On conformal Killing tensors of a Riemannian manifold of constant curvature

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Introduction. Recently S. Tachibana [2][1] has introduced a notion of a conformal Killing tensor field of degree 2 in a Riemannian manifold and T. Kashiwada [3] has given the definition of a conformal Killing tensor field of degree p ($p\geqq 2$) in a Riemannian manifold. They discussed such the tensor fields and obtained many interesting results.

In this paper, the author proves by the mathematical induction that a Riemannian manifold of constant curvature admitting a conformal Killing vector field admits necessarily a conformal Killing tensor field of degree p. §1 is devoted to give some preliminaries on a general Riemannian manifold R^n admitting a conformal Killing vector field. In §2 we give the definition of a conformal Killing tensor field of degree $p\geqq 2$.

Let us denote by M^n an n-dimensional Riemannian manifold of constant curvature which admits a conformal Killing vector field. We prove that M^n admits a conformal Killing tensor field of degree 2 in §3. Making use of the results in §3, in the last section §4 we shall show that M^n admits a conformal Killing tensor field of general degree.

The present author wishes to express his very sincere thanks to Professor Y. Katsurada for her many valuable advices and constant guidances.

§1. Preliminaries on a Riemannian manifold admitting a conformal Killing vector field. Let R^n ($n>2$) be an n-dimensional Riemannian manifold whose metric tensor is given by g_{ij}.

Let ξ^i be a vector field in R^n such that

$$\underset{\xi}{\pounds} g_{ij}=\xi_{i;j}+\xi_{j;i}=2\phi g_{ij} \tag{1.1}$$

where ϕ is a scalar field in R^n and the symbol $\underset{\xi}{\pounds}$ and ";" denote the operator of Lie derivation with respect to ξ^i and of covariant differentiation with respect to the Riemann connection determined by g_{ij} respectively. Then ξ^i is called a conformal Killing vector field. If ϕ vanishes identically in (1.1), then ξ^i is called a Killing vector field.

1) Numbers in brackets refer to the references at the end of the paper.

If P_{ij} is a covariant tensor field, then we have

$$\underset{\xi}{\pounds}(P_{ij;k})-(\underset{\xi}{\pounds}P_{ij})_{;k}=-\left(\underset{\xi}{\pounds}\left\{{l \atop ki}\right\}\right)P_{lj}-\left(\underset{\xi}{\pounds}\left\{{l \atop kj}\right\}\right)P_{il} \text{ (cf. [6])},$$

where $\left\{{i \atop jk}\right\}$ denotes the Christoffel symbol of the first kind.

Applying the above formula to the metric tensor g_{ij}, we obtain

$$\underset{\xi}{\pounds}\left\{{i \atop jk}\right\}=\frac{1}{2}g^{il}\cdot\left[(\underset{\xi}{\pounds}g_{kl})_{;j}+(\underset{\xi}{\pounds}g_{lj})_{;k}-(\underset{\xi}{\pounds}g_{jk})_{;l}\right]. \tag{1.2}$$

Substituting (1.1) into (1.2), we find

$$\underset{\xi}{\pounds}\left\{{i \atop jk}\right\}=\delta^i_j\phi_k+\delta^i_k\phi_j-g_{jk}\phi^i \tag{1.3}$$

where $\phi_i=\phi_{;i}$, $\phi^i=g^{ij}\phi_j$ and δ^i_j denotes the Kronecker deltas.
Substituting (1.3) into

$$\underset{\xi}{\pounds}R^i{}_{jkl}=\left(\underset{\xi}{\pounds}\left\{{i \atop jk}\right\}\right)_{;l}-\left(\underset{\xi}{\pounds}\left\{{i \atop lk}\right\}\right)_{;j}$$

where $R^i{}_{jkl}$ is the curvature tensor, we obtain

$$\underset{\xi}{\pounds}R^i{}_{jkl}=-\delta^i_l\phi_{j;k}+\delta^i_k\phi_{j;l}-g_{jk}\phi^i{}_{;l}+g_{jl}\phi^i{}_{;k}\,. \tag{1.4}$$

By contraction with respect to i and l, it follows from (1.4) that

$$\underset{\xi}{\pounds}R_{jk}=-(n-2)\phi_{k;j}-g_{jk}\phi^i{}_{;i} \tag{1.5}$$

where R_{jk} is the Ricci tensor.
Transvecting (1.5) with g^{jk}, we find

$$\underset{\xi}{\pounds}R=-2(n-1)\phi^i{}_{;i}-2\phi R \tag{1.6}$$

where R is the scalar curvature.

