A relation between the fibers of Milnor fiberings associated to polynomials $f(z)=z_0^{a_0}+\cdots+z_n^{a_n}$

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§ 0. Introduction

All manifolds will be oriented and differentiable of class C^∞. Let $a=(a_0, a_1, a_2, \cdots, a_n)$ be a set of integers, $a_i>1$ and consider a polynomial $f(z_0, z_1, \cdots, z_n)=z_0^{a_0}+z_1^{a_1}+\cdots+z_n^{a_n}$, $z_i\in \boldsymbol{C}$ $(i=0, 1, 2, \cdots, n)$. If K_a is the intersection of $f^{-1}(0)$ and the unit sphere S^{2n+1} in $\boldsymbol{C}^{n+1}$, then we have an associated Milnor fibering ϕ: $S^{2n+1}-K_a\to S^1$. It is well known that a fiber F_a of ϕ is a $(n-1)$-connected 2n-manifold and the closure $\bar{F}_a$ of F_a in S^{2n+1}, a manifold with boundary K_a (see [5]). The purpose of this paper is to give a relation between F_a and F_b, where b is another set of integers, $b=(b_0, b_1, \cdots, b_n)$, $a_i\leqq b_i$ $(i=0, 1, 2, \cdots, n)$.

By Pham's results [6] we can give a canonical basis $x_1, x_2, \cdots, x_{\mu_a}$ to H_n $(\bar{F}_a; \boldsymbol{Z})$ and also a basis $y_1, y_2, \cdots, y_{\mu_b}$ to H_n $(\bar{F}_b: \boldsymbol{Z})$, where $\mu_a=(a_0-1)(a_1-1)\cdots(a_n-1)$ and $\mu_b=(b_0-1)(b_1-1)\cdots(b_n-1)$. (see Theorem 1.6). Then we have

THEOREM A. *Let* F_a, F_b, $\{x_i\}_{i=1,2,\cdots,\mu_a}$ *and* $\{y_j\}_{j=1,2,\cdots,\mu_b}$ *be as above. If* $n\geqq 3$, *then there exists a smooth embedding* e: $\bar{F}_a\to\bar{F}_b$ *so that each* x_i *is mapped onto* y_i *by* $(e)_*$ *and that* $\bar{F}_b-e(F_a)$ *is a manifold with boundary* $(-K_a)\cup K_b$ $(i=0, 1, \cdots, \mu_a)$.

This is proved by considering the intersection pairing of $H_n(\bar{F}_a)$, $H_n(\bar{F}_b)$ and maps α: $\pi_n(\bar{F}_a)$ (and $\pi_n(\bar{F}_b))\to\pi_{n-1}(SO_n)$ which are defined in [7].

Let $a=(2, 2, \cdots, 2, s)$. If s odd, then we have well known results that K_a is a homotopy sphere which is determined in [1]. But if s is even, then K_a is not a homotopy sphere. As an application of Theorem A we have the following

THEOREM B. i) *If n is even, then* K_a *is diffeomorphic to* $D^n\times S^{n-1}\cup_{f_a} S^{n-1}\times D^n$, *where* f_a *is described as follows. Let* ∂: $\pi_n(S^n)\to\pi_{n-1}(SO_n)$ *be a boundary homomorphism associated to the fibration* $SO_n\to SO_{n+1}\to S^n$, $\iota_n=id_{S^n}$ *and* $\varphi_a=\partial([s/2]\,\iota_n)$. *Then a diffeomorphism* f_a: $S^{n-1}\times S^{n-1}\to S^{n-1}\times S^{n-1}$ *is given by* $f_a(x, y)=(x, \varphi_a(x), y)$.

ii) *If n is odd, then* K_a *is diffeomorphic to*

$$\begin{array}{lll} & S^{n-1}\times S^n & \textit{when} \quad s\equiv 0(8), \\ & \partial D(\tau_{S^n}) & \qquad\quad s\equiv 2(8), \\ & (S^{n-1}\times S^n)\#\Sigma & \qquad\quad s\equiv 4(8), \\ \textit{and} & \partial D(\tau_{S^n})\#\Sigma & \qquad\quad s\equiv 6(8), \end{array}$$

where $\partial D(\tau_{S^n})$ *is the boundary of the tangent disk bundle of* S^n *and* Σ, *a generator of* bP_{2n}.

