On the structure of the oriented cobordism ring modulo an equivalence

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§ 0. Introduction

Let $\mathbf{I}_n$ denote the subgroup of Ω_n^{so}, the oriented cobordism group of dim n, generated by all $[M'-M]\in\Omega_n^{so}$ such that M and M' have the same oriented homotopy type. Then $I_*=\sum_{n\geq 0} I_n$ is an ideal of Ω_*^{so}. In this paper we will determine the structure of Ω_*^{so}/I_* modulo 2-torsion.

THEOREM 0. 1. *The rank of* Ω_{4k}^{so}/I_{4k} *is one.*

THEOREM 0. 2. *For an odd prime* p, $\mathrm{Tor}(\Omega_*^{so}/I_*)\otimes \mathbf{Z}_p$ *is isomorphic to the polynomial ring* $\mathbf{Z}_p[\beta_{p-1},\cdots,\beta_{a\frac{p-1}{2}},\cdots]$, *where all a are positive integers so that* $a\left(\frac{p-1}{2}\right)$ *is not any form of* $\frac{p^j-1}{2}$ $(j=1,2,3,\cdots)$ *and the degree of* $\beta_{a\frac{p-1}{2}}$ *is* $2a(p-1)$.

Theorem 0. 1 has been proved in [4].

In § 1 we will show that there is an Ω_*^{so}-homomorphism $d_*;\ \Omega_*^{so}(F/0)\to\Omega_*^{so}$ so that $\mathrm{Cok}(d_*)$ is isomorphic modulo 2-torsion to $\mathrm{Tor}(\Omega_*^{so}/I_*)$. This homomorphism is originally found in [9, 10]. In § 2 we will compute $\mathrm{Cok}(d_*)$.

All manifolds will be compact, oriented and smooth.

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§ 1. Interpetation of Ω_n^{so}/I_n

Let M and M' be manifolds of dim n and a: $(M,\partial M)\to(M',\partial M')$, a homotopy equivalence of degree 1. (Both of $a|M$ and $a|\partial M$ are homotopy equivalences). We denote this by a *triple* (a, M, M'). If M and M' are closed, simply connected, or manifolds of dim n, then we call a triple (a, M, M') *closed, simply connected*, or *of dim n*. We define that closed triples of dim n (a, M, M') and (b, N, N') are *cobordant* if there exists a triple of dim $(n+1)$, (A, V, V') with $\partial V=M\cup(-N)$, $\partial V'=M'\cup(-N')$, $A|M=a$ and $A|N=b$. Then it is easily seen that this is an equivalence relation. As usual we define the cobordism group Ω_n^{h-eq}: an abelian group of the cobordism classes of closed, simply connected triples of dim n. In the definition we require for convenience that cobordism manifolds are also simply connected. The zero element is a triple cobordant to an empty set.

Then we can define a homomorphism $\bar{d}_n: \Omega_n^{h-eq} \to \Omega_n^{so}$ by mapping a triple of dim n, (a, M, M') into $[M-M'] \in \Omega_n^{so}$. Note that the image of d_n is contained in I_n. We will prove that its converse is also true.

LEMMA 1.1. *The image of* $\bar{d}_n$ *is* I_n.

(PROOF) If dim $n \leqq 3$, the lemma is trivial. So we may assume $n>3$. We will prove that a closed triple (a, M, M') is cobordant to a simply connected triple (a_0, M_0, M_0'). We can suppose that M, M' are connected. Let $\alpha_1, \cdots, \alpha_m$ be the finite generators of $\pi_1(M')$. We represent α_i by an embedding $\alpha_i': S^1 \to M'$ $(i=1, 2, \cdots, m)$ with a path combining the point of M' with an embedded circle $S_i' = \alpha_i'(S^1)$. Let $S_i' \times D^{n-1}$ be the normal disk bundle of S_i'. Since $n>3$, we may assume that $S_i' \times D^{n-1}$ $(i=1, \cdots, m)$ do not meet each other. Let a be transverse regular on $\bigcup_{i=1}^{m} S_i'$. Since $\pi_1(M) \cong \pi_1(M')$, we can make $a^{-1}(S_i')$ connected by the usual method. We denote $a^{-1}(S_i') = S_i$ $(i=1, 2, \cdots, m)$. Since a is a map of degree 1, $a|S_i: S_i \to S_i'$ is of degree 1. Therefore by changing a by the homotopy extension theorem, we may consider that $a|a^{-1}(S_i' \times D^{n-1}) = id$. We now surgery M, M' by these embeddings. Let W and W' be their surgery traces and M_0, M_0' the oposite boundaries respectively. Then we can extend $a: M \to M'$ to a map $A: W \to W'$ so that $A(M_0) \subset M_0'$. Then A is a homotopy equivalence of degree 1. Since $\pi_1(M_0) = \pi_1(M_0') = 0$, the isomorphism $A_*: H_*(W, M_0) \to H_*(W', M_0')$ shows that a_0: $M_0 \to M_0'$ is an homotopy equivalence of degree 1, where $a_0 = A|M_0$. Q.E.D.

