Remarks on boundary value problems for hyperbolic equations

By Kôji KUBOTA

§ 1. Introduction and results

Let $\boldsymbol{R}_+^{n+1}$ be the open half space $\{x=(x', x_n);\ x'=(x_0, x_1, \cdots, x_{n-1})\in \boldsymbol{R}^n,\ x_n>0\}$. We shall consider the boundary value problem (P, B_j) in $\boldsymbol{R}_+^{n+1}$:

$$\begin{cases} P(x, D)u=f & \text{in } \boldsymbol{R}_+^{n+1}, \\ B_j(x', D)u=f_j & \text{on } \boldsymbol{R}^n,\ j=1, \cdots, l, \end{cases}$$

where $D=(D_0, D_1, \cdots, D_{n-1}, D_n)$, $D_k=\dfrac{1}{i}\dfrac{\partial}{\partial x_k}$, $P(x, D)$ is a strictly x_0-hyperbolic operator of order m and $B_j(x', D)$ is a boundary operator of order m_j. Throughout this paper coefficients of differential operators are assumed to be C^∞-functions and constant outside a compact set of $\boldsymbol{R}^{n+1}$. Furthermore suppose that leading coefficients of P and B_j with respect to D_n are equal to 1 and $m_1<\cdots<m_l<m$.

DEFINITION 1. *The boundary value problem (P, B_j) with homogeneous boundary conditions is said to be L^2-well posed if and only if there exist positive constants γ_0, C_0 such that for any $\gamma\geqq\gamma_0$ and for any $f\in H_{0,0;,\gamma}$, the problem has a unique solution $u\in H_{m,-1;\gamma}$ which satisfies*

$$\|u\|_{m,-1;\gamma}\leqq\frac{C_0}{\gamma}\|f\|_{0,0;\gamma}. \tag{1.1}$$

For notations see § 2.

This definition was posed in a different form by R. Agemi and T. Shirota in researches [2] for hyperbolic mixed problems with vanishing initial data in the quadrant $\{x=(x_0, x_1, \cdots, x_{n-1}, x_n);\ x_0>0,\ x_n>0\}$. But they are equivalent to each other when P and B_j are homogeneous and of constant coefficients.

In this paper we study on L^2-well posedness of the dual problems and on the differentiability of solutions of L^2-well posed problems. (See Sakamoto [6], Rauch and Massey III [5]).

Let $P^*(x, D)$ be the formal adjoint of $P(x, D)$. By the assumptions of $P(x, D)$ and $B_j(x', D)$, $j=1, \cdots, l$, we see that there exist differential operators $B_{l+1}(x', D), \cdots, B_m(x', D)$; $B_1'(x', D), \cdots, B_m'(x', D)$ such that

$$(1.2)\qquad (Pu, v)-(u, P^*v) = \sum_{j=1}^{m} i\langle B_j u, B_j' v\rangle\,, \qquad u, v \in C_0^\infty(\overline{\boldsymbol{R}_+^{n+1}})\,,$$

$m_j+r_j=m-1$, $j=1, \cdots, m$, and $\{B_1, \cdots, B_m\}$, $\{B_1', \cdots, B_m'\}$ are Dirichlet sets, where $(\cdot, \cdot)$, $\langle\cdot, \cdot\rangle$ are the inner products of $L^2(\boldsymbol{R}_+^{n+1})$, $L^2(\boldsymbol{R}^n)$ respectively, and m_j, r_j are the orders of B_j, B_j' respectively (see Schechter [7]). From now on we mean by B_j, $j=l+1, \cdots, m$; B_j', $j=1, \cdots, m$, operators with the above properties.

Our main results are the following

THEOREM 1. *Let the problem* (P, B_j) *with homogeneous boundary conditions be* L^2*-well posed. Then for every integer* k, s $(k\geqq 0)$ *there exist positive constants* $\gamma_{k,s}$, $C_{k,s}$ *such that for any* $\gamma\geqq\gamma_{k,s}$ *and for any* $f\in H_{k,s;\gamma}$, $f_j\in H_{m-m_j-\frac{1}{2}+k+s;\gamma}$, $j=1, \cdots, l$, *the problem* (P, B_j) *has a unique solution* $u\in H_{m+k,s-1;\gamma}$ *which satisfies*

$$(1.3)\qquad \|u\|_{m+k,s-1;\gamma} \leqq \frac{C_{k,s}}{\gamma}\left(\|f\|_{k,s;\gamma} + \sum_{j=1}^{l}\langle f_j\rangle_{m-m_j-\frac{1}{2}+k+s;\gamma}\right).$$

