On some 3-dimensional Riemannian manifolds

By Kouei SEKIGAWA

1. Introduction. The Riemannian curvature tensor R of a locally symmetric Riemannian manifold (M, g) satisfies

$$(*) \qquad R(X, Y)\cdot R=0 \qquad \text{for all tangent vectors } X \text{ and } Y,$$

where $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M. Conversely, does this algebraic condition on the curvature tensor field R imply that $\nabla R=0$? K. Nomizu conjectured that the answer is positive in the case where (M, g) is complete irreducible and dim $M\geqq 3$. But, recently, H. Takagi [9] gave an example of 3-dimensional complete, irreducible real analytic Riemannian manifold (M, g) satisfying $(*)$ and $\nabla R\neq 0$ as a hypersurface in a 4-dimensional Euclidean space E^4. Furthermore, the present author proved that, in an $(m+1)$-dimensional Euclidean space $E^{m+1}(m\geqq 4)$, there exist some complete, irreducible real analytic hypersurfaces which satisfy $(*)$ and $\nabla R\neq 0$ ([6] in references). Let R_1 be the Ricci tensor of (M, g). Then, $(*)$ implies in particular

$$(**) \qquad R(X, Y)\cdot R_1=0 \qquad \text{for all tangent vectors } X \text{ and } Y.$$

In the present paper, with respect to this problem, we shall give an affirmative answer in the case where (M, g) is a certain 3-dimensional compact, irreducible real analytic Riemannian manifold, that is

THEOREM. *Let (M, g) be a 3-dimensional compact, irreducible real analytic Riemannian manifold satisfying the condition $(*)$ (or equivalently $(**)$). If the Ricci form of (M, g) is non-zero, positive semi-definite on M, then (M, g) is a space of constant curvature.*

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2. Lemmas. *Let (M, g) be a 3-dimensional real analytic Riemannian manifold. Let R^1 be a field of symmetric endomorphism satisfying $R_1(X, Y)=g(R^1X, Y)$. It is known that the curvature tensor of (M, g) is given by*

$$(2.1) \qquad R(X, Y)=R^1X\wedge Y+X\wedge R^1Y-\frac{\text{trace } R^1}{2}X\wedge Y,$$

for all tangent vectors X and Y.

At each point of M, we may choose an orthonormal basis $\{e_i\}$ such that $R^1e_i=K_ie_i$, $1\leqq i, j, k, h, \cdots\leqq 3$. Then, from (*) (or equivalently (**)) and (2. 1), we see that essentially the following cases are possible;

(I) $K_1=K_2=K_3=K$, $K\neq 0$,

(II) $K_1=K_2=K$, $K_3=0$, $K\neq 0$,

(III) $K_1=K_2=K_3=0$,

For (I), by [4], we have

PROPOSITION 2. 1. *If the rank of the Ricci form R_1 is* 3 *at least at one point of M, then (M, g) is a space of constant curvature.*

With respect to our problem, without loss of generality, we may assume that M is orientable (if necessarily, consider the orientable double covering space of M). Next, we shall assume that the rank of R_1 is at most 2 on M. Then, (II) or (III) is valid on M. If the rank of R_1 is 2 at some point of M, then the rank of R_1 is also 2n ear the point. Thus, let $W=\{x\in M;$ the rank of R_1 is 2 at $x\}$, which is an open set of M. For each point $x_0\in W$, let W_0 be the connected component of x_0 in W. Then, non-zero eigenvalue of R^1, say K, is a real analytic function on W_0 and we can take two real analytic distributions T_1 and T_0 corresponding to K and 0, respectively on W_0. Thus, for each point $x\in W_0$, we may choose a real analytic orthonormal frame field $\{E_i\}$ near x in such a way that $\{E_a\}$ and $\{E_3\}$ are bases for T_1 and T_0, respectively. Here, $a, b, c, \cdots=1, 2$. From (2. 1) and (II), we have

LEMMA 2. 2. *With respect to the above basis $\{E_i\}$,*

(2. 2) $$R(E_1, E_2)=KE_1\wedge E_2 \text{ and otherwise being zero.}$$

In general, for a local real analytic orthonormal frame field $\{E_i\}$ on an open set U in a real analytic Riemannian manifold (M, g), we may put

(2. 3) $$\nabla_{E_i}E_j=\sum_{k=1}^{m}B_{ijk}E_k,$$

where $m=\dim M$ and B_{ijk} $(i, j, k=1, 2, \cdots, m)$ are certain real analytic functions on U satisfying $B_{ijk}=-B_{ikj}$.

