A note on the subdegrees of finite permutation groups*)

By Eiichi BANNAI**) and Shiro IWASAKI

Let G be a transitive permutation group on a finite set Ω. Let G_a be the stabilizer of $a\in\Omega$ in G. Let $\Delta_1=\{a\}$, $\Delta_2, \cdots, \Delta_r$ be the orbits of G_a on Ω (these are called the suborbits of (G, Ω)). Then we say that the permutation group (G, Ω) is of rank r, and we call $|\Delta_i|$'s the subdegrees of (G, Ω) (From the transitivity of (G, Ω), the $|\Delta_i|$'s are independent of the choice of $a\in\Omega$). When (G, Ω) is given, it is sometimes required to obtain the subdegrees. The purpose of this short note is to give a *practical* method to calculate the subdegrees when the structure of the group G is fairly known.

The authors thank the members of "YUGENGUN TANSHIN" (a semi-private set of letters written in Japanese circulated periodically among the Japanese young people who are studying finite group theory), especially Mr. Hikoe Enomoto and Miss Yoko Usami, for valuable discussions. In particular, Usami calculated some important examples by applying our method (see Appendix).

NOTATION: Let G be a transitive permutation group on a set Ω, and let $H=G_a$ be the stabilizer of $a\in\Omega$. Let $\mathfrak{H}_1, \mathfrak{H}_2, \cdots, \mathfrak{H}_t$ be the sets of all H-conjugate subgroups of H (i. e., any subgroup $X\leqq H$ is contained in some and only one $\mathfrak{H}_i$). Moreover we fix an element $H_i\in\mathfrak{H}_i$ $(i=1, 2, \cdots, t)$. Let us define a partial order among $\mathfrak{H}_i$'s by $\mathfrak{H}_i\leqq\mathfrak{H}_j$ if there exist subgroups $X_i\in\mathfrak{H}_i$ and $X_j\in\mathfrak{H}_j$ such that $X_i\leqq X_j$. If $X_i\lneqq X_j$, we denote by $\mathfrak{H}_i<\mathfrak{H}_j$. Let us set $\Omega_i=H_i\backslash H$ (the right cosets of H by H_i), then H acts on Ω_i naturally. Let us set

$$I_\Omega(H_i) = \{b\in\Omega \mid b^h=b \quad \text{for any} \quad h\in H_i\}, \quad \text{and}$$
$$I_{\Omega_j}(H_i) = \{b\in\Omega_j \mid b^h=b \quad \text{for any} \quad h\in H_i\}.$$

(Note that the cardinality of these sets are independent of the choice of H_i in $\mathfrak{H}_i$ and of the choice of H_j in $\mathfrak{H}_j$.) Moreover let us set

$$A_{G,H}(H_i) = \{X\leqq H \mid \text{there exists } g\in G \text{ such that } X^g = H_i\}$$
$$(\text{where } X^g = g^{-1}Xg), \text{ and}$$

$$A_{H,H}(H_i) = \mathfrak{H}_i.$$

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We denote by y_i the number of suborbits Δ's such that (H, Δ) is isomorphic to (H, Ω_i) as a permutation group (i.e., $H_b(b\in\Delta)$ and $H_c(c\in\Omega_i)$ are conjugate in H). For a group X and its subgroup Y, $N_X(Y)$ denotes the normalizer of Y in X.

PROPOSITION 1. (This is nothing but a restatement of a result of Alperin [1]). *We have*

$$|I_\Omega(H_i)| = \frac{|A_{G,H}(H_i)|\cdot|N_G(H_i)|}{|A_{H,H}(H_i)|\cdot|N_H(H_i)|} = \frac{|A_{G,H}(H_i)|\cdot|N_G(H_i)|}{|H|}.$$