THEOREM 2. *Let the hypothesis of Theorem* 1 *be fulfilled. Then for every integer* k, $s(k\geqq 0)$ *there exist positive constants* $\gamma_{k,s}'$, $C_{k,s}'$ *such that for any* $\gamma\geqq\gamma_{k,s}'$ *and for any* $g\in H_{k,s;-\gamma}$, $g_j\in H_{m-r_j-\frac{1}{2}+k+s;-\gamma}$, $j=l+1, \cdots, m$, *the problem* (P^*, B_j'):

$$\begin{cases} P^*v = g & \text{in } \boldsymbol{R}_+^{n+1}\,, \\ B_j'v = g_j & \text{on } \boldsymbol{R}^n\,, \quad j=l+1, \cdots, m \end{cases}$$

has a unique solution $v\in H_{m+k,s-1;-\gamma}$ *which satisfies*

$$(1.4)\qquad \|v\|_{m+k,s-1;-\gamma} \leqq \frac{C_{k,s}'}{\gamma}\left(\|g\|_{k,s;-\gamma} + \sum_{j=l+1}^{m}\langle g_j\rangle_{m-r_j-\frac{1}{2}+k+s;-\gamma}\right).$$

In the above estimates it is important to notice that we can't replace $-\dfrac{1}{2}$ by -1. (See Rauch [4]).

The methods used in the present note are usual, but the necessary and sufficient condition for L^2-well posedness stated in Remark 1) of §5 is useful in further investigations, for example, see Agemi's forthcoming paper [1].

We can also obtain the similar results for first order systems.

The author would like to thank Professor T. Shirota who suggested the problem and gave constant encouragement to him.

§2. Preliminaries

We shall use some spaces. For real numbers p, q and a positive number

γ, we denote by $H_{p,q;\pm\gamma}(\boldsymbol{R}^{n+1})$ the spaces of $u\in\mathscr{D}'(\boldsymbol{R}^{n+1})$ such that $e^{\mp\gamma x_0}u\in H_{p,q}(\boldsymbol{R}^{n+1})$, the norms in $H_{,q;\pm\gamma}(\boldsymbol{R}^{n+1})$ are defined by

$$\|u\|^2_{p,q;\pm\gamma}=\int_{R^{n+1}}(\gamma^2+|\xi|^2)^p(\gamma^2+|\xi'|^2)^q|\widehat{e^{\mp\gamma x_0}u}(\xi)|^2d\xi\,,$$

where ξ, ξ' are the dual variables of x, x' respectively, $|\xi|^2=\sum_{j=0}^{n}\xi_j^2$, $|\xi'|^2=\sum_{j=0}^{n-1}\xi_j^2$, and where $\widehat{e^{\mp\gamma x_0}u}(\xi)$ are the Fourier transformations of $e^{\mp\gamma x_0}u$. Similarly we define $H_{q;\pm\gamma}(\boldsymbol{R}^n)$ with the norms

$$\langle u\rangle^2_{q;\pm\gamma}=\int_{R^n}(\gamma^2+|\xi'|^2)^q|\widehat{e^{\mp\gamma x_0}u}(\xi')|^2d\xi'\,.$$

By $H_{p,q;\pm\gamma}(\boldsymbol{R}^{n+1}_+)$ we mean the sets of all $u\in\mathscr{D}'(\boldsymbol{R}^{n+1}_+)$ respectively such that there exist distributions $U\in H_{p,q;\pm\gamma}(\boldsymbol{R}^{n+1})$ with $U=u$ in $\boldsymbol{R}^{n+1}_+$. The norms of u are defined respectively by

$$\|u\|_{p,q;\pm\gamma}=\inf_U\|U\|_{p,q;\pm\gamma}\,.$$

Finally, we set

$$\mathring{H}_{p,q;\pm\gamma}(\overline{\boldsymbol{R}^{n+1}_+})=\{u\,;\ u\in H_{p,q;\pm\gamma}(\boldsymbol{R}^{n+1}),\ \text{supp}\ u\subset\overline{\boldsymbol{R}^{n+1}_+}\}\,.$$

From now on, for simplicity we denote by $H_{q;\pm\gamma}$, $H_{p,q;\pm\gamma}$, $\mathring{H}_{p,q;\pm\gamma}$ the spaces $H_{q;\pm\gamma}(\boldsymbol{R}^n)$, $H_{p,q;\pm\gamma}(\boldsymbol{R}^{n+1}_+)$, $\mathring{H}_{p,q;\pm\gamma}(\overline{\boldsymbol{R}^{n+1}_+})$ respectively.

The following lemma can be proved in the same way as in Theorem 2.5.1 of Hörmander [3].

LEMMA 2.1. *$C_0^\infty(\overline{\boldsymbol{R}^{n+1}_+})$ is dense in $H_{p,q;\pm\gamma}$ and $C_0^\infty(\boldsymbol{R}^{n+1}_+)$ is dense in $\mathring{H}_{p,q;\pm\gamma}$. The spaces $H_{p,q;\pm\gamma}$ and $\mathring{H}_{-p,-q;\mp\gamma}$ are dual Hilbert spaces with respect to extensions of the sesquilinear form*

$$\int_{R^{n+1}_+}u\bar{v}dx\,;\ u\in C_0^\infty(\overline{\boldsymbol{R}^{n+1}_+}),\ v\in C_0^\infty(\boldsymbol{R}^{n+1}_+)\,.$$

The following lemma is a variant of Theorem 2.5.4 in [3], which can be proved in the same way as in the proof of the theorem by using Lemma 2.1.