Here we recall the results of D. Sullivan [9, 10]. Let M be a simply connected manifold with dim $M \geq 5$ and $hS(M)$, the concordance classes of homotopy smoothings. D. Sullivan has defined η: $hS(M) \to [M, F/0]$ and a surgery obstruction $\mathscr{S}$: $[M, F/0] \to \boldsymbol{Z}$ when dim $M \equiv 0(4)$. For the rest of the paper we often use the construction of $\mathscr{S}$. Here we recall it. The homotopy classes $[M, F/0]$ corresponds isomorphically to the equivalence classes of $F/0$-bundles over M. Let f: $M \to F/0$ and (E, t) be a corresponding $F/0$-bundle and a spherical trivialization t: $E \to D^n$ (D^n is the unit n-disk). If $\tilde{\gamma}$ is a universal $F/0$-bundle, then E is the associated disk bundle of $f^*(\tilde{\gamma})$. Let t be transversal regular on $0 \in D^n$. If we put $t^{-1}(0) = M'$, then $\mathscr{S}$ is defined by $\mathscr{S}(f) = 1/8\ (I(M) - I(M'))$. Note that $\tau_{M'} = a^*\tau_M + a^*f^*(\tilde{\gamma})$, where a: $M' \to M$ is a restriction of a projection $E \to M$.

Note that a triple (a, M, M') is a homotopy equivalence. We can define

$\bar{\eta}$: $\Omega_n^{h-eq} \to \Omega_n^{so}(F/0)$, by mapping a triple $\alpha=(a, M, M')$ into the cobordism class of $\eta(\alpha)$. Let (A, W, W') be a cobordism of (a, M, M') and (a_0, M_0, M_0'). Then the following diagram commutes,

$$\begin{array}{ccc} hS(W) & \longrightarrow & [W, F/0] \\ \downarrow r & & \downarrow r \\ hS(M\cup(-M_0)) & \longrightarrow & [M\cup(-M_0), F/0] \end{array}$$

where r is the restriction map. This shows that $\bar{\eta}$ is well defined. It is clear that $\bar{\eta}$ is a homomorphism. If we provide $\Omega_*^{h-eq}=\sum_{n\geq 0}\Omega_n^{h-eq}$ with an Ω_*^{so}-module structure by $[N]\times\alpha=(a\times id, M\times N, M'\times N)$ for $[N]\in\Omega_m^{so}$, $\alpha=(a, M, M')\in\Omega_n^{h-eq}$, then $\bar{\eta}_*$: $\Omega_*^{h-eq}\to\Omega_*^{so}(F/0)$ is an Ω_*^{so}-homomorphism. In fact $\eta([N]\times\alpha)=P_1\circ\eta(\alpha)$, where P_1 is a projection: $M'\times N\to M'$. The map $\#$; $bP_{n+1}\to hS(M)$ in [9, Theorem 3] also induces a homomorphism $\#$: $bP_{n+1}\to\Omega_n^{h-eq}$ by defining $\#(\Sigma)=$(a map of degree 1, Σ, S^n) for $\Sigma\in bP_{n+1}$, S^n standard sphere. Note that the connected sum $(a, \Sigma, S^n)\ \#\ (b, M', M)$ is defined and cobordant to $(a\cup b, \Sigma\cup M, s^n\cup M')$ for a simply connected triple (a, M, M') since we can change a: $M\to M'$ so that a is an identity map on some embedded small n-disks of M and M'. With these notations the following proposition is an easy consequence from [9, Theorem 3].