From (2. 2) and (2. 3), by considering the second Bianchi identity, we have

(2. 4) $$B_{33a}=0,$$

(2. 5) $$E_3K+K(B_{131}+B_{232})=0.$$

From (2. 4), we see that each trajectory of E_3 is a geodesic. For each point $x\in W_0$, let L_x^3 be the geodesic whose initial point is x and initial direction is $(E_3)_x$. And let s denote its arc-length parameter. Using the same symbol for convenience, we shall assume that L_x^3 denotes also the set of the points on L_x^3 and $x(s)$ denotes the point on L_x^3 corresponding to the value s of the parameter. For each point $x\in W_0$, we may choose a real analytic orthonormal frame field $\{E_i\}$ on a neighborhood $U_x(\subset W_0)$ of x in such a way that

(i) $\{E_a\}$ and $\{E_3\}$ are bases for T_1 and T_0, respectively,

(ii) $\nabla_{E_3}E_i=0$, $\quad i=1,2,3$.

From (2.3) and (ii), we have

$$B_{3ij}=0 \qquad \text{on } U_x. \tag{2.6}$$

From (2.2), (2.3) and (2.6), we have

$$\begin{aligned}R(E_a, E_3)E_3 &= \nabla_{E_a}\nabla_{E_3}E_3-\nabla_{E_3}\nabla_{E_a}E_3-\nabla_{[E_a,E_3]}E_3\\ &= -\sum_{i=1}^{3}(E_3B_{a3i}+\sum_{k=1}^{3}B_{a3k}B_{k3i})E_i\\ &= -\sum_{j=1}^{3}(E_3B_{a3i}+\sum_{b=1}^{2}B_{a3b}B_{b3i})E_i=0.\end{aligned}$$

Thus, from the above equation and (2.5), we have

$$\begin{aligned}&E_3B_{131}+(B_{131})^2+B_{132}B_{231}=0,\\ &E_3B_{232}+(B_{232})^2+B_{231}B_{132}=0,\end{aligned} \tag{2.7}$$

$$\begin{aligned}&B_{132}=C_1K, \qquad B_{231}=C_2K,\\ &B_{132}-B_{232}=DK,\end{aligned} \tag{2.8}$$

where C_1, C_2 and D are certain real analytic functions on U_x satisfying $E_3C_1=E_3C_2=E_3D=0$.

From (2.5) and (2.8), we have

$$\begin{aligned}B_{131}&=\frac{1}{2}(DK-E_3K/K),\\ B_{232}&=-\frac{1}{2}(DK+E_3K/K).\end{aligned} \tag{2.9}$$

Thus, from (2.5), (2.7), (2.8) and (2.9), putting $E_3=d/ds$ or $-d/ds$ along L_x^3, we have

if $K>0$, then

$$\frac{d^2}{ds^2}(1/\sqrt{K})=-H(\sqrt{K})^3, \tag{2.10}$$

if $K<0$, then

$$\frac{d^2}{ds^2}(1/\sqrt{-K}) = -H(\sqrt{-K})^3, \tag{2.11}$$

where $H=D^2/4+C_1C_2$.

Solving (2.10) ((2.11), resp.), we have

$$1/\sqrt{K} = \sqrt{(\alpha s-\beta)^2 - H/\alpha^2} \tag{2.12}$$
$$(1/\sqrt{-K} = \sqrt{(\alpha s-\beta)^2 - H/\alpha^2}, \quad \text{resp.}),$$

where α and β are certain real numbers.