LEMMA 2.2. *Let $u\in H_{p-1,q+1;\pm\gamma}$ and $D_n^m u\in H_{p-m,q;\pm\gamma}$. Then $u\in H_{p,q;\pm\gamma}$ and we have*

$$\|u\|_{p,q;\pm\gamma}\leqq m(\|u\|_{p-1,q+1;\pm\gamma}+\|D_n^m u\|_{p-m,q;\pm\gamma})\,.$$

From Lemma 2.2 it follows

COROLLARY 2.3. *Let $Q(x, D)$ be a differential operator of order m such that the coefficient of D_n^m is equal to 1. If $u\in H_{-1,q+1;\pm\gamma}$ and $Qu\in H_{p-m,q;\pm\gamma}$, then $u\in H_{p,q;\pm\gamma}$ and we have for $\gamma\geqq\gamma_0$*

$$\|u\|_{p,q;\pm\gamma}\leqq C(\|u\|_{p-1,q+1;\pm\gamma}+\|Qu\|_{p-m,q;\pm\gamma}),$$

where C depends on p and q but not on γ or u.

§3. Proof of Theorem 1

The following lemma can be proved in the same way as in Theorem 2.5.7 of [3].

LEMMA 3.1. *Let $E_j(x', D)$, $j=0, 1, \cdots, m-1$, be boundary operators of orders j such that the leading coefficients with respect to D_n are equal to 1. Let q be an integer. Then for any $\gamma>0$ and $g_j\in H_{m-j-\frac{1}{2}+q;\gamma}$, $j=0, 1, \cdots, m-1$, there exists a function $w\in H_{m,q;\gamma}$ such that*

$$E_jw|_{x_n=0}=g_j, \qquad j=0, 1, \cdots, m-1,$$

and

$$\|w\|_{m,q;\gamma}\leqq C_q\sum_{j=0}^{m-1}\langle g_j\rangle_{m-j-\frac{1}{2}+q;\gamma}, \qquad \gamma\geqq\gamma_0,$$

where C_q is independent of γ and g_j.

LEMMA 3.2. *Let the hypothesis of Theorem 1 be fulfilled. Then there exists a constant $C_1>0$ such that for any $\gamma\geqq\gamma_0$ and any $f\in H_{0,0;\gamma}$, $f_j\in H_{m-m_j-\frac{1}{2};\gamma}$, $j=1, \cdots, l$, (P, B_j) has a unique solution $u\in H_{m,-1;\gamma}$ which satisfies*

$$\|u\|_{m,-1;\gamma}\leqq\frac{C_1}{\gamma}\left(\|f\|_{0,0;\gamma}+\sum_{j=1}^{l}\langle f_j\rangle_{m-m_j-\frac{1}{2};\gamma}\right). \tag{3.1}$$

PROOF. We can assume that the leading coefficients of B_j, $j=1, \cdots, m$, with respect to D_n are equal to 1. If in Lemma 3.1 with $q=0$ we set

$$E_{m_j}=B_j, \qquad j=1, \cdots, m,$$

and

$$\begin{aligned} g_{m_j}&=f_j, & j&=1, \cdots, l, \\ &=0, & j&=l+1, \cdots, m, \end{aligned}$$

then we see that there exists a function $w\in H_{m,0;\gamma}$ such that for $\gamma\geqq\gamma_0$

$$B_jw|_{x_n=0}=f_j, \qquad j=1, \cdots, l,$$

and

(3. 2)
$$\|w\|_{m,0;\gamma} \leqq C \sum_{j=1}^{l} \langle f_j \rangle_{m-m_j-\frac{1}{2};\gamma}.$$

Since $Pw \in H_{0,0,;\gamma}$, by the existence of solutions to (P, B_j) with homogeneous boundary conditions and (1. 1) we find that for $\gamma \geqq \gamma_0$ there exists a solution $v \in H_{m,-1;\gamma}$ of the problem

$$\begin{cases} Pv = f - Pw & \text{in } \boldsymbol{R}_+^{n+1}, \\ B_j v = 0 & \text{on } \boldsymbol{R}^n, \quad j = 1, \cdots, l \end{cases}$$

which satisfies

(3. 3)
$$\|v\|_{m,-1;\gamma} \leqq \frac{C_0}{\gamma} \|f - Pw\|_{0,0,;\gamma}.$$

Set $u = v + w$. Then u is a solution of (P, B_j). Furthermore from (3. 2) and (3. 3) we obtain (3. 1) with another constant C_1, since

$$\|w\|_{m,-1;\gamma} \leqq \frac{1}{\gamma} \|w\|_{m,0;\gamma}.$$

The uniqueness follows immediately from the one of solutions to (P, B_j). The proof is complete.