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§ 1. Intersection pairings and maps α

Let $a=(a_0, a_1, a_2, \cdots, a_n)$, $a_i>1$. We write F, $\bar{F}$ and K in place of F_a, $\bar{F}_a$ and K_a for convenience. We shall study the diffeomorphism class of the manifold $\bar{F}$. For this we notice the following results by C. T. C. Wall [7].

(1. 1) Let N be a compact differentiable $(n-1)$-connected $2n$-manifold, $\partial N\neq\phi$ and $n\geqq 3$. Then the diffeomorphism class of N is determined by the intersection pairing: $H_n(N)\otimes H_n(N)\to \boldsymbol{Z}$ and a map α: $\pi_n(N)\to\pi_{n-1}(SO_n)$. α is defined by corresponding an embedded n-sphere to its normal bundle in N.

Now we recall results by F. Pham and J. Milnor.

(1. 2) Let Ω_a be the set of all a-th roots of unity. Then F is homotopy equivalent to $\Omega_{a_0}*\Omega_{a_1}*\cdots*\Omega_{a_n}$, where $*$ denotes join [6]. It follows that $H_n(F)\cong\tilde{H}_0(\Omega_{a_0})\otimes\tilde{H}_0(\Omega_{a_1})\otimes\cdots\otimes\tilde{H}_0(\Omega_{a_{n+1}})$ [5, p. 74].

We identify $H_n(F)$ with $\tilde{H}_0(\Omega_{a_0})\otimes\tilde{H}_0(\Omega_{a_1})\otimes\cdots\otimes\tilde{H}_0(\Omega_{a_n})$ by this isomorphism. Let $w_j=e^{(2\pi/a_j)i}$ $(j=0, 1, 2, \cdots, n)$, $\sqrt{-1}=i$. Then (w_j^i-1) $(i=1, 2, \cdots, a_j-1)$ is a basis of $\tilde{H}_0(\Omega_{a_j})$. We shall write $w_j^i-1<w_j^k-1$ if $0<i<k\leqq a_j$. We order the basis $\{(w_0^{i_0}-1)\otimes\cdots\otimes(w_n^{i_n}-1)\}$ of $H_n(F)$ by the lexicographic order. Let h_t be a trivialization of $\phi^{-1}(C)$ defined by $h_t(z_0, z_1, z_2, \cdots, z_n)=(e^{it/a_0}z_0, e^{it/a_1}z_1, \cdots, e^{it/a}z_n)$, where $C=\{e^{it}\in S^1|t\in[-\pi/2, \pi/2]\}$. [5, p. 73]

PROPOSITION 1. 3. *Consider* $H_n(F)$ *with the above ordered basis. Then the matrix of the linking number* $\{L((h_{-t})_* x_i, x_j)\}$ *is given by* $A_0\otimes A_1\otimes\cdots\otimes A_n$, *where* L *denotes the linking number,* $A_i=\begin{pmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & 1 \\ 0 & & \ddots & \vdots \\ & & & 1 \end{pmatrix}$, *the rank of which is* a_i-1 $(i=0, 1, 2, \cdots, n)$ *and* t *is a sufficiently small positive number.*