Now, for each point $x\in W_0$, let $\{E_i\}$ be a real analytic orthonormal frame field on a neighborhood U_x satisfying (i) and (ii). Then, $\{U_x\}_{x\in W_0}$ is an open covering of W_0.

Since M is orientable, if $U_x\cap U_{\bar{x}}\neq\emptyset$, $\{E_i\}$ and $\{\bar{E}_i\}$ are defined on U_x and $U_{\bar{x}}$, respectively, then we may put

$$\begin{aligned} \bar{E}_1 &= (\cos\theta)E_1 + (-\sin\theta)E_2, \\ \bar{E}_2 &= (\sin\theta)E_1 + (\cos\theta)E_2, \\ \bar{E}_3 &= E_3, \qquad \text{on} \quad U_x\cap U_{\bar{x}}, \end{aligned} \tag{2.13}$$

or

$$\begin{aligned} \bar{E}_1 &= (\cos\theta)E_1 + (\sin\theta)E_2, \\ \bar{E}_2 &= (\sin\theta)E_1 + (-\cos\theta)E_2, \\ \bar{E}_3 &= -E_3, \qquad \text{on} \quad U_x\cap U_{\bar{x}}, \end{aligned} \tag{2.14}$$

where $\cos\theta$ and $\sin\theta$ are certain real analytic functions on $U_x\cap U_{\bar{x}}$ satisfying $E_3\cos\theta = E_3\sin\theta = 0$.

Let $C_1(E)$, $C_2(E)$, $D(E)$ and $H(E)$ denote the ones defined as in (2.8) with respect to $\{E_i\}$ on $U_x(\subset W_0)$. Then, from (2.13) and (2.14), by direct computation, we have for (2.13)

$$\begin{aligned} C_1(\bar{E}) &= C_1(E)\cos^2\theta - C_2(E)\sin^2\theta + D(E)/2)\sin 2\theta, \\ C_2(\bar{E}) &= C_2(E)\cos^2\theta - C_1(E)\sin^2\theta + (D(E)/2)\sin 2\theta, \\ D(\bar{E}) &= D(E)\cos 2\theta - (C_1(E)+C_2(E))\sin 2\theta, \qquad \text{on} \quad U_x\cap U_{\bar{x}}, \end{aligned} \tag{2.15}$$

for (2.14)

$$\begin{aligned} C_1(\bar{E}) &= C_1(E)\cos^2\theta - C_2(E)\sin^2\theta - (D(E)/2)\sin 2\theta, \\ C_2(\bar{E}) &= C_2(E)\cos^2\theta - C_1(E)\sin^2\theta - (D(E)/2)\sin 2\theta, \\ D(\bar{E}) &= -D(E)\cos 2\theta - (C_1(E)+C_2(E))\sin 2\theta, \qquad \text{on} \quad U_x\cap U_{\bar{x}}. \end{aligned} \tag{2.16}$$

From (2.15) and (2.16), we have

$$C_1(\bar{E}) - C_2(\bar{E}) = C_1(E) - C_2(E), \tag{2.17}$$

$$(2.18)\qquad H(\bar{E})=D(\bar{E})^2/4+C_1(\bar{E})C_2(\bar{E}) \\ =D(E)^2/4+C_1(E)C_2(E)=H(E), \qquad \text{on } U_x\cap U_{\bar{x}}.$$

From (2.17), we see that $f=(C_1(E)-C_2(E))K$ for some $\{E_i\}$ on U_x, $x\varepsilon W_0$, is a real analytic functionon W_0.

3. Some results. In this section, furthermore, we shall assume that (M, g) is complete. Then, by (2.12) and (2.18), we have

LEMMA 3.1. *For each point $x\in W_0$, L_x^3 is infinitely extendible in W_0.*

By lemma 3.1, we see that $(1/K)|_{L_x^3}=(\alpha s-\beta)^2-H/\alpha^2$ must be defined for all real numbers s along L_x^3.