LEMMA 3. 3. *Let the hypothesis of Theorem* 1 *be fulfilled. Then for every integer s there exist positive constants $\gamma_{0,s}$ and $C_{0,s}$ such that for any $\gamma \geqq \gamma_{0,s}$ and any $f \in H_{0,s;\gamma}$, $f_j \in H_{m-m_j-\frac{1}{2}+s;\gamma}$ (P, B_j) has a unique solution $u \in H_{m,s-1;\gamma}$ which satisfies*

(3. 4)
$$\|u\|_{m,s-1;\gamma} \leqq \frac{C_{0,s}}{\gamma} \left(\|f\|_{0,s;\gamma} + \sum_{j=1}^{l} \langle f_j \rangle_{m-m_j-\frac{1}{2}+s;\gamma} \right).$$

PROOF. For $\gamma > 0$ and $u \in C_0^\infty(\boldsymbol{R}^n)$ we define $\Lambda_\gamma u$ as follows:

$$(\Lambda_\gamma u)(x') = e^{\gamma x_0} \mathscr{F}^{-1}\left[(\gamma^2 + |\xi'|^2)^{\frac{1}{2}} \mathscr{F}(e^{-\gamma x_0} u) \right],$$

where $\mathscr{F}$ and $\mathscr{F}^{-1}$ are the Fourier transform and the Fourier inverse transform respectively. Then for real number q

$$\langle \Lambda_\gamma u \rangle_{q-1;\gamma} = \langle u \rangle_{q;\gamma}, \qquad u \in C_0^\infty(\boldsymbol{R}^n).$$

Therefore Λ_γ is so extended on $H_{q;\gamma}$ that $H_{q;\gamma}$ and $H_{q-1;\gamma}$ are isomorphic to each other. Λ_γ is also regarded as a isomorphism from $H_{p,q;\gamma}$ to $H_{p,q-1;\gamma}$ for real numbers p and q such that

$$\|\Lambda_\gamma u\|_{p,q-1;\gamma} = \|u\|_{p,q;\gamma}, \qquad u \in H_{p,q;\gamma}.$$

Now we set for an integer s

$$u = \Lambda_\gamma^{-s} v .$$

Then (P, B_j) is equivalent to the problem

$$(3.5)\qquad \begin{cases} Pv + \Lambda_\gamma^s (P\Lambda_\gamma^{-s} - \Lambda_\gamma^{-s} P) v = \Lambda_\gamma^s f & \text{in } \boldsymbol{R}_+^{n+1}, \\ B_j v + \Lambda_\gamma^s (B_j \Lambda_\gamma^{-s} - \Lambda_\gamma^{-s} B_j) v = \Lambda_\gamma^s f_j & \text{on } \boldsymbol{R}^n, \quad j=1, \cdots, l. \end{cases}$$

If $f \in H_{0,s;\gamma}$ and $f_j \in H_{m-m_j-\frac{1}{2}+s;\gamma}$, then $\Lambda_\gamma^s f \in H_{0,0;\gamma}$ and $\Lambda_\gamma^s f_j \in H_{m-m_j-\frac{1}{2};\gamma}$. Therefore using Lemma 3.2 we find by a standard perturbation method that there exist positive constants $\gamma_{0,s}$ and $C_{0,s}$ such that for any $\gamma \geqq \gamma_{0,s}$ and any $f \in H_{0,s;\gamma}$, $f_j \in H_{m-m_j-\frac{1}{2}+s;\gamma}$ (3.5) has a unique solution $v \in H_{m,-1;\gamma}$ satisfying

$$(3.6)\qquad \|v\|_{m,-1;\gamma} \leqq \frac{C_{0,s}}{\gamma} \left(\|\Lambda_\gamma^s f\|_{0,0,;\gamma} + \sum_{j=1}^{l} \langle \Lambda_\gamma^s f_j \rangle_{m-m_j-\frac{1}{2};\gamma} \right),$$

because that there exist a positive constant C_s' such that for $\gamma \geqq \gamma_0$ and $v \in H_{m,-1;\gamma}$

$$\|\Lambda_\gamma^s (P\Lambda_\gamma^{-s} - \Lambda_\gamma^{-s} P) v\|_{0,0;\gamma} \leqq C_s' \|v\|_{m,-1;\gamma},$$
$$\langle \Lambda_\gamma^s (B_j \Lambda_\gamma^{-s} - \Lambda_\gamma^{-s} B_j) v \rangle_{m-m_j-\frac{1}{2};\gamma} \leqq C_s' \|v\|_{m,-1;\gamma}.$$

Since (3.6) is equivalent to (3.4), the proof is complete.

PROOF OF THEOREM 1. Let the hypothesis of the theorem be fulfilled. Since $H_{k,s;\gamma} \subset H_{0,k+s;\gamma}$ and the leading coefficient of P with respect to D_n is equal to 1, the assertion of the theorem follows immediately from Lemma 3.3 and Corollary 2.3.

§4. Proof of Theorem 2

LEMMA 4.1 (existence of solutions). *Suppose that for every $\gamma \geqq \gamma_0$ we have*

$$(4.1)\qquad \|u\|_{m,-1;\gamma} \leqq \frac{C_1}{\gamma} \left(\|Pu\|_{0,0;\gamma} + \sum_{j=1}^{l} \langle B_j u \rangle_{m-m_j-\frac{1}{2};\gamma} \right), \qquad u \in H_{m,0;\gamma}.$$

Then for every integer s there exists a positive constant γ_s' such that for any $\gamma \geqq \gamma_s'$ and any $g \in H_{0,s;-\gamma}$, $g_j \in H_{m-r_j-\frac{1}{2}+s;-\gamma}$, $j=l+1, \cdots, m$ (P^*, B_j') has a solution $v \in H_{m,s-1;-\gamma}$.

