# Logmodular parts of function algebras

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**1. Introduction** Let $A$ be a function algebra on a compact Hausdorff space $X$, that is, a closed subalgebra of the complex Banach algebra $C(X)$, which contains the constant functions and separates the points of $X$. Gleason discovered that $\|\varphi-\psi\|<2$ defines an equivalent relation on the set *Spec* $A$ of all multiplicative linear functionals of $A$ while Bishop introduced the other metric $\sigma(\varphi, \psi)$ on *Spec* $A$ such that $\sigma(\varphi, \psi)<1$ defines the same equivalent relations as Gleason's, and König established an algebraic relation between these two metrics (see [1; 143∼144] and [2, 3]). Importance of Gleason's equivalent classes, called Gleason parts, is shown by the fact that $\varphi$ and $\psi$ belong to the same Gleason part if and only if they admit mutually dominating representing measures in the sense that there are probability measures $\mu$ and $\nu$ on $X$ such that for all $f$ in $A$

$$\varphi(f)=\int f d\mu, \quad \psi(f)=\int f d\nu$$

and $\mu/k\leq\nu\leq k\mu$ for some positive constant $k$. The minimum $k$ can be determind in terms of $\sigma(\varphi, \psi)$ (see [2, 3]).

Let $E$ be a subset of $C(X)$, which is closed under multiplication and contains the constant functions. $M(E)$ is the set of all cotinuous multiplicative functions $\Phi$ of $E$ to the non-negative real numbers with $\|\Phi\|_E\leq 1$, where $\|\cdot\|_E$ denotes the supremum on the set of $f$ with $\|f\|\leqq 1$. We shall show that $\sigma_E(\Phi, \Psi)=\sup_{r>0}\|\Phi^r-\Psi^r\|_E<1$ defines an equivalent relation on $M(E)$, and that, in the case of a multiplicative group $E$, $\Phi$ and $\Psi$ belong to the same equivalent class if and only if there exist positive measures $\mu$ and $\nu$ such that $\mu/k\leq\nu\leq k\mu$ for some $k>0$ and for all $f$ in $E$

$$\log\Phi(f)=\int\log|f|\,d\mu, \quad \log\Psi(f)=\int\log|f|\,d\nu.$$

Applicability of these results to *Spec* $A$ is based on the observation that each $\varphi$ in *Spec* $A$ is completely determined by the values of its modulus $\Phi(f)=|\varphi(f)|$ on any set containing exp $A$, the set of $f$ with $f=\exp(g)$ for some $g$ in $A$. We can take as $E$ the whole space $A$, exp $A$ or $A^{-1}$, the set of invertible functions. Associating to $\varphi$ its modulus $\Phi$ we shall show that though $\sigma_A(\Phi, \Psi)$ is trivial, that is, $0-1$ valued, $\|\Phi-\Psi\|_A$ coincides with

$\sigma(\varphi, \psi)$. $\sigma_{\exp A}(\Phi, \Psi)<1$ is shown to define the same equivalent relation as Gleason's while $\sigma_{A^{-1}}(\Phi, \Psi)<1$ introduces new equivalent classes, called log-modular parts, in *Spec A*. We shall present several examples in which log-modular parts coincide with or are different from Gleason parts.

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**2. Metric on $M(E)$** $M(E)$ consists of all continuous functions $\Phi$ on $E$ such that $\Phi(f)\geq 0$, $\Phi(fg)=\Phi(f)\Phi(g)$ and $\|\Phi\|_E\leq 1$. Since the mapping $\alpha\to\Phi(\alpha)$ is multiplicative and continuous on the positive numbers, either $\Phi$ vanishes identically or $\Phi(\alpha)=\alpha^k$ for some constant $k$. $\|\Phi\|_E\leq 1$ implies that either $\Phi(f)=0$ or $k$ is non-negative and $\Phi(f)\leq\|f\|^k$. With respect to point-wise definition $M(E)$ becomes a multiplicative semigroup, and together with $\Phi$ all its non-negative exponents $\Phi^r$ belong to $M(E)$. The order relation $\Phi_1\preccurlyeq\Phi_2$ is introduced to mean that there is $\Psi\in M(E)$ with $\Phi_1=\Psi\Phi_2$. Then $\Phi_1\preccurlyeq\Phi_2$ is equivalent to $\Phi_1(f)\leq\Phi_2(f)$ for all $f\in E$ with $\|f\|\leq 1$.