PROPOSITION 1.2. *For $n\geq 5$, the sequence*

$$bP_{n+1}\xrightarrow{\#}\Omega_n^{h-eq}\xrightarrow{\bar{\eta}}\Omega_n^{si}(F/0)\xrightarrow{\mathscr{S}}P_n$$

is an exact sequence.

(PROOF) Let $\alpha=(a, M, M')\in\Omega_n^{h-eq}$ and $\bar{\eta}(\alpha)=0$. Then we have a map f: $W'\to F/0$ with $\partial W'=M'$ and $f|M'=\eta(\alpha)$. By the same argument as above we have a normal map of degree 1 A: $W\to W'$. (This is a Browder's notation [3, §2]). It follows form [3, (2.11)] that there exists an homotopy equivalence B: $(V, \partial V)\to(W', \partial W')$ so that $\partial V=M\#\Sigma$ for some $\Sigma\in bP_{n+1}$ and $B|\partial V=a\#$ (a map of degree 1). Other parts is immediate from [9, Theorem 3]. Q.E.D.

Let f: $M'\to F/0$ be a representative element of $x\in\Omega_n^{so}(F/0)$ and h: $M\to M'$ *a* normal map of degree 1 corresponding to f. If we define d_n by $d_n(x)=[M-M']\in\Omega_n^{so}$, then $d_*=\sum_{n\geq 0}d_n$: $\Omega_*^{so}(F/0)\to\Omega_*^{so}$ is well defined and an Ω_*^{so}-homomorphism. In fact let $f\circ P_1$ be the composition: $M'\times N\to M'\to F/0$. Then we can take $(h\times id, M\times N, M'\times N)$ as the corresponding normal map of degree 1. Since d_* is an Ω_*^{so}-homomorphism, Cok(d_*) has a ring structure induced from that of Ω_*^{so}. Then we have the following

THEOREM 1. 2. *The ring* Tor (Ω_*^{so}/I_*) *is isomorphic modulo* 2-*torsion to the ring* Cok (d_*).

(PROOF) We only need to consider $*\equiv 0(4)$. At first we prove this for $n=4$. There exists an almost parallelizable closed manifold M' of dim 4 with index 16. [5, Theorem 2]. It follows from [5, Lemma 1] that the stable normal bundle ν of an almost parallelizable manifold is trivial as a spherical fiber space. Let E be the associated disk bundle of ν. If we take a spherical trivialization t, we have an $F/0$-bundle (E, t). This is an element α of $\Omega_4^{so}(F/0)$. Let a: $M\to M'$ be a map of degree 1 which is constructed as above from (E, t). Then $\tau_M=a^*(\tau_{M'}\oplus\nu)$ which is a trivial bundle. Since M is a parallelizable manifold, $I(M)=0$. Therefore the index of $d(\alpha)=I(M)-I(M')=-I(M')=-16$. Note that Ω_4^{so} is characterized by index. So Cok (d_4) is a 2-torsion. On the other hand $I_4=0$ and $\Omega_4^{so}/I_4\cong \boldsymbol{Z}$, that is, Tor $(\Omega_4^{so}/I_4)=0$. For $n\equiv 0(4)$, $n>4$, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_n^{h-eq} & \xrightarrow{\bar{\eta}} & \Omega_n^{so}(F/0) & \xrightarrow{\mathscr{S}} & \boldsymbol{Z} & & \\
 & & \downarrow{\scriptstyle \bar{d}_n} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle \times 8} & & \\
0 & \longrightarrow & \operatorname{Ker} I & \longrightarrow & \Omega_n^{so} & \xrightarrow{I} & \boldsymbol{Z} & \longrightarrow & 0.
\end{array}$$