When R^n is an Einstein space, that is,

$$R_{jk}=\frac{R}{n}g_{jk}\,,\qquad R=\text{const.},$$

we have, for a conformal Killing vector field ξ^i,

$$\underset{\xi}{\pounds}R_{jk}=\frac{R}{n}\underset{\xi}{\pounds}g_{jk}=\frac{2R}{n}\phi g_{jk}\,,\qquad \underset{\xi}{\pounds}R=0\,.$$

Consequently, from (1.5) and (1.6), we get

$$\frac{2R}{n}\phi g_{jk}=-(n-2)\phi_{k;j}-g_{jk}\phi^{i}{}_{;i}\,,\qquad (n-1)\phi^{i}{}_{;i}+R\phi=0\,,$$

respectively. From these relations, it follows that

$$\phi_{;i;j}=-k\phi g_{ij}\,,\qquad k=\frac{R}{n(n-1)}\,. \tag{1.7}$$

Thus if an Einstein space of dimension $n>2$ admits a conformal Killing vector field, then it admits a non-zero scalar function ϕ which satisfies the above equation.

A space of constant curvature ($n>2$) is a Riemannian manifold satisfying

$$R^{i}{}_{jkl}=k(g_{jk}\delta_{l}^{i}-\delta_{k}^{i}g_{jl}) \tag{1.8}$$

and then k is a constant given by $k=\dfrac{R}{n(n-1)}$.

A space of constant curvature is necessarily an Einstein space.

§ 2. **Conformal Killing tensor field.** In this section, as the generalization of a conformal Killing vector field we shall show the definition of a conformal Killing tensor field which is given by S. Tachibana and T. Kashiwada.

We shall call a skew symmetric tensor T_{ij} a conformal Killing tensor field of degree 2 in R^{n} if there exists a vector field ρ_{i} such that

$$T_{ij;k}+T_{kj;i}=2\rho_{j}g_{ik}-\rho_{k}g_{ij}-\rho_{i}g_{jk}\,. \tag{2.1}$$

The vector ρ_{i} is called the associated vector field of T_{ij}. If ρ_{i} vanishes identically in (2.1), then T_{ij} is called a Killing tensor field of degree 2.

Furthermore, we shall generalize it to the case of degree p ($p\geqq 2$). A skew symmetric tensor field $T_{i_1\cdots i_p}$ is called a conformal Killing tensor field of degree p in R^{n}, if there exists a skew symmetric tensor field $\rho_{i_1\cdots i_{p-1}}$ such that

$$T_{i_1\cdots i_p;i}+T_{ii_2\cdots i_p;i_1}=2\rho_{i_2\cdots i_p}g_{i_1 i}-\sum_{h=2}^{p}(-1)^{h}\cdot(\rho_{i_1\cdots\hat{i}_h\cdots i_p}g_{ii_h}+\rho_{ii_2\cdots\hat{i}_h\cdots i_p}g_{i_1 i_h})\,, \tag{2.2}$$

where $\hat{i}_h$ means that i_h is omitted. We call $\rho_{i_1\cdots i_{p-1}}$ the associated tensor field of $T_{i_1\cdots i_p}$. If $\rho_{i_1\cdots i_{p-1}}$ vanishes identically in (2.2), then $T_{i_1\cdots i_p}$ is called a Killing tensor field of degree p.

Especially, if R^{n} is a space of constant curvature, then the associated tensor field of conformal Killing tensor field of degree p is a Killing tensor field (cf. [3]).

§ 3. Conformal Killing tensor field of degree 2. In the following sections, let M^n be an n-dimensional Riemannian manifold of constant curvature.

LEMMA 3. 1. *Let R^n $(n>2)$ be an Einstein space which admits a conformal Killing vector field ξ^i. Then R^n admits a Killing vector field.*

PROOF. We put

$$\rho_i = \xi_i + \frac{1}{k}\phi_i, \qquad k = \frac{R}{n(n-1)}.$$

Differentiating this covariantly, by means of (1. 1) and (1. 7) we get

$$\rho_{i;j} + \rho_{j;i} = 0. \tag{3. 1}$$

THEOREM 3. 2. *If M^n admits a conformal Killing vector field ξ^i, then M^n admits a conformal Killing tensor field of degree 2.*

