On almost complex structures on the products and connected sums of the quaternion projective spaces

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1. Introduction and results

It is known by F. Hirzebruch [3] and by T. Heaps [2]*) that the quaternion projective space $P_n(Q)$ of quaternion dimension n has no almost complex structure for $n\neq 3$.

In this note, we consider whether almost complex structures exist or not on the product spaces $P_{n_1}(Q)\times\cdots\times P_{n_r}(Q)$ of quaternion projective spaces, and the connected sums

$$\alpha P_n(Q)\#\left(-\beta P_n(Q)\right)$$
$$=\underbrace{P_n(Q)\#\cdots\# P_n(Q)}_{\alpha\text{ copies}}\#\underbrace{(-P_n(Q))\#\cdots\#(-P_n(Q))}_{\beta\text{ copies}},$$

where the sign - denotes the reversed orientation.

THEOREM A. *The product spaces* $P_{n_1}(Q)\times\cdots\times P_{n_r}(Q)$ *for* $r\geqq 2$ *admit no almost complex structures if* $n_i\neq 2, 3$ *for an integer* i, $1\leqq i\leqq r$.

THEOREM B. *The connected sums*

$$\alpha P_n(Q)\#\left(-\beta P_n(Q)\right),\quad \textit{where}\ \ n\leqq 10,$$

admit no almost complex structures if $n=1, 2, 4, 5, \cdots, 10$ *or* $n=3$, $\alpha\neq 3\beta+1$.

2. The product spaces $P_{n_1}(Q)\times\cdots\times P_{n_r}(Q)$.

Let $P_n(Q)$ be the quaternion projective space of quaternion dimension n. Let $p_i\in H^{4i}(P_n(Q);Z)$ be tthe ith Pontrjagin class. Let $c_i\in H^{2i}(P_n(Q);Z)$ be the ith Chern class if $P_n(Q)$ has an almost complex structure. Let $u\in H^4(P_n(Q);Z)$ be the canonical generator. By F. Hirzebruch [3] or A. Borel and F. Hirzebruch [1], we have the total Pontrjagin class of $P_n(Q)$,

$$p=\sum_{i=0}^{\infty}p_i=(1+u)^{2n+2}(1+4u)^{-1}.$$

*) The authors thank M. Adachi for his advice on the Heaps' theorm.

If $P_n(Q)$ has an almost complex structure, by the relation $\sum_{i=0}^{\infty}(-1)^i p_i = \left(\sum_{i=0}^{\infty} c_i\right)\left(\sum_{i=0}^{\infty}(-1)^i c_i\right)$, we obtain

$$(1-u)^{2n+2}(1-4u)^{-1} = c^2,$$

where $c = \sum_{i=0}^{\infty} c_i = \sum_{j=0}^{\infty} c_{2j}$ is the total Chern class of $P_n(Q)$.

Let s_n be the coefficient of x_n in the power series expansion of $(1-x)^{n+1}(1-4x)^{-\frac{1}{2}}$ and set $a_n = s_n/(n+1)$. By F. Hirzebruch [2], a_n are non-negative integers,

$$a_{n+1} \geqq 2a_n - 1,$$

and

$$a_n > 1 \quad \text{for} \quad n \geqq 4.$$

By this result, Hirzebruch proved that the quaternion projective spaces $P_n(Q)$ with the natural differentiable structure have no almost complex structures for $n \neq 2, 3$.

We consider almost complex structures of product spaces of quaternion projective spaces,

$$P_{n_1}(Q) \times \cdots \times P_{n_r}(Q),$$

and obtain the following theorem:

THEOREM A. *The product spaces $P_{n_1}(Q) \times \cdots \times P_{n_r}(Q)$ with the natural differentiable structures have no almost complex structures if $n_i \neq 2, 3$ for an integer i,* $1 \leqq i \leqq r$ and $r \geqq 2$.

PROOF. Let $u_i \in H^4(P_{n_i}(Q); Z)$ be the canonical generator and π_i; $P_{n_1}(Q) \times \cdots \times P_{n_r}(Q) \to P_{n_i}(Q)$, the natural projection. We denote the image $\pi_i^* u_i$ by the same letter u_i. Since $H^*(P_{n_i}(Q); Z)$ has no torsion, we have the total Pontrjagin class

$$\begin{aligned} &p\left(P_{n_1}(Q) \times \cdots \times P_{n_r}(Q)\right) \\ &\quad = p\left(P_{n_1}(Q)\right) \cdots p\left(P_{n_r}(Q)\right) \\ &\quad = (1+u_1)^{2n_1+2} \cdots (1+u_r)^{2n_r+2}(1+4u_1)^{-1} \cdots (1+4u_r)^{-1}. \end{aligned}$$