(PROOF) The proof shall rely on results of F. Pham and J. Milnor [5, §9 and 6]. The join of $(n+1)$ unit circle S^1, $S^1*S^1*\cdots*S^1$, is canonically embedded in C^{n+1}. Let g be a *p. l.* homeomorphism: $S^1 * S^1 * \cdots * S^1 \to S^{2n+1}$ which is given by $g((t_0z_0, t_1z_1, \cdots, t_nz_n)) = (t_0z_0/\sqrt{t_0^2+\cdots+t_n^2}, \cdots, t_nz_n/\sqrt{t_0^2+\cdots+t_n^2})$, where $t_i \in \boldsymbol{R}$, $z_i \in S^1$ and $t_0+t_1+\cdots+t_n=1$. Now consider the fibration $\phi\circ g$: $S^1 * \cdots * S^1 - g^{-1}(K) \to S^1$ in place of ϕ: $S^{2n+1}-K \to S^1$. Let ψ: $C^{n+1}-f^{-1}(0)\to S^1$ be $\psi(z)=f(z)/|f(z)|$. It is easy to see that $g(\Omega_{a_0} * \Omega_{a_1} * \cdots * \Omega_{a_n}) \subset \phi^{-1}(1)$, so $\Omega_{a_0} * \Omega_{a_1} * \cdots * \Omega_{a_n} \subset (\phi\circ g)^{-1}(1)$. At first we show that this inclusion is a homotopy equivalence. Let d: $\phi^{-1}(1)\times \boldsymbol{R} \to \psi^{-1}(1)$ be a diffeomorphism and r: $\psi^{-1}(1) \to \Omega_{a_0} * \Omega_{a_1} * \cdots * \Omega_{a_n}$, *a* deformation retract which are defined in [5, §9 and 6]. Then it follows from the construction of r that the composition map: $\Omega_{a_0} * \Omega_{a_1} * \cdots * \Omega_{a_n} \xrightarrow{g} \phi^{-1}(1)\times 0 \subset \phi^{-1}(1)\times \boldsymbol{R} \xrightarrow{d} \psi^{-1}(1) \xrightarrow{r} \Omega_{a_0} * \Omega_{a_1} * \cdots * \Omega_{a_n}$ is an identity map. It follows that $d\circ(g/\Omega_{a_0} * \cdots * \Omega_{a_n})$ is a homotopy equivalence. Therefore we may consider that $(g^{-1})_*((w_0^{i_0}-1)\otimes\cdots\otimes(w_n^{i_n}-1))$ of $H_n(g^{-1}(F))$ is represented by an embedded n-sphere $\begin{pmatrix}1\\ w_0^{i_0}\end{pmatrix} * \begin{pmatrix}1\\ w_1^{i_1}\end{pmatrix} * \cdots * \begin{pmatrix}1\\ w_n^{i_n}\end{pmatrix}$. There is a trivialization $\bar{h}_t$ of $(\phi\circ g)^{-1}(c)$ which comes from h_t; $\bar{h}_t(z_0, \cdots, z_n) = (e^{it/a_0}z_0, e^{it/a_1}z_1, \cdots, e^{it/a_n}z_n)$, where $(z_0, z_1, \cdots, z_n) \in \phi^{-1}(S^{2n+1}-K)$ and $t \in [-\pi/2, \pi/2]$. For convenience put $x=(g^{-1})_*((w_0^{i_0}-1)\otimes(w_1^{i_1}-1)\otimes\cdots\otimes(w_n^{i_n}-1))$, $y=(g^{-1})_*(w_0^{j_0}-1)\otimes(w_1^{j_1}-1)\otimes\cdots\otimes(w_n^{j_n}-1))$. Now we can calculate the linking number $(h_{-t})_*x$ and y by using these facts together with Lemma 1.5.

$$L\left((h_{-t})_* x, y\right) = L\left(\begin{pmatrix}e^{-it/a_0}\\ e^{-it/a_0}w_0^{i_0}\end{pmatrix} * \cdots * \begin{pmatrix}e^{-it/a_n}\\ e^{-it/a_n}w_n^{i_n}\end{pmatrix}, y\right)$$
$$= \prod_{l=0}^{n} L\left(\begin{pmatrix}e^{-it/a_l}\\ e^{-it/a_l}w_l^{i_l}\end{pmatrix}, \begin{pmatrix}1\\ w_l^{j_l}\end{pmatrix}\right),$$

where t is a sufficiently small positive number.

Since $$L\left(\begin{pmatrix}e^{-it}\\ e^{-it}e^{i\theta}\end{pmatrix}, \begin{pmatrix}1\\ e^{i\varphi}\end{pmatrix}\right) = \begin{cases}0, & \text{when } 0<\varphi<\theta<2\pi\\ 1, & \phantom{\text{when }} 0<\theta\leqq\varphi<2\pi,\end{cases}$$

the matrix of the linking number $\left\{L\left(\begin{pmatrix}e^{-it/a_l}\\ e^{-it/a_l}w_l^{i_l}\end{pmatrix}, \begin{pmatrix}1\\ w_l^{j_l}\end{pmatrix}\right)\right\}$ is $\begin{pmatrix}1 & 1 & \cdots & 1\\ & 1 & \cdots & 1\\ 0 & & \ddots & \vdots\\ & & & 1\end{pmatrix}$,

the rank of which is (a_l-1).

REMARK 1.4. We can calculate $L((h_t)_* x, y)$ similarly by using

$$L\left(\begin{pmatrix}e^{it/a_l}\\ e^{it/a_l}e^{i\theta}\end{pmatrix}, \begin{pmatrix}i\\ e^{i\varphi}\end{pmatrix}\right) = \begin{cases}0, & \text{when } 0<\varphi<\theta<2\pi\\ -1, & \phantom{\text{when }} 0<\theta\leqq\varphi<2\pi.\end{cases}$$

We will need this later.