Proof. Let q be an integer. By (4.1) we have for $u \in H_{m,q;\gamma}$

$$(4.2)\qquad \|\Lambda_\gamma^q u\|_{m,-1;\gamma} \leqq \frac{C_1}{\gamma} \left(\|P\Lambda_\gamma^q u\|_{0,0;\gamma} + \sum_{j=1}^{l} \langle B_j \Lambda_\gamma^q u \rangle_{m-m_j-\frac{1}{2};\gamma} \right).$$

Since $\|(P\Lambda_\gamma^q - \Lambda_\gamma^q P)u\|_{0,0;\gamma}$ and $\langle (B_j \Lambda_\gamma^q - \Lambda_\gamma^q B_j) u \rangle_{m-m_j-\frac{1}{2};\gamma}$ are estimated by $\tilde{C}_q$ $\|u\|_{m,q-:;\gamma}$ $(\gamma \geqq \gamma_0)$, where $\tilde{C}_q$ is independent of γ and u, it follows from (4.2) there exist positive constants γ_q and C_q such that for $\gamma \geqq \gamma_q$ we have

$$(4.3)\qquad \|u\|_{m,q-1;\gamma}\leqq\frac{C_q}{\gamma}\Big(\|Pu\|_{0,q;\gamma}+\sum_{j=1}^{l}\langle B_ju\rangle_{m-m_j-\frac{1}{2}+q;\gamma}\Big),\qquad u\in H_{m,q;\gamma}.$$

Set

$$D_q=\{u;\ u\in H_{m,q;\gamma},\ B_ju=0\quad \text{on } \boldsymbol{R}^n,\ j=1,\cdots,l\}.$$

Then according to (4.3) D_q is a pre-Hilbert space with the norm $\|Pu\|_{0,q;\gamma}$ for $\gamma\geqq\gamma_q$. We denote by $\mathscr{H}_{q;\gamma}$ the completion of D_q with respect to $\|Pu\|_{0,q;\gamma}$ and denote by $[u,w]_{q;\gamma}$ the inner product of $\mathscr{H}_{q;\gamma}$. Notice that $\mathscr{H}_{q;\gamma}\subset H_{m,q-1;\gamma}$ and

$$(4.3)'\qquad \|u\|_{m,q-1;\gamma}\leqq\frac{C_q}{\gamma}\sqrt{[u,u]_{q;\gamma}},\qquad u\in\mathscr{H}_{q;\gamma}.$$

Let $g\in H_{0,s;-\gamma}$ and $g_j\in H_{m-r_j-\frac{1}{2}+s;-\gamma}$ (s: integer). Furthermore set for $u\in\mathscr{H}_{-(m+s-1);\gamma}$

$$F(u)=(u,g)+\sum_{j=l+1}^{m}i\langle B_ju,g_j\rangle.$$

Then by (4.3)$'$ with $q=-(m+s-1)$, $F(u)$ is continuous in $\mathscr{H}_{-(m+s-1);\gamma}$. Hence we see by Riesz's theorem that there exists $w\in\mathscr{H}_{-(m+s-1);\gamma}$ such that for all $u\in\mathscr{H}_{-(m+s-1);\gamma}$

$$F(u)=[u,w]_{-(m+s-1);\gamma}.$$

Notice that

$$[u,w]_{q;\gamma}=\int_{\boldsymbol{R}^{n+1}_+}(\gamma^2+|\xi'|^2)^{\frac{q}{2}}\mathscr{F}[e^{-\gamma x_0}Pu]\cdot\overline{(\gamma^2+|\xi'|^2)^{\frac{q}{2}}\mathscr{F}[e^{-\gamma x_0}Pw]}\,d\xi'dx_n.$$

Set

$$v=e^{-\gamma x_0}\mathscr{F}^{-1}[(\gamma^2+|\xi'|^2)^{-(m+s-1)}\mathscr{F}(e^{-\gamma x_0}Pw)].$$

Then $v\in H_{0,m+s-1;-\gamma}$ and we have for $u\in D_{-(m+s-1)}$

$$(4.4)\qquad (Pu,v)=(u,g)+\sum_{j=l+1}^{m}i\langle B_ju,g_j\rangle,$$

which implies that

$$(4.5)\qquad P^*v=g\quad\text{in } \mathscr{D}'(\boldsymbol{R}^{n+1}_+).$$

Since $v\in H_{0,m+s-1;-\gamma}$ and $g\in H_{0,s;-\gamma}$, by (4.5) and Corollary 2.3 we find that $v\in H_{m,s-1;-\gamma}$. Hence using (4.4), (4.5) and (1.2) extended by continuity, we have for $u\in D_{-(m+s-1)}$

$$\sum_{j=l+1}^{m} \langle B_j u,\ B'_j v - g_j\rangle = 0\,. \tag{4.6}$$

Since by Lemma 3.1 $B_j u$ are arbitrary, we have

$$B'_j v = g_j \qquad \text{in } \mathscr{D}'(\boldsymbol{R}^n),\quad j=l+1, \cdots, m.$$

Thus the proof is complete, if we set $\gamma'_s = \gamma_{-(m+s-1)}$.