Let us introduce a functional:

$$\rho_E(\Phi, \Psi)=\sup\left\{\left|\log\left[\frac{\log\Phi(f)}{\log\Psi(f)}\right]\right|;\quad f\in E\quad \|f\|<1\right\}$$

with convention $-\infty/-\infty=1$. Clearly $\rho_E(\Phi, \Psi)<\infty$ defines an equivalent relation on $M(E)$ and is given by

$$\rho_E(\Phi, \Psi)=\inf\{\log k:\ \Phi^k\preccurlyeq\Psi\preccurlyeq\Phi^{1/k},\qquad k>1\}\ .$$

Now a metric on $M(E)$ is defined by

$$\sigma_E(\Phi, \Psi)=\sup_{r>0}\|\Phi^r-\Psi^r\|_E\,.$$

THEOREM 1. *$\sigma_E(\Phi, \Psi)$ and $\rho_E(\Phi, \Psi)$ are connected to each other by the relation*

$$\sigma_E(\Phi, \Psi)=H\big(\rho_E(\Phi, \Psi)\big),$$

*where $H(x)$ $(x\geq 0)$ is the monotone inceasing function defined by*

$$H(x)=(e^x-1)\exp\left[\frac{xe^x}{1-e^x}\right].$$

*In particular, $\sigma_E(\Phi, \Psi)<1$ defines an equivalent relation on $M(E)$.*

PROOF. Since $H(x)$ is monotone increasing, it suffices to prove that for $1>\alpha\geq\beta\geq 0$.

$$\sup_{r>0}(\alpha^r-\beta^r) = H\left(\log\left(\frac{\log\beta}{\log\alpha}\right)\right).$$

If $F(r)=\alpha^r-\beta^r$, $r>0$ and $r_0=\log\left(\frac{\log\beta}{\log\alpha}\right)\Big/\log\frac{\alpha}{\beta}$, $F'(r_0)=0$.

$$F(r_0) = \left[_{\alpha}\ \frac{1}{\log\alpha-\log\beta}\times\log\left(\frac{\log\beta}{\log\alpha}\right)\right]\times\left[\frac{\log\beta-\log\alpha}{\log\beta}\right]$$

and

$$F(r_0) = \left[_{\beta}\ \frac{1}{\log\alpha-\log\beta}\times\log\left(\frac{\log\beta}{\log\alpha}\right)\right]\times\left[\frac{\log\beta-\log\alpha}{\log\alpha}\right]$$

$$\sup_{r>0} F(r) = \sqrt{F(r_0)F(r_0)}$$

$$= \frac{\dfrac{\log\beta}{\log\alpha}-1}{\sqrt{\dfrac{\log\beta}{\log\alpha}}}\exp\left[\frac{1+\dfrac{\log\beta}{\log\alpha}}{1-\dfrac{\log\beta}{\log\alpha}}\times\frac{1}{2}\log\frac{\log\beta}{\log\alpha}\right]$$

$$= H\left(\log\left(\frac{\log\beta}{\log\alpha}\right)\right).$$

This completes the proof.

A positive measure $\mu$ on $X$ is called a Jensen measure for $\Phi\in M(E)$ if for all $f\in E$

$$\log\Phi(f)\leq\int\log|f|\,d\mu\,.$$

If $E$ becomes a multiplicative group, inequality in the above definition becomes equality. Existence of a Jensen measure can be shown just as in Bishop (c.f. [1; 33–34]). In fact, if $\Phi(f)=0$ for all $f\in E$, it is trivial. Let $\Phi(f)\leq\|f\|^k$ for a positive constant $k$. In the real Banach space $C_R(X)$ of all real valued continuous functions the convex cone of negative functions is disjoint from the convex cone of functions $u$ such that $nu>k\log|f|-\log\Phi(f)$ for some $f\in E$ and some positive integer $n$, and a positive measure, which represents a functional separating these two convex cones, is a Jensen measure for $\Phi$.