PROOF. Since M^n admits a Killing vector field ρ^i by Lemma 3. 1, differentiating (3. 1) covariantly, we obtain

$$\rho_{i;j;k} + \rho_{j;i;k} = 0.$$

From the above equation, we have

$$\rho_{i;j;k} + \rho_{j;i;k} + \rho_{i;k;j} + \rho_{k;i;j} - (\rho_{j;k;i} + \rho_{k;j;i}) = 0.$$

Then by virtue of Ricci's identity, we get

$$2\rho_{i;j;k} - \rho_h(R^h{}_{jik} + R^h{}_{kij} + R^h{}_{ikj}) = 0.$$

In consequence of Bianchi's identity the above equation reduces to

$$\rho_{i;j;k} + \rho_h R^h{}_{kji} = 0.$$

Then by means of (1. 8) the last equation turns to

$$\rho_{i;j;k} = k(\rho_j g_{ki} - \rho_i g_{jk}).$$

We put $T_{ij} = \rho_{i;j}$, then the above equation is rewritten as follows:

$$T_{ij;k} = k(\rho_j g_{ki} - \rho_i g_{jk}), \tag{3. 2}$$

and hence we obtain

$$T_{ij;k} + T_{kj;i} = k(2\rho_j g_{ki} - \rho_i g_{jk} - \rho_k g_{ji}). \tag{3. 3}$$

This equation shows that T_{ij} is a conformal Killing tensor field of degree 2 whose associated vector field is given by $k\rho_i$.

§ 4. Conformal Killing tensor field of degree $p \geqq 3$. At the first, we shall show that a conformal Killing tensor field of degree 3 can be constructed by a conformal Killing tensor field of degree 2 and the vector ϕ_i.

By virture of Theorem 3. 2, we have shown that constant Riemannian curvature space M^n admits a conformal Killing tensor field T_{ij} of degree 2. Put

$$T_{ijk}=T_{ij}\phi_k+T_{jk}\phi_i+T_{ki}\phi_j. \tag{4. 1}$$

Then it is clear that T_{ijk} is skew symmetric with respect to all indices.

Differentiating (4. 1) covariantly, by means of (1. 7) and (3. 2) we have

$$T_{ijk;l}=k\Big[(\rho_j\phi_k-\rho_k\phi_j-\phi T_{jk})g_{il}-(\rho_i\phi_k-\rho_k\phi_i-\phi T_{ik})g_{jl} +(\rho_i\phi_j-\rho_j\phi_i-\phi T_{ij})g_{kl}\Big].$$

Hence we put

$$\rho_{jk}=\rho_j\phi_k-\rho_k\phi_j-\phi T_{jk},$$

then the last equation turns to

$$T_{ijk;l}=k(\rho_{jk}g_{il}-\rho_{ik}g_{jl}+\rho_{ij}g_{kl}), \tag{4. 2}$$

and hence we get

$$T_{ijk;l}+T_{ljk;i}=k(2\rho_{jk}g_{il}-\rho_{ik}g_{jl}-\rho_{lk}g_{ji}+\rho_{ij}g_{kl}+\rho_{lj}g_{ki}).$$

This equation shows that T_{ijk} is a conformal Killing tensor field of degree 3 whose associated tensor field is given by $k\rho_{ij}$. Therefore we have

THEOREM 4. 1. *Let M^n be an n-dimensional Riemannian manifold of constant curvature which admits a conformal Killing vector field ξ^i. Then M^n admits a conformal Killing tensor field of degree* 3.

Next, we prove that M^n admits a conformal Killing tensor field of degree p, under the assumption that M^n admits a conformal Killing tensor field of degree $p-1\geqq 3$.

We assume that M^n admits a skew symmetric tensor field $T_{i_1\cdots i_{p-1}}$ such that

$$T_{i_1\cdots i_{p-1};i}=-k\sum_{h=1}^{p-1}(-1)^h\rho_{i_1\cdots\hat{i}_h\cdots i_{p-1}}g_{i_hi}, \tag{4. 3}$$

where $\rho_{i_2\cdots i_{p-1}}$ denotes a Killing tensor field of degree $p-2$.

Putting $p=2$ and $p=3$ in (4. 3), we obtain (3. 2) and (4. 2) respectively. Then we have

$$T_{i_1\cdots i_{p-1};i}+T_{ii_2\cdots i_{p-1};i_1}=k\cdot\Big[2\rho_{i_2\cdots i_{p-1}}g_{i_1i} -\sum_{h=2}^{p-1}(-1)^h\cdot(\rho_{i_1\cdots\hat{i}_h\cdots i_{p-1}}g_{i_hi}+\rho_{ii_2\cdots\hat{i}_h\cdots i_{p-1}}g_{i_hi_1})\Big],$$

where $\rho_{i_2\cdots i_{p-1}}$ denotes the associated tensor field of $T_{i_1\cdots i_{p-1}}$. Thus this equation shows that $T_{i_1\cdots i_{p-1}}$ is a conformal Killing tensor field of degree $p-1$.