PROPOSITION 3.2. *If the distribution T_1 is involutive on W_0, then (M, g) is reducible.*

PROOF. Assume that T_1 is involutive. Then, it follows that $[E_1, E_2]\in T_1$, that is

$$(3.1)\qquad B_{1\,32}-B_{2\,31}=0\,.$$

Thus, from (3.1), we have $H=H(E)=D(E)^2/4+C_1(E)^2\geqq 0$. Thus, from lemma 3.1. and (2.12), by the similar arguments as in [7], we can show that $H=0$ and furthermore K is constant along L_x^3, $x\in W_0$. Therefore, from (2.9), (3.1) and the fact $H=H(E)=0$, we have $B_{1\,31}=B_{1\,32}=B_{2\,31}=B_{2\,32}=0$. Thus, we see that T_1 and T_0 are parallel on W_0, that is to say, the open subspace $(W_0, g|_{W_0})$ is reducible. Since (M, g) is real analytic, we can conclude that (M, g) is reducible. Q. E. D.

Next, furthermore, we shall assume that M is compact and the rank of the Ricci form R_1 is different from 0 everywhere on M. Then, it follows that $W_0=M$. Then, α can not be 0 in (2.12). Since $1/K$ is continuous on M, it must be bounded on M. But, since $1/K$ coincides with $(\alpha s-\beta)^2-H/\alpha^2$ or $-((\alpha s-\beta)^2-H/\alpha^2)$ along L_x^3, $x\in M$, it can not be bounded on $L_x^3\subset M$. This is a contradiction. Thus, we see that $H=H(E)=0$ at every point $x\in M$ with respect to any $\{E_i\}$ on U_x. Thus, from (2.10) and (2.11), by the similar arguments as in [5], we can see that K is constant along each L_x^3, $x\in M$. That is

PROPOSITION 3.3. *If M is compact and the rank of the Ricci form R_1 is different from 0 everywhere on M, then K is constant along each L_x^3, $x\in M$.*

4. Proof of the main theorem. In the sequel, we shall assume that M is compact and the rank of R_1 is different from 0 everywhere on M.

The purpose of this section is to prove the reducibility of (M, g) under these circumstances. Now, we assume that there exists a point $z\in M$ such that $f(z)\neq 0$. Let $V=\{x\in M; f(x)\neq 0\}$, which is an open set of M. For any point $x_0\in V$, let V_0 be the connected component of x_0 in V. Now, since $H=H(E)=0$ for any $\{E_i\}$ on sufficiently small $U_x(\subset V_0)$, we see that $\wedge(E)=\sqrt{D(E)^2+(C_1(E)+C_2(E))^2}>0$. Thus, we can define a real analytic orthonormal frame field $\{E_i^*(E)\}$ on U_x in such a way that

$$\begin{aligned} E_1^*(E) &= (\cos\xi)E_1+(-\sin\xi)E_2\,, \\ E_2^*(E) &= (\sin\xi)E_1+(\cos\xi)E_2\,, \\ E_3^*(E) &= E_3\,, \end{aligned} \tag{4.1}$$

where ξ is a certain real analytic function on U_x satisfying $\cos 2\xi=(C_1(E)+C_2(E))/\wedge(E)$ and $\sin 2\xi=D(E)/\wedge(E)$.

Next, if $U_x\cap U_{\bar{x}}\neq\emptyset$, $\{E_i\}$ and $\{\bar{E}_i\}$ are defined on U_x and $U_{\bar{x}}$, respectively, then, by the similar way as in (4.1), we may obtain an orthonormal frame field $\{E_i^*(\bar{E})\}$ with respect to $\{\bar{E}_i\}$ on $U_{\bar{x}}(\subset V_0)$. Then we have

LEMMA 4.1. *On* $U_x\cap U_{\bar{x}}$, *we have*

$$E_i^*(\bar{E}) = \pm E_i^*(E)\,, \qquad i=1,2,3, \tag{4.2}$$

where the plus sign or minus sign in (4.2) *is determined by the orientation of* M.