The following lemma is derived immediately from (1.2) extended by continuity.

LEMMA 4.2 (uniqueness of solutions). *Suppose that for any* $f\in C_0^\infty(\boldsymbol{R}_+^{n+1})$ (P, B_j) *with homogeneous boundary conditions has a solution* $u\in H_{m,-(m+s-1);\gamma}$. *Then the solution of* (P^*, B'_j) *is unique in* $H_{m,s-1;-\gamma}$.

The following lemma is used to obtain (1.4).

LEMMA 4.3 (a priori estimate). *Suppose that for any* $\gamma\geqq\gamma_0$ *and any* $f\in C_0^\infty(\boldsymbol{R}_+^{n+1})$ (P, B_j) *with homogeneous boundary conditions has a solution* $u\in H_{m,-1;\gamma}$ *satisfying* (1.1). *Then for every integer s there exist positive constants* γ''_s *and* C''_s *such that for any* $\gamma\geqq\gamma''_s$ *we have*

$$\|v\|_{m,s-1;-\gamma} \leqq \frac{C''_s}{\gamma}\left(\|P^*v\|_{0,s;-\gamma} + \sum_{j=l+1}^{m}\langle B'_j v\rangle_{m-r_j-\frac{1}{2}+s;-\gamma}\right),\quad v\in H_{m,s;-\gamma}. \tag{4.7}$$

PROOF. It follows from the hypothesis that for any $\gamma\geqq\gamma_0$ and $f\in C_0^\infty(\boldsymbol{R}_+^{n+1})$ there exists a function $u\in H_{m,-1;\gamma}$ which satisfies (1.1) and

$$\begin{cases} Pu = f & \text{in } \boldsymbol{R}_+^{n+1}, \\ B_j u = 0 & \text{on } \boldsymbol{R}^n,\quad j=1, \cdots, l\,. \end{cases} \tag{4.8}$$

Let $w\in H_{m,1-m;-\gamma}$. Then (4.8) and (1.2) imply

$$(f, w) - (u, P^*w) = \sum_{j=l+1}^{m} i\langle B_j u, B'_j w\rangle\,. \tag{4.9}$$

From (4.9) we have by (1.1) and the trace inequality

$$\begin{aligned} |(f, w)| &\leqq \|u\|_{0,m-1;\gamma}\|P^*w\|_{0,1-m;-\gamma} + C\|u\|_{m,-1;\gamma}\sum_{j=l+1}^{m}\langle B'_j w\rangle_{-r_j+\frac{1}{2};-\gamma} \\ &\leqq \frac{C_0(1+C)}{\gamma}\|f\|_{0,0;\gamma}\left(\|P^*w\|_{0,1-m;-\gamma} + \sum_{j=l+1}^{m}\langle B'_j w\rangle_{-r_j+\frac{1}{2};-\gamma}\right), \end{aligned} \tag{4.10}$$

where C is independent of γ, f and w. Using the duality in Lemma 2.1 we obtain from (4.10)

$$\|w\|_{0,0;-\gamma} \leqq \frac{C_0(1+C)}{\gamma}\left(\|P^*w\|_{0,1-m;-\gamma} + \sum_{j=l+1}^{m}\langle B'_j w\rangle_{-r_j+\frac{1}{2};-\gamma}\right). \tag{4.11}$$

Furthermore from (4.11) and Corollary 2.3 we have

$$(4.12)\qquad \|w\|_{m,-m;-\gamma} \leqq \frac{C'}{\gamma}\left(\|P^*w\|_{0,1-m;-\gamma} + \sum_{j=l+1}^{m} \langle B_j'w\rangle_{-r_j+\frac{1}{2};-\gamma}\right),$$

$$w \in H_{m,1-m;-\gamma},$$

where C' is independent of γ and w.

Let $v\in H_{m,s;-\gamma}$ and set $w=\Lambda_{-\gamma}^{m+s-1}v$. Then in the same way as (4.3) was derived from (4.1), from (4.12) we obtain (4.7) with constants γ_s'' and C_s'' independent of γ and v.

PROOF OF THEOREM 2. By virtue of Corollary 2.3 it is sufficient to prove the theorem for $k=0$. Let the hypothesis of Theorem 1 be fulfilled. Furthermore let s be an integer. The existence of solutions follows from Lemmas 3.2 and 4.1. The uniqueness of solutions follows from Lemmas 3.3 and 4.2.