If we put

$$T_{i_1\cdots i_p}=\sum_{h=1}^{p}(-1)^h T_{i_1\cdots\hat{i_h}\cdots i_p}\phi_{i_h}. \tag{4.4}$$

Then it is clear that $T_{i_1\cdots i_p}$ is skew symmetric with respect to all indices.

Differentiating (4.4) covariantly we have

$$T_{i_1\cdots i_p;i}=\sum_{h=1}^{p}(-1)^h T_{i_1\cdots\hat{i_h}\cdots i_p;i}\phi_{i_h}+\sum_{h=1}^{p}(-1)^h T_{i_1\cdots\hat{i_h}\cdots i_p}\phi_{i_h;i}.$$

Substituting (1.7) and (4.3) into this equation, we find

$$\begin{aligned}T_{i_1\cdots i_p;i}&=-\sum_{h=1}^{p}(-1)^h\cdot k\cdot\sum_{\substack{k=1\\(h\neq k)}}^{p}(-1)^k\rho_{i_1\cdots\hat{i_h}\cdots\hat{i_k}\cdots i_p}\phi_{i_h}g_{i_k i}\\&\quad -k\phi\sum_{h=1}^{p}(-1)^h T_{i_1\cdots\hat{i_h}\cdots i_p}g_{i_h i}\\&=-k\sum_{h=1}^{p}(-1)^h\Big[\sum_{\substack{k=1\\(h\neq k)}}^{p}(-1)^k\rho_{i_1\cdots\hat{i_h}\cdots\hat{i_k}\cdots i_p}\phi_{i_k}+\phi T_{i_1\cdots\hat{i_h}\cdots i_p}\Big]g_{i_h i}.\end{aligned}$$

Hence if we put

$$\rho_{i_1\cdots\hat{i_h}\cdots i_p}=\sum_{\substack{k=1\\(h\neq k)}}^{p}(-1)^k\rho_{i_1\cdots\hat{i_h}\cdots\hat{i_k}\cdots i_p}\phi_{i_k}+\phi T_{i_1\cdots\hat{i_h}\cdots i_p},$$

then the last equation turns to

$$T_{i_1\cdots i_p;i}=-k\sum_{h=1}^{p}(-1)^h\rho_{i_1\cdots\hat{i_h}\cdots i_p}g_{i_h i}, \tag{4.5}$$

and hence we get

$$\begin{aligned}&T_{i_1\cdots i_p;i}+T_{ii_2\cdots i_p;i_1}\\&=-k\sum_{h=1}^{p}(-1)^h\rho_{i_1\cdots\hat{i_h}\cdots i_p}g_{i_h i}-k\sum_{\substack{h=1\\(h\neq 1)}}^{p}(-1)^h\rho_{ii_2\cdots\hat{i_h}\cdots i_p}g_{i_h i_1}\\&=-k\Big[-\rho_{i_2\cdots i_p}g_{i_1 i}+\sum_{h=2}^{p}(-1)^h\rho_{i_1\cdots\hat{i_h}\cdots i_p}g_{i_h i}\Big]\\&\quad -k\Big[-\rho_{i_2\cdots i_p}g_{i_1 i}+\sum_{h=2}^{p}(-1)^h\rho_{ii_2\cdots\hat{i_h}\cdots i_p}g_{i_h i_1}\Big]\\&=k\Big[2\rho_{i_2\cdots i_p}g_{i_1 i}-\sum_{h=2}^{p}(-1)^h\cdot(\rho_{i_1\cdots i_h\cdots i_p}g_{i_h i}+\rho_{ii_2\cdots\hat{i_h}\cdots i_p}g_{i_h i_1})\Big].\end{aligned}$$

This equation shows that $T_{i_1\cdots i_p}$ is a conformal Killing tensor field of degree

p whose associated tensor field is given by $k\rho_{i_1\cdots i_p}$. Therefore we have

THEOREM 4. 2. *Let M^n be an n-dimensional Riemannian manifold of constant curvature which admits a conformal Killing vector field ξ^i. Then M^n admits a conformal Killing tensor field of degree p.*

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(Received February 28, 1973)