PROOF. By the definition of $\{E_i^*(\bar{E})\}$, we have

$$\begin{aligned} E_1^*(\bar{E}) &= (\cos\bar{\xi})\bar{E}_1+(-\sin\bar{\xi})\bar{E}_2\,, \\ E_2^*(\bar{E}) &= (\sin\bar{\xi})\bar{E}_1+(\cos\bar{\xi})\bar{E}_2\,, \\ E_3^*(\bar{E}) &= \bar{E}_3\,, \end{aligned} \tag{4.3}$$

where $\bar{\xi}$ is a certain real analytic function on $U_{\bar{x}}$ satisfying $\cos 2\bar{\xi}=(C_1(\bar{E})+C_2(\bar{E}))/\wedge(\bar{E})$ and $\sin 2\bar{\xi}=D(\bar{E})/\wedge(\bar{E})$.

First, for the case (2.13), from (2.15), (4.1) and (4.3), we have $\wedge(\bar{E})=\wedge(E)$ and furthermore

$$\begin{aligned} \cos 2\bar{\xi} &= \big(C_1(\bar{E})+C_2(\bar{E})\big)/\wedge(\bar{E}) \\ &= \big(1/\wedge(E)\big)\big((\cos^2\theta)C_1(E)-(\sin^2\theta)C_2(E) \\ &\quad +(\sin\theta\cos\theta)D(E)+(\cos^2\theta)C_2(E)-(\sin^2\theta)C_1(E) \\ &\quad +(\sin\theta\cos\theta)D(E)\big) \\ &= \big(1/\wedge(E)\big)\big((\cos 2\theta)\big(C_1(E)+C_2(E)\big)+(\sin 2\theta)D(E)\big)=\cos 2(\xi-\theta)\,, \end{aligned}$$

similarly

$$\sin 2\bar{\xi} = \sin 2(\xi - \theta).$$

Thus, we have

(4. 4) $$\xi - \theta = \bar{\xi} + n\pi \qquad (n = 1, 2, \cdots).$$

Again, from (2. 13), (2. 15), (4. 1) and (4. 3), we have

$$E_1^*(\bar{E}) = (\cos\bar{\xi})\big((\cos\theta)E_1 + (-\sin\theta)E_2\big) + (-\sin\bar{\xi})\big((\sin\theta)E_1 + (\cos\theta)E_2\big)$$
$$= \big(\cos(\bar{\xi}+\theta)\big)E_1 + \big(-\sin(\bar{\xi}+\theta)\big)E_2.$$

Thus, from (4. 4), we see that $E_1^*(\bar{E}) = E_1^*(E)$ or $E_1^*(\bar{E}) = -E_1^*(E)$. Furthermore, we see that $E_2^*(\bar{E}) = E_2^*(E)$ corresponding to $E_1^*(\bar{E}) = E_1^*(E)$ or $E_2^*(\bar{E}) = -E_2^*(E)$ corresponding to $E_1^*(\bar{E}) = -E_1^*(E)$. Similarly, considering the case (2. 14), we see that (4. 2) is valid. Q. E. D.

For each $\{E_i^* = E_i^*(E)\}$ on $U_x(\subset V_0)$, let $T_{ij} = \text{span}\{E_i^*, E_j^*\}$ $(i \leqq j)$. Then, by the definition of $\{E_i^*(E)\}$, we see that

(4. 5) $$C_1(E^*)C_2(E^*) = 0 \quad \text{and} \quad D(E^*) = 0.$$

Thus, we may assume, for example

(4. 6) $$C_1(E^*) \neq 0, \quad C_2(E^*) = 0, \quad D(E^*) = 0, \qquad \text{on } V.$$

Thus, from (2. 9) (4. 6) and proposition 3. 3, we have

(4. 7) $$B_{1\,32}^* \neq 0, \quad B_{2\,31}^* = B_{1\,31}^* = B_{2\,32}^* = 0 \qquad \text{on } U_x,\ x \in V_0,$$

where $B_{i\,jk}^*$ $(i, j, k = 1, 2, 3)$ denote the ones defined as before corresponding to $\{E_i^*\}$. Then, from (4. 7), we have