In fact, $bP_{n+1}=0$ for $n\equiv 0(4)$ [6]. If $\alpha=(a, M, M')\in\Omega_n^{h-rq}$, then $d_n\circ\bar{\eta}=[M-M']$. $[M-M']$ is an element of Ker I. This is the first vertical map. The identity $8\mathscr{S}=I\circ d_n$ follows from the definition of $\mathscr{S}$. This diagram leads us to the exact sequence

$$0\longrightarrow \operatorname{Ker} I/I_n\longrightarrow \operatorname{Cok}(d_n)\longrightarrow \operatorname{Cok}(8\cdot\mathscr{S})\longrightarrow 0\,.$$

It follows from Lemma 1. 3 that Cok (d_n) is isomorphic modulo 2-torsion to Ker I/I_n. Since the ring structure of Cok(d_*) and Ker I/I_* is induced from that of Ω_*^{so}, it is clear that the isomorphism: Ker $I/I_*\to$Cok (d_*) is a ring isomorphism. Lemma 1. 4 completes the proof. Q. E. D.

LEMMA 1. 3. Cok $(8\cdot\mathscr{S})$*is a* 2-*torsion.*

(PROOF) As above we have an element α of $\Omega_4^{so}(F/0)$ with $\mathscr{S}(\alpha)=2$. Since $\mathscr{S}$ is an Ω_*^{so}-homomorphism and the index of $2n$ dimensional complex projective space is one, Cok $(8\cdot\mathscr{S})$ is a 2-torsion group. Q. E. D.

LEMMA 1. 4. Cok (d_n) *is a torsion group.*

(PPOOF) Let f: $M'\to F/0$ and a:$M\to M'$ be as above. It follows from the construction of (a, M, M') that $\tau_M = a^* \circ f^* \bar{\gamma} + a^* \tau_{M'}$. Let γ be a universal oriented bundle over BSO, $i: F/0 \to BSO$ the inclusion. Then the universal $F/0$-bundle $\bar{\gamma}$ is $i^*\gamma$ with a spherical trivialization. Let $\mu: F/0 \times BSO \to BSO$ be the classifying map of $\bar{\gamma} \times \gamma$, $P_2: F/0 \times BSO \to BSO$ the projection on the second factor and $c: M' \to BSO$ the classifying map of $\tau_{M'}$. Then $a^* \circ f^*(\bar{\gamma}) \oplus a^* \tau_{M'} = a^* \circ \Delta^* \circ (f \times c)^* (\bar{\gamma} \times \gamma) = a^* \circ \Delta^* \circ (f \times c)^* \circ \mu^*(\gamma) = (\mu \circ (f \times c) \circ \Delta \circ a)^*(\gamma)$, where $\Delta: M' \to M' \times M'$ is a diagonal map. This shows the following diagram commutes:

$$\begin{array}{ccc}
\varinjlim_k \Pi_{n+k}(F/0^+ {}_\wedge MSO(k)) & \xrightarrow{\bar{h}_n} & H_n(F/0 \times BSO:\ \boldsymbol{Z}) \\
\cong \uparrow \varphi & & \downarrow (\mu_* - P_{2*})((-1) \times id)_* \\
\Omega_n^{so}(F/0) \xrightarrow{d_n} \Omega_n^{so} & \xrightarrow{h_n} & H_n(BSO:\ \boldsymbol{Z})
\end{array}$$

where $h_n, \bar{h}_n$ are the Thom homomorphism and (-1) denotes the inverse map. Let $\bar{c}: M' \to BSO$ be the classifying map of the stable normal bundle of $\nu_{M'}$. If α represents (M', f), then it is well known that $\bar{h}_n \cdot \varphi(\alpha) = (f \times c)_* \circ \Delta_*([M'])$, where $[M']$ is the fundamental class of M' ([2]). $P_{2_*} \circ ((-1) \times id)_* \circ (f \times \bar{c})_* \circ \Delta_*([M']) = \bar{c}_*([M'])$. $\mu_* \circ ((-1) \times id)_* (f \times \bar{c})_* \circ \Delta_*([M']) = \mu_*((-f) \times \bar{c})_* \circ \Delta_* \circ a_*([M]) = (-1)_* \circ \mu_* \circ (f \times c)_* \circ \Delta_* \circ a_*([M]) = (-1)_* \circ (\mu \cdot (f \times c) \circ \Delta \circ a)_*([M])$. It follows from the definition of h_n that $h_n \circ d_n(\alpha) = (-1)_* (\mu \circ (f \times c) \circ \Delta \circ a)_*([M]) - c_*([M'])$.

