO}(\bigvee_{i=1}^{l} S_i^{4n}) \cong \bigoplus_{i=1}^{l} \widetilde{KO}(S_i^{4n}) \cong \boldsymbol{Z} \oplus \cdots \oplus \boldsymbol{Z}$, we write an element $k^*(i_w)^*(\xi)$ by $(b_1, b_2, \cdots, b_l)$. Let β: $\bigvee_{i=1}^{l} S_i^{4n} \to S^{4n}$ be a map represented by degree $(b_1, b_2, \cdots, b_l)$. Then $\beta^*(1)=(b_1, \cdots, b_l)$. We define α as in Lemma 3. 1 by using β. α induces a map $\hat{\alpha}$; $\hat{W} \to S^{4n} \underset{\alpha}{\cup} e^{8n}$.

Consider the following two exact sequences,

$$\begin{array}{ccccccc}
0 \longrightarrow & \widetilde{KO}(S^{8n}) & \xrightarrow{p_*} & \widetilde{KO}(S^{4n} \underset{\alpha}{\cup} e^{8n}) & \xrightarrow{(i_{S^{4n}})^*} & \widetilde{KO}(S^{4n}) & \longrightarrow 0 \\
& \| & & \downarrow (\hat{\alpha})^* & (i_w)^* & \downarrow h^* \circ \beta^* & \\
0 \longrightarrow & \widetilde{KO}(S^{8n}) & \xrightarrow{p_*} & \widetilde{KO}(\hat{W}) & \longrightarrow & \widetilde{KO}(\bigvee_{i=1}^{l} S^{4n}) & \longrightarrow 0 .
\end{array}$$

Let x be an element of $\widetilde{KO}(S^{4n} \underset{\alpha}{\cup} e^{8n})$ so that $(i_{S^{4n}})^* x=1$. Then $(i_w)^*(\xi-(\hat{\alpha})^* x) =0$, so there exsits an element $y \in \widetilde{KO}(S^{8n})$ so that $\xi-(\hat{\alpha})^* y=p^* y$. $\langle ph(\xi), [\hat{W}]\rangle = \langle ph\ (\hat{\alpha}^* x), [\hat{W}]\rangle + \langle ph(p^* y), [\hat{W}]\rangle = \langle phx, (\hat{\alpha})_*[\hat{W}]\rangle + \langle phy, [S^{8n}]\rangle \equiv a_n$ (e invariant of α) (mod 1). It follows from Lemma 3. 1 that e invariant of $\alpha=0(1)$. Q. E. D.

LEMMA 3. 3 *If W is a simply connected F-parallelizable manifold whose boundary is a homotopy sphere Σ, and* $\dim W \geqq 6$, *then* r; $[\hat{W}, F/0] \to [W, F/0]$ *is onto.*

(PROOF) Consider the exact sequence; $[\hat{W}, F/0] \to [W, F/0] \to [\Sigma, F/0]$. The image d ($[W, F/0]$) is contained in bF_m, that is, in bP_m. The commutativity of the following diagram shows that the map $[W, F/0] \to [\Sigma, F/0]$ is a zero map

$$\begin{array}{ccc}
hs(W) & \longrightarrow & [W, F/0] \\
\downarrow & & \downarrow \\
bP_m \longrightarrow hs(\Sigma)=\theta_{m-1} & \longrightarrow & [\Sigma, F/0] .
\end{array}$$
Q. E. D.

For the rest of this section we will prove the following

THEOREM 3. 4 *Let* (*C*) *hold for W and* $l \neq 0$. *If* $k=2n$, *then the image*

d ($[\hat{W}, F/0]$) consists of all $\left\{2^{2n-1}(2^{2n-1}-1)\cdot a_n \text{ numerator}\left(\frac{B_n}{4n}\right)\right\}^2\cdot\ \varphi(b_1, \cdots, b_l)$ (mod $\sigma_k/8$), *where φ is a quadratic from associated with the pairing $H_{2k}(\hat{W})\otimes H_{2k}\hat{W}(\ )\to H_{4k}(\hat{W})$ and b_i are integers. If k; odd, then the image d* ($[\hat{W}, F/0]$)$=0$.

(PROOF) According to [1, Theorem 3. 7], the image of i_*; $\pi_{4n}(F/0)\to\pi_{4n}(BSO)\cong \boldsymbol{Z}$ is generated by $m(2n)$. In the diagram

$$\begin{array}{ccccc}
[S^{8n}, F/0] & \xrightarrow{P_*} & [\hat{W}, F/0] & \xrightarrow{r_*} & [W, F/0] \cong \bigoplus_1^l \pi_{4n}(F/0) \\
\downarrow i_* & & \downarrow i_* & & \downarrow i_* \\
[S^{8n}, BSO] & \xrightarrow{P_*} & [\hat{W}, BSO] & \xrightarrow{r_*} & [W, BSO] \cong \bigoplus_1^l \pi_{4n}(BSO)
\end{array}$$