LEMMA 4. 2. *T_{23} is involutive on V_0.*

Now, from (2. 2), (2. 3), (2. 4), (2. 6) and (4. 7), we have

$$R(E_1^*, E_2^*)E_3^* = \nabla_{E_1^*}\nabla_{E_2^*}E_3^* - \nabla_{E_2^*}\nabla_{E_1^*}E_3^* - \nabla_{[E_1^*, E_2^*]}E_3^*$$
$$= -\big((E_2^* B_{1\,32}^*) + B_{1\,21}^* B_{1\,32}^*\big)E_2^* - (B_{2\,32}^* B_{2\,21}^*)E_1^* = 0.$$

Thus, we have

(4. 8) $$B_{2\,21}^* = 0,$$

(4. 9) $$E_2^* B_{1\,32}^* + B_{1\,21}^* B_{1\,32}^* = 0.$$

From (4. 8) and (4. 9), we see that $\nabla_{E_2^*}E_2^* = 0$, that is, each trajectory of E_2^* is a geodesic. From (4. 7), since $\nabla_{E_2^*}E_3^* = \nabla_{E_3^*}E_2^* = \nabla_{E_3^*}E_3^* = 0$, consequently, we have

LEMMA 4. 3. *Let $M_{23}(x)$ be the maximal integral submanifold of T_{23} through $x \in V_0$. Then $M_{23}(x)$ becomes totally geodesic subspace with respect to the induced metric and hence locally flat.*

Now, let L_x^2 be the geodesic whose initial point is x, $x \in V_0$, and whose tangent vector is E_2^* or $-E_2^*$ at each point of L_x^2. And let t denote its arc-length parameter. Using the same symbol for convenience, we shall assume that L_x^2 denotes also the set of the points on L_x^2 and $x(t)$ denotes the point on L_x^2 corresponding to the value t of the parameter. Again, from (2. 2), (2. 3), (2. 4), (2. 6) and (4. 7), we have

$$\begin{aligned} &R(E_1^*, E_3^*)E_2^* \\ &\quad = -\sum_{i=1}^{3} E_3^* B_{1\ 2i}^{*} E_i^* = 0 , \\ &R(E_1^*, E_2^*)E_1^* \\ &\quad = -\left((E_2^* B_{1\ 12}^{*}) + (B_{1\ 21}^{*} B_{1\ 12}^{*})\right) E_2^* \\ &\quad = -K E_2^* . \end{aligned}$$

Thus, we have

$$E_3^* B_{1\ 21}^{*} = 0 , \tag{4. 10}$$

$$E_2^* B_{1\ 21}^{*} + B({}_{1\ 21}^{*})^2 = -K . \tag{4. 11}$$

From (4. 9) and (4. 11), we have

$$\frac{d^2}{dt^2}(B_{1\ 32}^{*}) + \left(-K - 2(B_{1\ 21}^{*})^2\right) B_{1\ 32}^{*} = 0 \quad \text{along } L_x^2 . \tag{4. 12}$$

(4. 12) is equivalent to

$$\frac{d^2 f}{dt^2} + (-K - 2G^2) f = 0 \quad \text{along } L_x^2 ,$$

where $G^2 = (B_{1\ 21}^{*})^2$.

Now, if we put $f^* = f^2$, then, from (4. 9) and (4. 11), we have

$$\frac{d^2 f^*}{dt^2} + 2\left(-K - 3(G^2)\right) f^* = 0 \quad \text{along } L_x^2 . \tag{4. 13}$$

We can easily see that $f=0$ on the complement of V_0 in M. Then we have

LEMMA 4. 4. *For each point $x \in V_0$, L_x^2 is infinitely extendible in V_0.*

PROOF. Since (M, g) is complete, as a geodesic in (M, g), L_x^2 is infinitely extendible. If this geodesic does not lie in V_0, let t_0 be a point such that $x(t)\in V_0$ for $t<t_0$ but $x(t_0)\notin V_0$. Then, we see that $f(x(t_0))=0$. Now, we put $y=f(t)=f(x(t))$, $x(t)\in L_x^2$, where, using the same symbol for convenience, we shall assume that L_x^2 denotes also the extention of L_x^2. Then, $f(t)$ is a real analytic function defined for all real numbers t. Since f is not identically 0, we may put