THEOREM 2. *The following assertions for $\Phi$, $\Psi\in M(E)$ are equivalent*:

(1) $\Phi^k\preccurlyeq\Psi\preccurlyeq\Phi^{1/k}$ *for some* $k>1$.

(2) *there exist Jensen measures $\mu$ for $\Phi$ and $\nu$ for $\Psi$ such that $\mu/k\leq\nu\leq k\mu$ and for all $f\in E$*

$$\frac{1}{k}\left[\int \log |f|\, d\mu - \log \Phi(f)\right] \leq \int \log |f|\, d\nu - \log \Psi(f)$$
$$\leq k\left[\int \log |f|\, d\mu - \log \Phi(f)\right],$$

*with convention* $-\infty+\infty=0$. *If E becomes a multiplicative group the last integral inequalities are redundant.*

Proof is almost parallel to Bishop (c.f. [1; 143]). We may assume $\Phi \not\equiv 0$ and $\Psi \not\equiv 0$. Suppose $\Phi^k \prec \Psi \prec \Phi^{1/k}$ for some $k>1$. Then, there exist $F$ and $G$ in $M(E)$ such that $\Phi^k = F\Psi$ and $\Psi^k = G\Phi$. There exist Jensen measures $\sigma$ for $F$ and $\tau$ for $G$. Put $\mu=(k\sigma+\tau)/(k^2-1)$ and $\nu=(k\tau+\sigma)/(k^2-1)$, then $\mu$ and $\nu$ are Jensen measures for $\Phi$ and $\Psi$ respectively. Clearly $0\leq\mu\leq k\nu$ and $0\leq\nu\leq k\mu$. From $\sigma=k\mu-\nu$ and $\tau=k\nu-\mu$, we get the latter half of (2). This shows that (1) implies (2).

If (2) is valid, for all $f\in E$

$$k \log \Psi(f) - \log \Phi(f) \leq \int \log |f|\, d(k\nu-\mu)$$
$$k \log \Phi(f) - \log \Psi(f) \leq \int \log |f|\, d(k\mu-\nu).$$

Put $F(f)=\Psi(f)^k/\Phi(f)$ if $\Phi(f)\neq 0$, $F(f)=0$ if $\Phi(f)=0$ and put $G(f)=\Phi(f)^k/\Psi(f)$ if $\Psi(f)\neq 0$, $G(f)=0$ if $\Psi(f)=0$. Then $F\in M(E)$ and $G\in M(E)$, thus we get (1).

**3. Gleason parts and log-modular parts** Let $A$ be a function algebra on $X$. Let us denote for $\varphi, \psi, \cdots$ in the set *Spec* $A$ of all multiplicative linear functionals of $A$ their moduluses by corresponding capitals $\Phi, \Psi, \cdots$; $\Phi(f)=|\varphi(f)|$.

Gleason showed that $\|\varphi-\psi\|<2$ defines an equivalent relation on *Spec* $A$. Bishop and König introduced functions

$$\sigma(\varphi, \psi) = \sup\left\{|\psi(f)|\,;\ \ \varphi(f)=0\,,\ \ \|f\|<1\,,\ \ f\in A\right\}$$

and

$$G(\varphi, \psi) = \sup\left\{\left|\log Re\varphi(f) - \log Re\psi(f)\right|\,;\ \ Re f>0\ \ f\in A\right\}$$

respectively, and König showed the relations;

$$G(\varphi, \psi) = \log\frac{1+\sigma(\varphi, \psi)}{1-\sigma(\varphi, \psi)} = 2\log\frac{2+\|\varphi-\psi\|}{2-\|\varphi-\psi\|}.$$

To apply the results of §2, let us first take as $E$ the whole space $A$. Since for $\varphi\neq\psi$ in *Spec* $A$ there is $f\in A$ with $\psi(f)\neq 0$ and $\varphi(f)=0$, $\sigma_A(\Phi, \Psi)=0$

or 1 according as $\varphi=\psi$ or not. Thus each equivalent class with respect to $\sigma_A$ reduces to a singleton. In this case, however, the metric $\|\Phi-\Psi\|_A$ itself gives rise to the Gleason parts.