Now we complete the proof. It is well known that the kernel of h_n is a 2-torsion and that h_n and $\bar{h}_n$ are isomorphism modulo a torsion. These facts show that $\mathrm{Cok}(d_n)$ is isomorphic modulo a torsion to $\mathrm{Im}(h_n)/\mathrm{Im}(\mu_* - P_{2*}) \circ ((-1) \times id)_*$. If we consider a map: $F/0 \xrightarrow{j} F/0 \times BSO$, then $P_2 \circ i = *$ and $\mu \circ j = i$. Since $i_*: H_*(F/0: \boldsymbol{Z}) \to H_*(BSO: \boldsymbol{Z})$ is an isomorphism modulo torsion, the above module is a torsion. Therefore $\mathrm{Cok}(d_n)$ is a torsion. Q. E. D.

It follows from this lemma that $\mathrm{Tor}(\Omega_n^{so}/I_n) = \mathrm{Ker}\, I/I_n$.

§2. On the structure of Tor $(\Omega_*^{so}/I_*) \otimes Z_p$: p an odd prime

In this section p is always an odd prime. We denote the mod p total Pontrjagin class $1 + p_1 + p_2 + \cdots + p_n + \cdots$, $p_i \in H^{4i}(BSO: \boldsymbol{Z}_p)$ by $\prod_{j=1}^{n} (1 + x_j^2)$, dim $x_j = 2$. For any partition $\omega = (i_1, \cdots, i_r)$, let S_ω denote the elements of $H^{**}(BSO: \boldsymbol{Z}_p)$ defined by the functions $\sum x_1^{2i_1} x_2^{2i_2} \cdots x_r^{2i_r}$, where the sum denotes the smallest symmetric functions containing the monomial $x_1^{2i_1} x_2^{2i_2} \cdots x_r^{2i_r}$.

Let $\beta_\omega \in H_*(BSO: \boldsymbol{Z}_p)$ be the dual element of S_ω. Let $S_n = S_{(n)}$, $\beta_n = \beta_{(n)}$. Then $H_*(BSO: \boldsymbol{Z}_p)$ is the polynomial ring $\boldsymbol{Z}_p[\beta_1, \beta_2, \cdots, \beta_n, \cdots]$.

Recall that $\Omega_*^{so}/\mathrm{Tor}$ is the polynomial ring over the generators $[M_{4n}]$ $(n=1, 2, \cdots)$ where if n is not of the form $\frac{q^j-1}{2}$ for any prime q, then $S_n([M_{4n}])=1$, $S_n \in H^{4n}(BSO: \boldsymbol{Z})$ and if n is $\frac{q^j-1}{2}$ for a prime q, then all the Pontrjagin numbers of M_{4n} are divisible by q. Let P denote the projection: $\boldsymbol{Z}_p[\beta_1, \beta_2, \cdots, \beta_n, \cdots] \to \boldsymbol{Z}_p[\cdots, \beta_v, \cdots]$, where v are not any $\frac{p^j-1}{2}$ $(j=1, 2, \cdots)$. Then P induces an isomorphism of Image h_* onto $\boldsymbol{Z}_p[\cdots, \beta_v, \cdots]$, where h_* is the Thom homomorphism $\Omega_*^{so} \to H_*(BSO: \boldsymbol{Z}_p)$. If we prove the following proposition, $\mathrm{Cok}(d_*)$ is isomorphic to $\boldsymbol{Z}_p[\cdots, \beta_{a\frac{p-1}{2}}, \cdots]$ since Kernel h_* is a 2-torsion and p is an odd prime. We will need the following Quillen's result to prove the proposition.

(2. 1) (Quillen) Let J be the J-homomorphism. Then Kernel (J) coincides with the subgroup of $KO(\mathrm{X})$ generated by the elements $k^{e(k)}(\psi^k-1)(\xi)$, where $e(k)$ is a sufficiently large integer, ψ^k is the Adams operation, and $\xi \in KO(\mathrm{X})$ [7].