PROOF. Set $|A_{G,H}(H_i)|=d$, and set $A_{G,H}(H_i)=\{H_i^{g_1},\cdots,H_i^{g_d}\}$. Let us set $A=\{g\in G|H_i^g\leqq H\}$. Then $g\in A\Leftrightarrow H_i^g=H_i^{g_j}$ for some $j\Leftrightarrow gg_j^{-1}\in N_G(H_i)$ $\Leftrightarrow g\in N_G(H_i)g_j$. Therefore, $|A|=|N_G(H_i)|d$. While, let us set $A'=\{g\in G|$ $H_i\leqq H^g\}$. Clearly $|A'|=|A|$. Moreover we have that $Hg\in I_\Omega(H_i)\Leftrightarrow Hgx$ $=Hg$ for any $x\in H_i\Leftrightarrow gxg^{-1}\in H$ for any $x\in H_i\Leftrightarrow x\in H^g$ for any $x\in H_i\Leftrightarrow H_i$ $\leqq H^g\Leftrightarrow g\in A'$. Thus $|I_\Omega(H_i)|=|A|/|H|$. Thus we have completed the proof of Proposition 1.

PROPOSITION 2. *We have*

$$y_i\cdot|I_{\Omega_i}(H_i)| = |I_\Omega(H_i)|-\sum_j y_j|I_{\Omega_j}(H_i)|,$$

where the summation ranges over all j's such that $\mathfrak{H}_j>\mathfrak{H}_i$.

PROOF. Since $|I_\Omega(H_i)|=\sum_{j=1}^t y_j|I_{\Omega_j}(H_i)|$ and, for $i\neq j$, $|I_{\Omega_j}(H_i)|\neq 0$ if and only if $\mathfrak{H}_j>\mathfrak{H}_i$, the assertion follows at once.

REMARK. The number of suborbits of length 1 is equal to $|N_G(H)|/|H|$. Also, $|I_{\Omega_i}(H_i)|=|N_H(H_i)|/|H_i|$.

COROLLARY. *If H_i is maximal in H, then we have*

$$y_i\cdot\frac{|N_H(H_i)|}{|H_i|} = |I_\Omega(H_i)| - \frac{|N_G(H)|}{|H|}.$$

In addition, if (G,Ω) is primitive and H_i is not normal in H, then

$$y_i = |I_\Omega(H_i)|-1.$$

CONCLUDING REMARK. If we know the number $|A_{G,H}(H_i)|$ and $|N_G(H_i)|$ for all i, then we can calculate every $|I_\Omega(H_i)|$ by Proposition 1. Moreover, if we know $|I_{\Omega_j}(H_i)|$ for any $H_j>H_i$ (this is equal to $|A_{H,H_j}(H_i)|\cdot|N_H(H_i)|/|H_j|$ by Proposition 1), then, by Proposition 2 we can calculate every y_i by induction on the order $\leqq$. In this way, (at least theoretically) we can know the suborbits and subdegrees of a given permutation group (G,Ω).

APPENDIX Y. Usami (Ochanomizu University at present) calculated completely the subdegrees of $G=PSL(2,p^n)$ ($p=5$ or $p^{2n}-1\equiv 0 \pmod 5$) acting on the cosets by $H=A_5$ (the alternating group of degree 5) in [2], 1972 May, upon the authors' request. Here we only give some examples (the following table will be self-explanatory):

subdegrees	1	5	6	10	12	15	20	30	60
PSL (2, 11)	1	0	0	1	0	0	0	0	0
PSL (2, 19)	1	0	1	0	0	0	1	1	0
PSL (2, 29)	1	0	0	0	1	0	2	3	1
PSL (2, 31)	1	1	0	0	1	0	1	3	2
PSL (2, 41)	1	1	1	0	1	0	2	3	7
PSL (2, 5^2)	2	0	0	0	4	0	1	2	0

The complete table (which is a little complicated because the result varies according to some congruence properties of p^n) is now available upon request to any one of the authors.

Department of Mathematics, Faculty of Sciences,
University of Tokyo, Hongo, Tokyo, 113, Japan.

Department of Mathematics, Faculty of Sciences,
Hokkaido University, Sapporo, 060, Japan.

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[2] Some articles in "YUGENGUN TANSHIN" vol. 2–5, 6 and 3–1, 1972, January, March, May.

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