THEOREM 3. *For $\varphi, \psi \in Spec\ A$*

$$\|\Phi-\Psi\|_A=\sigma(\varphi, \psi).$$

*In particular, $\|\Phi-\Psi\|_A<1$ defines the same equivalent relation as Gleason's.*

PROOF. Let $A_\varphi=\{g\in A;\ \varphi(g)=0\ \|g\|<1\}$. For any $f\in A$ and $\|f\|<1$, we can write $f=\dfrac{g+\varphi(f)}{1+\overline{\varphi(f)}g}\ g\in A_\varphi$. For any $g\in A_\varphi$, let $f=\dfrac{g+\lambda}{1+\bar{\lambda}g}\ |\lambda|<1$, then $f\in A$ and $\varphi(f)=\lambda$. Then,

$$\|\Phi-\Psi\|_A=\sup_{g\in A_\varphi}\sup_{|\lambda|<1}\left\{\left||\lambda|-\left|\frac{\psi(g)+\lambda}{1+\bar{\lambda}\psi(g)}\right|\right|\right\}$$

$$=\sup_{g\in A_\varphi}\sup_{0\le t<1}\sup_{\theta}\left\{\left|t-\left|\frac{\psi(g)+e^{i\theta}t}{1+e^{-i\theta}t\psi(g)}\right|\right|\right\}.$$

When $t\geq|\psi(g)|$, we can get by simple computation,

$$\sup_{|\psi(g)|\le t<1}\sup_{\theta}\left\{\left|t-\left|\frac{\psi(g)+e^{i\theta}t}{1+e^{-i\theta}t\psi(g)}\right|\right|\right\}=\sup_{|\psi(g)|\le t<1}\left\{\frac{|\psi(g)|(1-t^2)}{1-t|\psi(g)|}\right\}=|\psi(g)|.$$

When $t\leq|\psi(g)|$, we can get similarly

$$\sup_{0\le t\le|\psi(g)|}\sup_{\theta}\left\{\left|t-\left|\frac{\psi(g)+e^{i\theta}t}{1+e^{-i\theta}t\psi(g)}\right|\right|\right\}$$

$$=\sup_{0\le t\le|\psi(g)|}\max\left\{\frac{-t^2|\psi(g)|+2t-|\psi(g)|}{1-t|\psi(g)|},\ \frac{|\psi(g)|(1-t^2)}{1+t|\psi(g)|}\right\}=|\psi(g).$$

Thus,

$$\|\Phi-\Psi\|_A=\sup_{g\in A_\varphi}|\psi(g)|=\sigma(\varphi, \psi).$$

Secondly, let us take as $E$ the set $\exp A$. The metric $\sigma_{\exp A}(\Phi, \Psi)$ coincides with $\|\Phi-\Psi\|_{\exp A}$ for $\varphi, \psi\in Spec\ A$. Since

$$G(\varphi, \psi)=\sup\left\{\log\left|\frac{\log|\varphi(\exp f)|}{\log|\psi(\operatorname{evp} f)|}\right|,\ \|\exp f\|<1\right\},$$

Theorem 1 and König's result yields;

THEOREM 4. *For $\varphi, \psi\in Spec\ A$*

$$\sigma_{\exp A}(\Phi, \Psi)=H\left(\log\frac{1+\sigma(\varphi, \psi)}{1-\sigma(\varphi, \psi)}\right),$$

*where* $H(x)=(e^x-1)\exp\left[\frac{xe^x}{1-e^x}\right]$.

*In particular,* $\sigma_{\exp A}(\Phi, \Psi)<1$ *defines the same equivalent relation as Gleason's.*

Jensen measures for exp $A$ are merely representing measures.