PROPOSITION 2. 2. *The image of $P \circ h_* \circ d_*$ is the ideal of $\boldsymbol{Z}_p[\cdots, \beta_v, \cdots]$ generated by all β_i, where i is not any multiple of $\frac{p-1}{2}$.*

(PROOF) We shall prove this by induction on degree. The statement of degree 0 is trivial. Suppose that the proposition is valid in degree less than n. Let $a \in \boldsymbol{Z}_p[\cdots, \beta_v, \cdots]$, b be an element of the above ideal and degree of $a \cdot b = n$, degree of $b \neq 0$. Then there exists $x \in \Omega_*^{so}$ and $y \in \Omega_*^{so}(F/0)$ so that $P \circ h_*(x) = a$ and $P \circ h_* \circ d_*(y) = b$. Since d_* is an Ω_*^{so}-homomorphism $P \circ h_* \circ d_*(x \cdot y) = a \cdot b$. Hence all the decomposable element of degree n of the above ideal are contained in the image of $P \circ h_* \circ d_*$. If $n \equiv 0\,(2(p-1))$, then the proposition is true. If $n \not\equiv 0\,(2(p-1))$ and $n = 4l$, then we only need to show that β_l is contained in the ideal. Let η be the canonical complex line bundle over $\boldsymbol{CP}^{2l}$. The associated spherical fiber space of $k^e(\psi_R^k-1)\eta$ (we denote this by ξ for convenience) becomes trivial by (2. 1) if e is sufficienty large integer compairing with an integer k. We now choose a spherival trivialization t of ξ. This is an $F/0$-bundle. So we have a normal map of degree 1: a: $M \to \boldsymbol{CP}^{2l}$ which corresponds to the $F/0$-bundle (ξ, t). Recall that the stable tangent bundle τ_M is $a^*\xi \oplus a^*\tau_{CP^{2l}}$. Let f: $\boldsymbol{CP}^{2l} \to F/0$ be the classifying map of (ξ, t) and $\alpha = (\boldsymbol{CP}^{2l}, f) \in \Omega_{4l}^{so}(F/0)$. Then $\langle S_l, h_{4l} \circ d_{4l}(\alpha) \rangle = \langle S_l(\tau_M), [M] \rangle - \langle S_l(\tau_{CP}), [\boldsymbol{CP}^{2l}] \rangle = \langle S_l(a^*\xi \oplus a^*\tau_{CP}), [M] \rangle - \langle S_l(\tau_{CP}), [\boldsymbol{CP}^{2l}] \rangle =$

$\langle S_l(\xi \oplus \tau_{CP}), [\boldsymbol{CP}^{2l}]\rangle - \langle S_l(\tau_{CP}), [\boldsymbol{CP}^{2l}]\rangle = \langle S_l(\xi), [\boldsymbol{CP}^{2l}]\rangle + \langle S_l(\tau_{CP}), [\boldsymbol{CP}^{2l}]\rangle - \langle S_l(\tau_{CP}), [\boldsymbol{CP}^{2l}]\rangle = \langle S_l(\xi), [\boldsymbol{CP}^{2l}]\rangle = \langle S_l(k^e(\psi_R^k - 1)\eta), [\boldsymbol{CP}^{2l}]\rangle$. Since $S_l(k^e(\psi_R^k - 1)\eta) = k^e(k^{2l}-1)S_l(\eta)$ and $\langle S_l(\eta), [\boldsymbol{CP}^{2l}]\rangle = 1$, $\langle S_l, h_{4l} \circ d_{4l}(\alpha)\rangle = k^e(k^{2l}-1)$. It is known that the greatest common divisor of all $k^e(k^{2l}-1)$ divides $m(2l)$ [1, Theorem 2.7]. On the other hand $m(2l)$ is not divisible by p if $2l \not\equiv 0$ (mod $(p-1)$) [1, P 139]. Hence if $2l \not\equiv 0\,(p-1)$, then there exists $k^e(k^{2l}-1)$ for some k so that $k^e(k^{2l}-1) \not\equiv 0(p)$. That is $h_n \circ d_n(\alpha) = a \cdot \beta_l +$ decomposable terms, where $a \not\equiv 0\,(p)$. This completes the proof.

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