(4.14) $$y=f(t)=u^n f_1(u), \qquad \text{for some integer } n\geqq 1,$$

where $u=t-t_0, |u|<\varepsilon$ for sufficiently small $\varepsilon>0$, and f_1 is a certain real analytic function defined for $|u|<\varepsilon$ satisfying $f_1(0)\neq 0$. We see that G^2 is a real analytic function on V_0. Teen, from (4.9) and (4.14), we have

(4.15) $$G(u)=-(1/u)\Big(\big(u(df_1/du)+nf_1\big)/f\Big) \qquad \text{for } E_2^*=d/dt$$

or

$$G(u)=(1/u)\Big(\big(u(df_1/du)+nf_1\big)\Big)/f\Big) \qquad \text{for } E_2^*=-d/dt \text{ along } L_x^2,$$

where $-\varepsilon<u<0$, for sufficiently small $\varepsilon>0$.

From (4.11) and (4.15), by direct computing, we have

(4.16) $$(1/u)^2G_1(u)=-K\big(x(u)\big), \qquad -\varepsilon<u<0, \qquad \text{for sufficienfly}$$

small $\varepsilon>0$, where G_1 is a real analytic function defined for $-\varepsilon<u<\varepsilon$ such that

$$G_1(u)=(1/f_1)^2\big(n+n^2)f_1^2+2nuf_1(df_1/du)+2u^2(df_1/du)^2-u^2f_1(d^2f_1/du^2)\big),$$

and hence $G_1(0)=n+n^2$.

Thus, for the left hand side of (4.16), we have $\lim_{u\to-0}(1/u)^2G_1(u)=+\infty$, and for the right hand side of (4.16), we have $\lim_{u\to-0}-K(x(u))=-K(x(t_0))$. This is a contradiction. Q.E.D.

From (4.9) and (4.11), we have

(4.17) $$d^2(1/f)/dt^2+K(1/f)=0, \qquad \text{along } L_x^2.$$

Next, we shall assume that $K>0$ on M. Since M is compact, there exists a point $x_0\in V\subset M$ such that $f^*(x_0)=\operatorname{Max}_{x\in M} f^*(x)>0$. Let V_0 be the connected component of x_0 in V. And consider $L_{x_0}^2$. Then, from (4.13), since $K>0$, we see that $d^2f^*/dt^2>0$ for all real numbers t. But, this is a contradiction. Thus, we can conclude that $f=0$ on M. Thus, by the same arguments as in the proof of proposition 3.2, we can see that (M, g) is reducible. Therefore, we have the main theorem.

5. Some remarks. Let (M, g) be a 3-dimensional complete, irreducible real analytic Riemannian manifold satisfying the condition (*) (or equivalently (**)). Now, we shall assume that the scalar curvature, S, of (M, g) is a non-zero constant. If the rank of the Ricci form R_1 of (M, g) is 3 at some point of M, then (M, g) is a space of constant curvature, $S/6$. In the sequel, we shall assume that the rank of the Ricci form R_1 of (M, g) is 2 everywhere on M. Then, from the constancy of $K=S/2$, we may apply the similar arguments to (M, g) in consideration which are independent on compactness of the manifold treated in the previous sections. First, we assume that $S>0$. Then, from (4. 17), we have

$$1/f(t)=c_1 \sin(\sqrt{S/2})t+c_2\cos(\sqrt{S/2})t\,, \quad \text{along } L_x^2,\ x\in V_0, \tag{5. 1}$$

where c_1 and c_2 are certain real numbers.