LEMMA 1.5. *Let* $\bar{S}^p$, $\bar{S}^{p'} \subset S^{n+1}$ *and* $\bar{S}^q$, $\bar{S}^{q'} \subset S^{m+1}$ *be piecewise linearly embedded spheres of dimension* p, p' *and* q, q' *respectively. Let* $p+p'=n$,

$q+q'=m$, $\bar{S}^p \cap \bar{S}^{p'}=\phi$ and $\bar{S}^q \cap \bar{S}^{q'}=\phi$. Moreover we assume that $S^{n+1}-\bar{S}^p$ and $S^{m+1}-\bar{S}^q$ are homotopy equivalent to $S^{p'}$ and $S^{q'}$ respectively. Then

$$L(\bar{S}^p, \bar{S}^{p'})\cdot L(\bar{S}^q, \bar{S}^{q'}) = L(\bar{S}^p * \bar{S}^q, \bar{S}^{p'} * \bar{S}^{q'})$$

(PROOF) We can think of the linking number, for example, $L(\bar{S}^p, \bar{S}^{p'})$ as the degree of the map c_p: $\bar{S}^p \subset S^{n+1}-\bar{S}^{p'} \xrightarrow{h_p} S^p$, where h_p is an oriented homotopy equivalence so that the orientations of $\bar{S}^p$ and $\bar{S}^{p'}$ are compatible with that of S^{n+1}. Similarly we have a map c_q for $\bar{S}^q$, $\bar{S}^{q'} \subset S^{m+1}$. Consider a map $c_p * c_q$: $\bar{S}^p * \bar{S}^q \to (S^{n+1}-\bar{S}^{p'}) * (S^{m+1}-\bar{S}^{q'}) \subset S^{n+m+3}-\bar{S}^{p'} * \bar{S}^{q'} \to S^p * S^q$. Then $\deg(c_p * c_q) = \deg(c_p) \cdot \deg(c_q)$. This completes the proof. (Q. E. D.)

Now we have the following

THEOREM 1.6. *We consider $H_n(F)$ with the above ordered basis. Then* i) *the matrix of the intersection pairing is given by $A_0 \otimes A_1 \otimes \cdots \otimes A_n - (-1)^{n+1}({}^tA_0) \otimes ({}^tA_0) \otimes \cdots \otimes ({}^tA_n)$.* ii) *$\alpha(x_i)$ $(i=0, 1, 2, \cdots, n)$ is an element of $\pi_{n-1}(SO_n)$ which corresponds to a tangent bundle of S^n.*

(PROOF) The first part follows from the fact that $\langle x, y\rangle = L((h_{-t})_* x, y) - L((h_t)_* x, y)$ [3, 2.5]. Now we prove the second part. If n is even, then it is easy to see $\langle x_i, x_i\rangle = 2$ $(i=0, 1, 2, \cdots, n)$. Therefore the Euler number of $\alpha(x_i)$ is 2. By [4, p. 51] $\alpha(x_i)$ is as stated above. If n is odd, then we have $L((h_{-t})_* x_i, x_i)=1$. It follows from [3, 3.3] that $\alpha(x_i)$ is a unique non-trivial element of $\mathrm{I}_m \partial$, where ∂: $\pi_n(S^n) \to \pi_{n-1}(SO_n)$. (Q. E. D).

The intersection pairing of $H_n(F)$ is given and stated in other form in [2, p. 88].

§ 2. Embeddings

In this section we give a proof of Theorem A. Let $a=(a_0, a_1, a_2, \cdots, a_n)$, $b=(b_0, b_1, b_2, \cdots, b_n)$ and $a_i \leqq b_i$ $(i=0, 1, 2, \cdots, n)$. Let F_a, F_b be fibers of Milnor fiberings for the polynomials $f_a(z)=z_0^{a_0}+z_1^{a_1}+\cdots+z_n^{a_n}$ and $f_b(z)=z_0^{b_0}+z_1^{b_1}+\cdots+z_n^{b_n}$.

