On the decompositions of function algebras

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Introduction. We shall be concerned with the decompositions of function algebras which are finer than the maximal antisymmetric decomposition. This fact was pointed out by Arenson [1] and Nishizawa [7], who respectively used the methods of Glicksberg [4] and Bishop [2]. Throughout the paper, underlying space X is a compact Hausdorff space and $C(X)$ denotes the algebra of all continuous complex-valued functions on X. We aim at the more systematic investigations of such decompositions of closed subspaces of $C(X)$ and of function algebras on X. Now we state our results in more detail, and define some usual notations which are used in this paper.

In §1, we consider a closed subspace B of $C(X)$. We show that the decompositions by the Glicksberg-Arenson method are always the decompositions by the Bishop-Nishizawa method, and that there exists the finest decomposition for each of the two methods. In §2, we consider a function algebra A on X. We show that there exists a one-to-one correspondence between p-sets in the base space X and p-sets in the maximal ideal space $\mathcal{M}(A)$. In virtue of this correspondence, we investigate the relations between the decompositions on X and those on $\mathcal{M}(A)$. In §3, we consider the rational function algebra $R(X)$ on a compact plane set X. In §4, we show that the difference between the maximal antisymmetric decomposition and the finer decomposition is of topological character. In §5, we shall construct three examples. Especially, Example 1 indicates that there must exists a decomposition which consists of more elementary components instead of the maximal antisymmetric components: Nevertheless, elementary components will not make simple algebras in general treatments.

Notations. $M(X)$ denotes the usual Banach space of all complex finite regular Borel measures on X. For μ in $M(X)$, we shall employ the notational abuse: $\mu(f)=\int f d\mu$. Let B be a closed subspace of $C(X)$, and we denote by $B^{\perp}$, $b(B^{\perp})$, and $b(B^{\perp})^e$ the total of anihilating measures of B, the unit ball of $B^{\perp}$, and the total of extreme points of $b(B^{\perp})$, respectively. Let E be a closed subset of X, and we denote by $f|E$ the restriction of the function f to E and $B|E=\{f|E: f\in B\}$. Let B_E denote the uniform closure of B/E in $C(E)$, and μ_E the restriction of μ to E: $\mu_E(K)=\mu(K\cap E)$. $M(E)$ can be considered as the closed subspace of $M(X)$ as the usual way.

1. Decompositions of closed subspaces of $C(X)$. Let B be a closed subspace of $C(X)$. We consider the family $\mathcal{E}$ of closed subsets of X, which satisfies the following condition:

(D) *If* $f\in C(X)$ *and* $f|E\in B_E$ *for all* $E\in\mathcal{E}$, *then* $f\in B$.

This condition suggests that B is obtained by connecting the elements of B_E continuously on X. In this sense, we say B is decomposed to $\{B_E\}_{E\in\mathcal{E}}$. This condition is equivalent to the following:

(D_1) $B^\perp =$ *weak* closed linear span of* $\bigcup_{E\in\mathcal{E}} B_E^\perp$.

For if $\mathcal{E}$ satisfies (D_1), and if $f\in C(X)$ and $f|E\in B_E$ for all $E\in\mathcal{E}$, then $f\perp B_E^\perp$ for all $E\in\mathcal{E}$. Hence $f\perp B^\perp$, or $f\in B$. To prove the converse, suppose $\mathcal{E}$ does not satisfy the condition (D_1). Then there exist $f\in C(X)$ and $\mu\in B^\perp$ such that $\mu(f)\neq 0$ and $f\perp B_E^\perp$ for all $E\in\mathcal{E}$, thus $f|E\in B_E$ for all $E\in\mathcal{E}$ and $f\notin B$. Hence (D) is not satisfied.

Here we give some stronger conditions which define the decompositions of B.

(Sc) *For any closed subset* S *of* X, $\mathcal{E}|S=\{E\cap S\colon E\in\mathcal{E}\}$ *satisfies the condition* (D) *for closed subspace* B_S *of* $C(S)$.

(BN) *For any* $\mu\in b(B^\perp)$ *and* $f\in C(X)$ *such that* $\mu(f)\neq 0$, *there exist* $E\in\mathcal{E}$ *and* $\nu\in b(B^\perp)$ *such that*

$$|\nu(f)|\geq|\mu(f)|, \quad \text{and} \quad \operatorname{supp}(\nu)\subset\operatorname{supp}(\mu)\cap E.$$

(GA) *If* $\mu\in b(B^\perp)^e$, *then there exists* $E\in\mathcal{E}$ *such that* $\operatorname{supp}(\mu)\subset E$.