, we can choose elements v_i $(i=1,2,\cdots,l)$ so that $i_*r_*(v_i)=m(2n)$ since r_* is onto [Lemma 3. 3]. For any element x of $[\hat{W}, F/0]$ there exists elements y of $\pi_{8n}(F/0)$, z of $[\hat{W}, F/0]$ and integers b_l $(i=1,2,\cdots,l)$ so that $i_*x=P_*y+i_*(\sum b_iv_i)+i_*z$ and $i_*r_*z=0$. Then $\mathscr{S}(x)=1/8\langle L(\hat{W})\ (1-L(i_*x), [\hat{W}\rangle=-1/8\langle L_{2n}(i_*x), [\hat{W}]\rangle=-1/8\langle\sum_{i=1}^{l} L_{2n}(b_iv_i)+L_{2n}(P_*y)+L_{2n}(i_*z)+\sum_{i<j} b_ib_jL_n(v_i)L_n(v_j),$ $[\hat{W}]\rangle\equiv-1/8\ \langle\sum_{i=1}^{l}\frac{1}{2}S_n^2b_i^2p_n^2(v_i)+\sum_{i<j} b_ib_jS_n^2p_n(v_i)p_n(v_j), [\hat{W}]\rangle=-\left\langle\left\{\frac{1}{4}S_n\cdot m(2n)\right.\right.$ $\left.\left.\cdot a_n\cdot(2n-1)\,!\left(\sum_{i=1}^{l} b_iu_i\right)\right\}^2, [\hat{W}]\right\rangle=-\left\{\frac{1}{4}S_n\cdot m(2n)\cdot a_n(2n-1)\,!\right\}^2\ \varphi(b_1, \cdots, b_l)=-$ $\left\{2^{2n-1}(2^{2n-1}-1)\,a_n\cdot\text{numevator}\left(\frac{B_n}{4n}\right)\right\}^2\varphi(b_1, \cdots, b_l)$. Here we used the following facts and Lemma 3. 5.

(1) $L_{2n}=s_{2n}p_{2n}+1/2(s_n^2-s_{2n})p_n^2+\text{other terms}$, where $s_n=\dfrac{2^{2n}(2^{2n-1}-1)}{(2n)\,!}B_n$ $(n\geqq 1)$. [4]

(2) Let v_i and u_i be the generators of $\widetilde{K}0(S^{4n})$ and $H^{4n}(S^{4n})$ respectively. Then $p_n(v_i)=a_n\cdot(2n-1)\,!\,u_i$. If k is odd, then i_*; $\pi_{2k}(F/0)\to\pi_{2k}(BSO)$ is zero. And the similar argument show the assertion. Q. E. D.

LEMMA 3. 5 *For an element ξ of $[\hat{W}, F/0]$, $1/8\langle L_{2n}(i_*\xi), [\hat{W}]\rangle\equiv 1/8$ $\langle 1/2\ s_n^2p_n^2(\xi), [W]\rangle$ (mod $\sigma_{2n}/8$), where W is in Proposition* 3. 2.

(PROOF) It follows from (1) that we need to prove $1/8\ \langle s_{2n}(p_{2n}(\xi)-1/2p_n^2(\xi)), [\hat{W}]\rangle\equiv 0$ (mod $\sigma_{2n}/8$). By [7], we may write $i_*\xi=\sum_i k_i^{ei}(\psi_R^{k_i}-1)(\xi_i)$ for some integers k_i, e_i and $\xi_i\in\widetilde{K}0(\hat{W})$. $1/8\ \langle s_{2n}(p_{2n}(\xi)-1/2p_n^2(\xi)), [\hat{W}]\rangle=$ $\langle 2^{4n-2}(2^{4n-1}-1)\dfrac{B_{2n}}{8n}\cdot ph(\xi), [\hat{W}]\rangle=2^{4n-2}(2^{4n-1}-1)\dfrac{B_{2n}}{8n}\sum_i k_i^{ei}(k_i^{4n}-1)\langle ph(\xi_i), [\hat{W}]\rangle$.

It follows from [1, p. 139] and Proposition 3. 2. Q. E. D.

PROOF OF THEOREM 0. 2 The first part of Theorem 0. 2 follows from Theorem 3. 4. since there exists $(a_1, \cdots, a_l)$ so that φ $(a_1, \cdots, a_l)=2$. It is sufficient to prove the latter part to show that any homotopy sphere of bF_8^0 is represented in the image d. Let a homotopy sphere Σ bound an F-parallelizable manifold with $I(W)=0$. Then we may consider W 3-connected, In fact, we can make W 3-connected by framed surgery, since a spherical trivialization over 3-skelton reduces to a framing. Now we have a h-smoothing h; $(W', \partial W') \to (W\ \partial W)$ so that W' is parallelizable, 3-connected and $I(W)=0$. So $\sum = \partial W \# (-\partial W')$ since $I(W')=0$. Q. E. D.

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