PROOF OF THEOREM A. We shall prove the special case of Theorem A. Let $b_0=a_0+1$, and $b_i=a_i$, $1\leqq i\leqq n$. The proof of other cases are much the same. This inductively prove the general case. By C. T. C. Wall [7] and Theorem 1.6, $\bar{F}_b$ is diffeomorphic to a unit $2n$-disk with attached μ_b n-handles $D^{2n} \cup (\bigcup_{f_i} D_i^n \times D^n)$, where the attaching map f_i is determined by an element of $\pi_{n-1}(SO_n)$ associated to the tangent bundle of S^n and matrix (c_{ij}) which is that in Theorem 1.6. c_{ij} denotes the linking number of $f_i(\partial D_i^n \times 0)$ and $f_j(\partial D_j^n \times 0)$. Now we consider a manifold $\bar{F}_b$ with the last n-handles

$D_{\mu_{a+1}} \times D^n, \cdots, D^n_{\mu_b} \times D^n$ removed. We denote this by N. Then the intersection pairing and the map α of N coincide with them of $H_n(\bar{F}_a)$. Therefore if $n \geqq 3$, $\bar{F}_a$ is diffeomorphic to N by (1. 1). It is trivial from the construction that x_i is mapped onto y_i by this embedding. (Q. E. D.)

Let h be a characteristic diffeomorphism of a fiber F of a Milnor fibering ϕ. Let $h_*: H_n(\bar{F}) \to H_n(\bar{F})$ be represented by a matrix (h_{ij}). Then we have the following proposition. We use the notations in Theorem 1. 6.

PROPOSITION 2. 1. ${}^t(h_{ij}) = (-1)^{n+1}({}^tA_0 \otimes {}^tA_1 \otimes \cdots \otimes {}^tA_n) \cdot (A_0 \otimes \cdots \otimes A_n)^{-1}$.

(PROOF) Notice $h_{2\pi} = h$. Then

$$\langle h_* x_i, x_j \rangle = L\big((h_{-t})_* h_* x_i, x_j\big) - L\big((h_t)_* h_* x_i, x_j\big)$$
$$= L\big((h_{2\pi - t})_* x_i, x_j\big) - L\big((h_t)_* \textstyle\sum h_{ki} x_k, x_j\big).$$

So, we have

$$\begin{aligned} {}^t(h_{ij})(\langle x_i, x_j \rangle) &= (-1)^{n+1}({}^tA_0 \otimes \cdots \otimes {}^tA_n) \\ &\quad -(-1)^{n+1}\,{}^t(h_{ij})({}^tA_0 \otimes \cdots \otimes {}^tA_n), \end{aligned}$$
$${}^t(h_{ij})\Big\{(A_0 \otimes \cdots \otimes A_n) - (-1)^{n+1}({}^tA_0 \otimes \cdots \otimes {}^tA_n)\Big\}$$
$$= (-1)^{n+1}({}^tA_0 \otimes \cdots \otimes {}^tA_n) - (-1)^{n+1}\,{}^t(h_{ij})({}^tA_0 \otimes \cdots \otimes {}^tA_n).$$

Hence, ${}^t(h_{ij}) = (-1)^{n+1}({}^tA_0 \otimes \cdots \otimes {}^tA_n) \cdot (A_0 \otimes \cdots \otimes A_n)^{-1}$.

REMARK 2. 2. Let h_a, h_b be the characteristic diffeomorphism of F_a, F_b which are associated to the polynomials f_a, f_b respectively. Although there is an embedding $e: \bar{F}_a \to \bar{F}_b$, $h_b \circ e$ is not homotopic to $e \circ h_a$. This follows from the consideration of $(h_a)_*$ and $(h_b)_*$ by the above proposition.

§ 3. Examples

In this section we will prove Theorem B. Let s be always even.

PROPOSITION 3. 1. *If n is even, then $H_{n-1}(K_a) \cong \boldsymbol{Z}_s$ and $H_n(K_a) \cong 0$. If n is odd, then $H_{n-1}(K_a) \cong H_n(K_a) \cong \boldsymbol{Z}$.*

(PROOF) We have the exact sequence,

$$0 \longrightarrow H_n(K_a) \longrightarrow H_n(\bar{F}_a) \xrightarrow{\Psi} H_n(\bar{F}_a, K_a) \xrightarrow{\partial} H_{n-1}(K_a) \longrightarrow 0.$$

Theorem 1. 6 shows that if n is odd,then $\Psi = \begin{pmatrix} 0 & 1 & 1 \cdots & 1 \\ -1 & 0 & 1 \cdots & 1 \\ -1 & -1 & 0 \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \;\; 1 \\ -1 & -1 \cdots & -1 & 0 \end{pmatrix}$

and that if n is even, then $\Psi=\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 2 & 1 \\ 1 & 1 & \cdots & 1 & 2 \end{pmatrix}$. Since s is even, we can transform them into $\begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 0 & 1 \\ & & & s \end{pmatrix}$ by integer elementary matrices respectively.