Now we shall prove (1.4) with $k=0$. Set

$$\gamma_{0,s}' = \max\{\gamma_s',\ \gamma_{s+1}',\ \gamma_{0,-(m+s-2)},\ \gamma_{0,-(m+s-1)},\ \gamma_s''\},$$

where γ_q', $\gamma_{0,q}$ and γ_q'' are the constants in Lemmas 4.1, 3.3 and 4.3, and let $\gamma\geqq\gamma_{0,s}'$. Furthermore let $g\in H_{0,s;-\gamma}$, $g_j\in H_{m-r_j-\frac{1}{2}+s;-\gamma}$, $j=l+1,\cdots,m$, and $v\in H_{m,s-1;-\gamma}$ be the unique solution of (P^*, B_j'). Then there exist $g^\nu\in H_{0,s+1;-\gamma}$ and $g_j^\nu\in H_{m-r_j-\frac{1}{2}+s+1;-\gamma}$ such that when $\nu\to\infty$

$$g^\nu \longrightarrow g \qquad \text{in } H_{0,s;-\gamma},$$

$$g_j^\nu \longrightarrow g_j \qquad \text{in } H_{m-r_j-\frac{1}{2}+s;-\gamma}.$$

Let $v^\nu\in H_{m,s;-\gamma}$ be the unique solution of the problem

$$\begin{cases} P^*v^\nu = g^\nu & \text{in } \boldsymbol{R}_+^{n+1}, \\ B_j'v^\nu = g_j^\nu & \text{on } \boldsymbol{R}^n,\quad j=l+1,\cdots,m. \end{cases}$$

Then by Lemma 4.3 we find a function $w\in H_{m,s-1;-\gamma}$ such that when $\nu\to\infty$

$$v^\nu \longrightarrow w \qquad \text{in } H_{m,s-1;-\gamma}.$$

Therefore w is a solution of (P^*, B_j') which satisfies (1.4) with $k=0$ and $C_{0,s}'=C_s''$. The uniqueness of solutions to (P^*, B_j') implies that $w=v$. The proof is complete.

COROLLARY 4.4. *Let C_j' be a linear operator such that*

$$C_j' = \sum_{k=0}^{r_j-1} \Gamma_{jk} D_n^k,$$

where for every real number ν, Γ_{jk} is a bounded operator from $H_{r_j-1-k+\nu;-\gamma}$ to $H_{\nu;-\gamma}$ whose operator norm has a bound independent of sufficiently large γ.

If we replace B'_j by $B'_j+C'_j$ in Theorem 2, then also the assertion of the theorem is valid.

§ 5. Remarks

1). Suppose that there exist positive constants γ_0 and C_0 such that for every $\gamma \geqq \gamma_0$

$$\|u\|_{m,-1;\gamma} \leqq \frac{C_0}{\gamma}\left(\|Pu\|_{0,0;\gamma} + \sum_{j=1}^{l} \langle B_j u\rangle_{m-m_j-\frac{1}{2};\gamma}\right), \qquad u \in H_{m,0;\gamma}$$

and

$$\|v\|_{m,-1;-\gamma} \leqq \frac{C_0}{\gamma}\left(\|P^*v\|_{0,0;-\gamma} + \sum_{j=l+1}^{m} \langle B'_j v\rangle_{m-r_j-\frac{1}{2};-\gamma}\right), \qquad v \in H_{m,0;-\gamma}.$$

Then the problem (P, B_j) with homogeneous boundary conditions is L^2-well posed.

In fact, by Lemma 4.1 the latter inequality implies the existence theorem for (P, B_j) and the former one derives (4.3) from which it follows the uniqueness theorem.

2). We can also prove Theorem 2 without Lemmas 3.1, 3.2 and 3.3.

Let the hypothesis of Theorem 1 be fulfilled. In the proof of Lemma 4.1, instead of D_q and (4.3) we use respectively

$$D = \left\{u;\ u \in C_0^\infty(\overline{\boldsymbol{R}_+^{n+1}}),\quad B_j u = 0 \quad \text{on } \boldsymbol{R}^n,\ j=1,\cdots,l\right\}$$

and

$$(1.1)' \qquad \|u\|_{m,-1;\gamma} \leqq \frac{C_0}{\gamma}\|Pu\|_{0,0;\gamma}, \qquad u \in D.$$

Then (4.3)′ is valid, if we set $q=0$ (consequently $s=1-m$). Furthermore under (4.6), instead of Lemma 3.1 we use the fact that $B_j u|_{x_n=0}$, $u \in D$, $j=l+1,\cdots,m$ can become arbitrary C_0^∞-functions, because that $\{B_1,\cdots,B_m\}$ is a Dirichlet set. (See [7]). Then we see that the assertion of Lemma 4.1 with $s=1-m$ is valid. The uniqueness of solutions in $H_{m,-m;-\gamma}$ follows from (4.7) with $s=-m$. Thus in the same way as in the proof of Theorem 2 we find by (4.7) with $s=-m$ and $s=-m-1$ that the assertion of Theorem 2 with $k=0$ and $s=-m$ is valid. Therefore we can prove Theorem 2 in the same way in the proofs of Lemma 3.3 and Theorem 1.