Since (M, g) is complete, from lemma 4. 4. and (5. 1), we see that there exists a real number t_0 such that $1/f(t_0)=0$. But, this is a contradiction. Thus, we have

PROPOSITION 5. 1. *Let (M, g) be a 3-dimensional complete, irreducible real analytic Riemannian manifold satisfying* (*) (*or equivalently* (**)). *If the scalar curvature S of (M, g) is constant and positive, then (M, g) is a space of constant curvature $S/6$.*

Next, we assume that $S<0$. From lemma 4. 3, for each point $x\in V_0$, we may choose a local coordinate system $(U_x;\ (u_1, u_2, u_3))$ with origin x, $U_x\subset V_0$ such that

$$\begin{aligned} E_1^* &= \lambda(\partial/\partial u_1)\,, \\ E_2^* &= a_{22}(\partial/\partial u_2)+a_{23}(\partial/\partial u_3)\,, \\ E_3^* &= a_{32}(\partial/\partial u_2)+a_{33}(\partial/\partial u_3)\,, \qquad -\varepsilon<u_1,\ u_2,\ u_3<\varepsilon\,. \end{aligned} \tag{5. 2}$$

where λ, a_{22}, a_{23}, a_{32} and a_{33} are certain real analytic functions on U_x, $\lambda>0$, and $a_{22}=a_{33}=1$, $a_{23}=a_{32}=0$ along $M_{23}(x)$ in U_x.

By considering $B_1^*{}_{31}=B_2^*{}_{31}=B_2^*{}_{32}=B_3^*{}_{ij}=0$, $i, j=1, 2, 3$, we see that a_{22}, a_{23}, a_{32} and a_{33} depend only on u_1. By (5. 2), the Riemannian metric tensor g is represented by

$$(g)\,;\quad \begin{pmatrix} 1/\lambda^2 & 0 & 0 \\ 0 & g_{22} & g_{23} \\ 0 & g_{32} & g_{33} \end{pmatrix} \quad \text{on } U_x\,, \tag{5. 3}$$

where $g_{pq}=g(\partial/\partial u_p,\ \partial/\partial u_q)$, $p, q=2, 3$.

Then we have

(5. 4) $$f=\lambda\Phi\,, \qquad t=a_{22}u_2+a_{23}u_3\,,$$

where $\Phi=a^{22}\left(\partial a_{32}/\partial u_1+\begin{Bmatrix}2\\1\,2\end{Bmatrix}a_{32}+\begin{Bmatrix}2\\1\,3\end{Bmatrix}a_{33}\right)+a^{32}\left(\partial a_{33}/\partial u_1+\begin{Bmatrix}3\\1\,2\end{Bmatrix}a_{32}+\begin{Bmatrix}3\\1\,3\end{Bmatrix}a_{33}\right)$,

(a^{pq}) denotes the inverse matrix of (a_{pq}), $p, q=2, 3$ and $\begin{Bmatrix}i\\j\,k\end{Bmatrix}$ denote the Christoffel symbols formed with $g_{ij}=g(\partial/\partial u_i, \partial/\partial u_j)$, $i, j, k=1, 2, 3$.

Then, by direct computing, we see that Φ depends only on u_1. Now, especially, we put $a_{22}=\cos u_1$, $a_{23}=-\sin u_1$, $a_{32}=\sin u_1$, $a_{33}=\cos u_1$ in (5. 2). Then, frsm (5. 4), we see that $\Phi=1$. Thus, the following Riemannian manifold (M, g) is an example of 3-dimensional complete, irreducible real analytic Riemannian manifolds satisfying (*) and $\nabla R\neq 0$:

$M=R^3$ (3-dimensional real number space),

$$(g):\quad \begin{pmatrix} 1/\lambda^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \text{with respect to}$$

a canonical coordinate system (u_1, u_2, u_3) on R^3, where

$$1/\lambda=c_1e^{(\sqrt{-S/2})t}+c_2e^{-(\sqrt{-S/2})t}\,, \qquad t=(\cos u_1)u_2+(-\sin u_1)u_3\,,$$

c_1, c_2, S are certain real constant.

The above Riemannian manifold is of the form $E^2\times_f E^1$, and the scalar curvature is S, where $f=1/\lambda$, (see [5], [10]). Some results concerning R(X, $Y)\cdot R=0$ may be founded inreferences.

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