If $P_{n_1}(Q) \times \cdots \times P_{n_r}(Q)$ has an almost complex structure, by the relation $\sum_{i=0}^{\infty}(-1)^i p_i = \left(\sum_{i=0}^{\infty} c_i\right)\left(\sum_{i=0}^{\infty}(-1)^i c_i\right)$, we obtain

$$(1-u_1)^{2n_1+2} \cdots (1-u_r)^{2n_r+2}(1-4u_1)^{-1} \cdots (1-4u_r)^{-1} = \left(\sum_{j=0}^{\infty} c_{2j}\right)^2 = c^2,$$

since $H^{2i}(P_{n_1}(Q)\times\cdots\times P_{n_r}(Q); Z)=0$ for odd i. It follows that

$$c=(1-u_1)^{n_1+1}\cdots(1-u_r)^{n_r+1}(1-4u_1)^{-\frac{1}{2}}\cdots(1-4u_r)^{-\frac{1}{2}}.$$

It is obvious that

$$u_1^{n_1+1}=\cdots=u_r^{n_r+1}=0\,,$$

and we have

$$\begin{aligned}c_{2(n_1+\cdots+n_r)}&=s_{n_1}\cdots s_{n_r}u_1^{n_1}\cdots u_r^{n_r}\\&=(n_1+1)\cdots(n_r+1)a_{n_1}\cdots a_{n_r}u_1^{n_1}\cdots u_r^{n_r}.\end{aligned}$$

If $n_i\neq 2, 3$ for some i, $1\leqq i\leqq r$, we have $a_{n_i}>1$ or 0 by the result of F. Hirzebruch [2], and hence $a_{n_1}\cdots a_{n_r}$ is greater than 1 or 0 since $a_{n_1}\cdots a_{n_i-1}a_{n_i+1}\cdots a_{n_r}$ are integers $\geqq 0$.

On the other hand, the Euler characteristic $E(P_{n_1}(Q)\times\cdots\times P_{n_r}(Q))$ is obviously

$$(n_1+1)\cdots(n_r+1)\,,$$

and we have by [5, Theorem 1. 1],

$$\begin{aligned}&(n_1+1)\cdots(n_r+1)a_{n_1}\cdots a_{n_r}u_1^{n_1}\cdots u_r^{n_r}\\&\quad=c_{2(n_1+\cdots+n_r)}\\&\quad=(n_1+1)\cdots(n_r+1)u_1^{n_1}\cdots u_r^{n_r}\\&\qquad\qquad\text{or}\quad -(n_1+1)\cdots(n_r+1)u_1^{n_1}\cdots u_r^{n_r}\,,\end{aligned}$$

which is impossible because $a_{n_1}\cdots a_{n_r}>1$ or 0 by the above arguments. Thus our theorem is proved.

3. The connected sums $\alpha P_n(Q)\#(-\beta P_n(Q))$.

THEOREM B. *The connected sums*

$$\alpha P_n(Q)\#\big(-\beta P_n(Q)\big)\,,\qquad \text{where}\ \ n\leqq 10\,,$$

admit no almost complex structures if $n=1, 2, 4, 5, \cdots, 10$ or $n=3$, $\alpha\neq 3\beta+1$.

PROOF. First of all, we have that homomorphism Φ:

$$\begin{aligned}&H^i\big(\alpha P_n(Q)\#(-\beta P_n(Q)); Z\big)\longrightarrow\\&\qquad H^i\big(P_n(Q)/\mathring{D}_1^{4n}; Z\big)\oplus\cdots\oplus H^i\big(P_n(Q)/\mathring{D}_\alpha^{4n}; Z\big)\oplus\\&\qquad H^i\big(-P_n(Q)/\mathring{D}_1^{4n}; Z\big)\oplus\cdots\oplus H^i\big(-P_n(Q)/\mathring{D}_\beta^{4n}; Z\big)\end{aligned}$$

is isomorphism for $1\leqq i\leqq 4n-1$, where $\Phi(u)=\sum_\lambda \iota_\lambda^*(u)+\sum_\mu \iota_\mu^*(u)$

$$\iota_\lambda:\quad P_n(Q)/\mathring{D}^{4n}_\lambda \longrightarrow \alpha P_n(Q)\#\bigl(-\beta P_n(Q)\bigr)$$
$$\iota_\mu:\quad -P_n(Q)/\mathring{D}^{4n}_\mu \longrightarrow \alpha P_n(Q)\#\bigl(-\beta P_n(Q)\bigr),$$

denoting by $P_n(Q)/\mathring{D}^{4n}_\lambda$ the complement of the open disk $\mathring{D}^{4n}_\lambda \subsetneq P_n(Q)$, and

$$\begin{aligned}\Phi&\left(p_i(\alpha P_n(Q))\#(-\beta P_n(Q))\right)\\ &=\sum_\lambda \iota_\lambda^*\left(p_i(\alpha P_n(Q))\#(-\beta P_n(Q))\right)+\sum_\mu \iota_\mu^*\left(p_i(\alpha P_n(Q))\#(-\beta P_n(Q))\right)\\ &=\sum_\lambda \bar{\iota}_\lambda^*\left(p_i(P_n(Q))\right)+\sum_\mu \bar{\iota}_\mu^*\left(p_i(-P_n(Q))\right),\qquad \text{for } i<n\end{aligned}$$

where

$$\bar{\iota}_\lambda:\quad P_n(Q)/\mathring{D}^{4n}_\lambda \subsetneq P_n(Q)$$
$$\bar{\iota}_\mu:\quad -P_n(Q)/\mathring{D}^{4n}_\mu \subsetneq -P_n(Q)\,.$$