PROPOSITION 3.2. *Let n be odd. Then a generator of $H_n(K_a)$ is $-x_1+x_2-x_3+x_4+\cdots+(-1)^{s-1}x_{s-1}$. If we represent it by an embedded sphere, then its normal bundle is trivial when $s\equiv 0\,(4)$ and a tangent bundle of S^n when $s\equiv 2\,(4)$.*

(PROOF) The proof is as follows from [7, Lemma 2],

$$\begin{aligned}\alpha\left(-x_1+x_2+\cdots+(-1)^{s-1}x_{s-s}\right) &\equiv \sum_{i=1}^{s-1}\alpha(x_i)+\sum_{i<j}\langle x_i, x_j\rangle\,(\partial\iota_n) \\ &\equiv 1+(1/2)(s-2)(s-3) \\ &\equiv \begin{cases} 0, & s\equiv 0\,(4) \\ 1 & s\equiv 2\,(4). \end{cases} \pmod 2.\end{aligned}$$

PROOF OF THEOREM B. At first we shall give a proof for n, even. Let $s\geqq 4$. We consider a manifold $K_{a'}$, $\bar{F}_{a'}$ associated to a set of integers $a'=(2,2,\cdots,2,s-1)$. Then it is well known that $K_{a'}$ is a standard sphere. Consider a manifold $\bar{F}_a-e(F_{a'})\underset{K_{a'}=\partial D}{\cup}D^{2n}$. Then this manifold is a unit disk with attached one n-handle, the boundary of which is K_a by Theorem A. If $N_a=D^{2n}\underset{f_a}{\cup}D^n\times D^n$, where $f_a(x,y)=(x,\varphi_a(x)\,y)$ and $(x,y)\in D^n\times D^n$, then $H_n(\partial N_a)\cong 0$ and $H_{n-1}(\partial N_a)\cong \boldsymbol{Z}_s$. By using this fact and (1.1), we know that N_a is diffeomorphic to $\bar{F}_a-e(F_{a'})\underset{K_{a'}=\partial D}{\cup}D^{2n}$. Especially K_a is diffeomorphic to ∂N_a. When $s=2$, it is easy to see that N_a is diffeomorphic to $\bar{F}_a$.

Next we shall prove ii). As above we consider a manifold $\bar{F}_a-e(F_{a'})$. By [1] $K_{a'}$ is diffeomorphic to a standard sphere S^{2n-1} when $s-1\equiv\pm 1\,(8)$ and to Σ when $s-1\equiv\pm 3\,(8)$. Now we consider the cases $s\equiv 0$ or $2\,(8)$. Since $K_{a'}$ is a standard sphere then, we can consider $\bar{F}_a-e(F_{a'})\underset{K_{a'}=\partial D}{\cup}D^{2n}$. Since the image of $\partial:\pi_n(S^n)\rightarrow\pi_{n-1}(SO_n)$ consists of only 2-elements, K_a must be diffeomorphic to either $S^{n-1}\times S^n$ or $\partial D(\tau_{S^n})$. If K_a is diffeomorphic to

$S^{n-1}\times S^n$, then $\alpha(\Sigma(-1)^i x_i)=0$. And if K_a is diffeomorphic to $\partial D(\tau_{S^n})$, then $\alpha(\Sigma(-1)^i x_i)=1$. Using these facts together with Proposition 3.2 and (1.1), we know that K_a is diffeomorphic to $S^{n-1}\times S^n$ when $s\equiv 0\,(8)$ and to $\partial D(\tau_{S^n})$ when $s\equiv 2\,(8)$. When $s\equiv 4, 6\,(8)$, the arguments are much the same. But since $K_{a'}$ is Σ, we must extinguish Σ before constructing $\overline{F}_a - e(F_{a'})\cup D^{2n}$. It is as usual to proceed as follows. At first Embed $D^{2n-1}\times I$ in $\overline{F}_a - e(F_{a'})$ and D_2^{2n-1} in Σ so that $D_1^{2n-1}\times 0\subset K_{a'}$ and $D_1\times 1\subset K_a$. Then we remove $\mathring{D}_1^{2n-1}\times I$ and $\mathring{D}_2^{2n-2}\times I$ from $\overline{F}_a - e(F_{a'})$ and $\Sigma\times I$ respectively. Then we attach these two manifolds with each open tube removed, on $\partial D_1\times I$ and $\partial D_2\times I$. It is easy to see that the boundary of the manifold constructed as above is $(-S^{2n-1})\cup K_a\#(-\Sigma)$. $K_a\#(-\Sigma)$ is determined as above. Therefore we have determined K_a.

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