3). Adding to the hypothesis of Theorem 1, suppose that $s \geqq 0$ and $f=f_j=0$ for $x_0<0, j=1,\cdots,l$. Then $u=0$ for $x_0<0$.

Proof. Let $u_\gamma \in H_{m,-1;\gamma}$ be the unique solution of (P, B_j). By virtue of

(1.3) it is sufficient to show that u_γ is independent of γ. We say that u is a weak solution of (P, B_j+C_j) in $H_{0,0;\gamma}$ if and only if $u\in H_{0,0;\gamma}$, $f\in H_{0,0;\gamma}$, $f_j\in H_{0;\gamma}$ and it holds

$$(f, v)-(u, P^*v) = \sum_{j=1}^{l} i\langle f_j, (B'_j+C'_j)v\rangle$$

for all $v\in C_0^\infty(\overline{\boldsymbol{R}_+^{n+1}})$ with $(B'_j+C'_j)v|_{x_n=0}=0$, $j=l+1, \cdots, m$, where $C_j=C_j(x', D)$ and $C'_j=C'_j(x', D)$ are operators of orders m_j-1 and r_j-1 respectively such that (1.2) holds when we replace $B_j, j=1, \cdots, l$, and B'_j, $j=1, \cdots, m$ by B_j+C_j and $B'_j+C'_j$ respectively. Using Corollary 4.4 we see that the weak solution of (P, B_j+C_j) is unique in $H_{0,0;\gamma}$ for $\gamma\geqq\gamma'_0$. Therefore we find by Proposition 1.3 in [6] that u_γ is independent of $\gamma\geqq\gamma''_0$.

4). For first order systems the similar results are valid. We consider the boundary value problem (L, B):

$$\begin{cases} L(x, D) \equiv \sum_{j=0}^{n} A_j(x)D_ju + C(x)u = f & \text{in } \boldsymbol{R}_+^{n+1}, \\ B(x')u = g & \text{on } \boldsymbol{R}^n, \end{cases}$$

where $A_j(x)$ and $C(x)$ are $m\times m$ matrix-valued functions and $B(x')$ is a $l\times m$ matrix-valued function. Suppose that $A_n(x)$ is the unit matrix and rank $B(x')=l$ for every $x'\in\boldsymbol{R}^n$. Let $b_1(x'), \cdots, b_l(x')$ be the rows of $B(x')$. For every $x'\in\boldsymbol{R}^n$ we denote $N(x')$ the orthogonal complement of the subspace generated by $b_1(x'), \cdots, b_l(x')$. Furthermore suppose that there exists a smooth basis $b_{l+1}(x'), \cdots, b_m(x')$ of $N(x')$. Set

$$T(x) = T(x', x_n) = \begin{pmatrix} b_1(x') \\ \vdots \\ b_m(x') \end{pmatrix}$$

and

$$\tilde{u}(x) = T(x)u(x).$$

Then the problem (L, B) is equivalent to the problem

$$\begin{cases} \tilde{L}\tilde{u} \equiv \sum_{k=0}^{n} (TA_kT^{-1})D_k\tilde{u} + (TLT^{-1})\tilde{u} = Tf & \text{in } \boldsymbol{R}_+^{n+1}, \\ \tilde{u}_j = g_j \quad \text{on } \boldsymbol{R}^n, & j=1, \cdots, l. \end{cases}$$

Let $\tilde{L}^*$ be the formal adjoint of $\tilde{L}$. Then the Green's formula for $\tilde{L}$ and $\tilde{L}^*$ is

$$(\tilde{L}\tilde{u}, \tilde{v}) - (\tilde{u}, \tilde{L}^*v) = i\langle\tilde{u}, \tilde{v}\rangle, \qquad \tilde{u}, \tilde{v}\in C_0^\infty(\overline{\boldsymbol{R}_+^{n+1}}),$$

and the adjoint boundary conditions are

$$\tilde{v}_j = h_j \qquad \text{on } \boldsymbol{R}^n, \quad j=l+1, \cdots, m.$$

Therefore we can prove the corresponding theorems by the same argument as in the preceding.

Department of Mathematics
Hokkaido University

References

[1] R. AGEMI: Remarks on L^2-well-posed mixed problems for hyperbolic equations of second order, to appear.

[2] R. AGEMI and T. SHIROTA: On necessary and sufficient conditions for L^2-well-posedness of mixed problems for hyperbolic equations, J. Fac. Sci. Hokkaido Univ., **21** (1970), 133–151.

[3] L. HÖRMANDER: Linear partial differential operators, Chapter II, Springer-Verlag, (1963).

[4] J. B. RAUCH: Energy inequalities for hyperbolic initial boundary value problems, Thesis, New York University, (1971).

[5] J. B. RAUCH and F. J. MASSEY III: Differentiability of solutions to hyperbolic initial-boundary value problems, to apper.

[6] R. SAKAMOTO: Mixed problems for hyperbolic equations II, J. Math. Kyoto Univ., **10** (1970), 403–417.

[7] M. SCHECHTER: General boundary value problems for elliptic partial differential equations, Comm. Pure Appl. Math., **12** (1959), 457–486.

(Received August **30**, 1972)