We also obtain the cohomology ring of $\alpha P_n(Q)\#(-\beta P_n(Q))$ to the following effect.

$$H^*\bigl(\alpha P_n(Q)\#(-\beta P_n(Q));Z\bigr)=Z[u_1,\cdots,u_\alpha,\ v_1,\cdots,v_\beta]$$
$$\begin{cases}\dim u_i=\dim\ v_j=4\\ u_i^n=-v_j^n\,,\quad u_i^{n+1}=v_j^{n+1}=0\\ u_i\cdot u_j=0\,,\qquad \text{for } i\neq j\\ v_k\cdot v_l=0\,,\qquad \text{for } k\neq l\\ u_i\cdot v_k=0\,.\end{cases}$$

Moreover, we have the Euler-Poinearé characteristic

$$\chi\bigl(\alpha P_n(Q)\#(-\beta P_n(Q))\bigr)=(n-1)(\alpha+\beta)+2\,,$$

and index

$$\sigma\bigl(\alpha P_n(Q)\#(-\beta P_n(Q))\bigr)=\begin{cases}\alpha-\beta & n;\ \text{even}\\ 0 & n;\ \text{odd.}\end{cases}$$

Setting $v=\sum_{i=1}^{\alpha}u_i+\sum_{j=1}^{\beta}v_j$, we obtain from the relation in the cohomology ring,

$$v^2=\sum_{i=1}^{\alpha}u_i^2+\sum_{j=1}^{\beta}v_j^2,\ \ v^3=\sum u_i^3+\sum v_j^3,\ \cdots.$$

It is almost obvious that for $i<n$, the coefficient t_i of v^i in $p_i(\alpha P_n(Q))\#(-\beta P_n(Q))=t_iv^i$ equals that of u^i in $p_i(P_n(Q))$. Then, we have from the

index theorem, that the coefficient t_n of v^n in

$$p_n\left(\alpha P_n(Q)\#(-\beta P_n(Q))\right) = t_n v^n = t_n(\sum u_i^n + \sum v_j^n)$$
$$= t_n(\alpha-\beta)m$$

equals that of u^n in $p_n(P_n(Q))$, where m is the canonical generator of $H^{4n}(\alpha P_n(Q)\#(-\beta P_n(Q)); Z)$.

Therefore, using $\sum_{i=0}^{\infty}(-1)^i p_i = (\sum_{j=1}^{\infty} c_j)^2$, we obtain

$$c_{2n} = (n+1)a_n v^n$$
$$= (n+1)a_n(\alpha-\beta)m .$$

Now, if the connected sum $\alpha P_n(Q)\#(-\beta P_n(Q))$ admits almost complex structure, we have from [5, Theorem 1.1]

$$(n+1)a_n(\alpha-\beta) = (n-1)(\alpha+\beta)+2 .$$

This equation is written in the form

$$(*) \qquad \alpha\left\{(n+1)a_n-(n-1)\right\} = \beta\left\{(n+1)a_n+(n-1)\right\}+2 .$$

When $n=1$, 4, 5, we have that the equation has no solution for α, β natural number. For $n=3$, we have that $\alpha=3\beta+1$. For $n=2$, by the theorem of T. Heaps [2], $\alpha P_2(Q)\#(-\beta P_2(Q))$ have no almost complex structures. For $n\geqq 6$, we obtain, for example, $a_6=5\times 3$, $a_7=6\times 6$, $a_8=7\times 13$, $a_9=8\times 29$, $a_{10}=9\times 67$. If a_n is divisible by $(n-1)$ for $n\geqq 6$, $(n-1)$ has to divide 2 in the equation (*), therefore we have that the equation has also no solution and $\alpha P_n(Q)\#(-\beta P_n(Q))$ does not admit almost complex structure. Q. E. D.

Now, as proposition, we consider the case that the quaternion projective space is replaced by the complex projective space.

PROPOSITION. *The connected sum $\alpha P_n(C)\#(-\beta P_n(C))$ of complex projective space $P_n(C)$ admits almost complex structure if and only if*

$$\alpha = n\beta+1 .$$

PROOF. According to J. Kahn [4], the connected sum of manifolds which admit weakly complex structure also admit weakly complex structure. Therefore, we compute the cohomology ring of $\alpha P_n(C)\#(-\beta P_n(C))$ and nth Chern class, similar to the above theorem, then we have that $\alpha=n\beta+1$. Q. E. D.

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