Finally let us take as $E$ the set $A^{-1}$ of all invertible functions. Then $\sigma_{A^{-1}}(\Phi, \Psi)<1$ defines an equivalent relation on *Spec* $A$. The equivalent class, containg $\varphi$, is called the log-modular part of $\varphi$. Then Theorem 2 yields;

THEOREM 5. *Multiplicative linear functionals* $\varphi$ *and* $\psi$ *belong to the same log-modular part if and only if there are positive measures* $\mu$ *and* $\nu$ *on* X *such that* $\mu/k\leq\nu\leq k\mu$ *for some* $k>0$ *and*

$$\log|\varphi(f)|=\int\log|f|\,d\mu \quad \text{and} \quad \log|\psi(f)|=\int\log|f|\,d\nu\,.$$

**4. Examples** Let $A$ be a function algebra on a compact Hausdorff set X. The log-modular part of a multiplicative linear functional sometimes coincides with its Gleason part, but sometimes not.

If exp $A$ is (uniformly) dense in $A^{-1}$, $\sigma_{A^{-1}}(\Phi, \Psi)$ coincides with $\sigma_{\exp A}(\Phi, \Psi)$, hence every log-modular part is a Gleason part. Moreover, if $A^{-1}$ is uniformly dense in $A$, every Gleason part (and hence every log-modular part) reduces to a singleton.

If the set of all representing measures for a multiplicative linear functional $\varphi$ is finite dimensional, it is known [1; 113–114] that $\varphi$ admits a Jensen measure $\mu$ (with respect to $A^{-1}$) such that every representing measure is majorated by a scalar multiple of $\mu$. If $\psi$ belongs to the Gleason part of such $\varphi$, the set of representing measures for $\psi$ is also finite dimensional. Since $\varphi$ and $\psi$ admit mutually dominating representing measures, they admit mutually dominating Jensen measures, hence by Theorem 5, $\psi$ belongs to the log-modular part of $\varphi$.

THEOREM 6. *If for some* $f\in A$ $\varphi(f)$ *lies on the boundary of the set* $\{\psi(f);\ \psi\in Spec\ A\}$ *in the complex plane, then the log-modular part of* $\varphi$ *is contained in the set* $\{\psi\in Spec\ A;\ \psi(f)=\varphi(f)\}$.

PROOF. Let $\{s_n\}$ be a complex number sequence such that $s_n\notin\{\psi(f);\ \psi\in Spec\ A\}$ $n=1,2,\cdots$ and $s_n\to\varphi(f)$, which lies on the boundary of the set $\{\psi(f);\ \psi\in Spec\ A\}$. Let $\sup\{|s_n-\psi(f)|;\ \psi\in Spec\ A\}=c_n/2$ and $g_n=\frac{s_n-f}{c_n}$, then $g_n\in A^{-1}$ and $\|g_n\|<1$. We can assume $c_n\not\to 0$, so $\psi(g_n)\to 0$ for every $\psi\in Spec\ A$ such that $\psi(f=\varphi(f)$, while $\psi(g_n)\not\to 0$ for every $\psi\in Spece\ A$ such that $\psi(f)\neq\varphi(f)$. Thus if $\psi(f)\neq\varphi(f)$, $\sigma_{A^{-1}}(\Psi, \Phi)=1$.

To apply these results, let X be a compact plane set and $A=R(X)$ the subspace of functions in $C(X)$, which are uniformly approximated by rational functions with poles off $X$. It is well known [1; 27] that any multiplicative linear functional $\varphi$ on $R(X)$ is realized by point evaluation at some $x\in X$; $\varphi(f)=f(x)$. Under this identification, the log-modular part of $x$ on the boundary of $X$ is a singleton. For the proof, apply Theorem 6 with $f$, the coordinate function: $f(x)=x$. Further the set of representing measures for $x$ on the boundary is finite dimensional only if the Dirac measure at $x$ is the unique representing measure for $x$. In fact, since the finite dimensionality leads to coincidence of Gleason and log-modular parts, the Gleason part of $x$ reduces to a singleton. On the other hand, Wilken (c.f. [1; 146]) showed that the Gleason part of $x$ reduces to a singleton only if the Dirac measure at $x$ is the unique representing measures for $x$.

A Swiss cheese $X$ [1; 25–26] of the complex plane shows an example that the Gleason parts are different from the log-modular parts in $R(X)$. In fact, there exist the points which are not peak point, while each point of $X$ is one point log-modular part.

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