DEFINITION 1.1. *Let* $\mathcal{E}$ *be a family of closed subsets of* X. *If* $\mathcal{E}$ *satisfies a condition* (C), *where* (C) *denotes* (D), (Sc), (BN), *and* (GA), *then we say* $\mathcal{E}$ *has the property* (C), *or* $\mathcal{E}$ *is a* (C)*-family for* B. *Moreover, if* $\mathcal{E}$ *is a partition of* X (*i.e., a pairwise disjoint, closed covering of* X), *then we say* $\mathcal{E}$ *has a property* (C*), *or* $\mathcal{E}$ *is a* (C)*-partition for* B. *And we shall use the notation* $\mathcal{E}|S$ *throughout the paper.*

THEOREM 1.2. *Let* $\mathcal{E}$ *be a family of closed subsets of* X. *Then, for the properties of* $\mathcal{E}$, *the following relations hold*:

$$(\mathrm{GA})\Rightarrow(\mathrm{BN})\Rightarrow(\mathrm{Sc})\Rightarrow(\mathrm{D}).$$

To prove the theorem, we need two lemmas.

LEMMA 1.3. *For any closed subset* S *of* X, *the following hold*:

(i) $B_S^\perp = B^\perp\cap M(S)$.

(ii) $b(B_S^\perp) = b(B^\perp)\cap M(S)$.

(iii) $b(B_S^{\perp})^e = b(B^{\perp})^e \cap M(S)$.

PROOF: (i) is clear by the definition of B_S, and (ii) follows from (i). To prove (iii), we suppose $\mu \in b(B_S^{\perp})^e$, then $\mu \in b(B^{\perp})$ by (ii). Thus we take $0<t<1$, and ν, $\lambda \in b(B^{\perp})$ such that $\mu = t\nu + (1-t)\lambda$. Then

$$\mu = \mu_S = t\nu_S + (1-t)\lambda_S .$$

Therefore

$$1 = \|\mu\| \leq t\|\nu_S\| + (1-t)\|\lambda_S\| \leq 1 .$$

Thus we must have $\|\nu_S\| = \|\lambda_S\| = 1$, and this implies $\nu_S = \nu$, $\lambda_S = \lambda$. If $\mu \in b(B^{\perp})^e \cap M(S)$, then (ii) implies immediately $\mu \in b(B_S^{\perp})^e$. This proves the lemma.

LEMMA 1.4. *If a family $\mathcal{E}$ of closed sets has the property* (Sc) (*or* (BN), (GA)) *for B, then, for any closed subsets S of X, $\mathcal{E}|S$ has also the property* (Sc) (*respectively* (BN), (GA)) *for B_S.*

PROOF: Since $(\mathcal{E}|S)|T = \{(E\cap S)\cap T\colon E \in \mathcal{E}\} = \mathcal{E}|T$ always holds for any closed subset T of S, the case (Sc) is clear. Suppose $\mathcal{E}$ is a (GA)-family. If $\mu \in b(B_S)^e$, then $\mu \in b(B^{\perp})^e$ by Lemma 1.3; therefore there exists $E \in \mathcal{E}$ such that $\operatorname{supp}(\mu) \subset E$. Since μ is a measure on S, we have $\operatorname{supp}(\mu) \subset E \cap S$. This shows the case (GA). Now we assume that $\mathcal{E}$ is a (BN)-family; and $f \in C(S)$, $\mu \in b(B_S^{\perp})$ such that $\mu(f) \neq 0$. Let g be a continuous extension of f on X, then we have

$$\mu(g) = \int g\, d\mu = \int_E f\, d\mu = \mu(f) \neq 0 ,$$

and $\mu \in b(B^{\perp})$ by Lemma 1.3, (ii). Now, there exist $\nu \in b(B^{\perp})$ and $E \in \mathcal{E}$ such that $|\nu(g)| \geq |\mu(g)|$, and $\operatorname{supp}(\nu) \subset \operatorname{supp}(\mu) \cap E$. Since μ is a measure on S, it follows $\operatorname{supp}(\nu) \subset \operatorname{supp}(\mu) \cap (E \cap S)$. Thus we have $\nu \in b(B_S^{\perp})$, and $|\nu(f)| = |\nu(g)| \geq |\mu(g)| = |\mu(f)|$. This shows that $\mathcal{E}|S$ has the property (BN).

PROOF OF THE THEOREM: In the condition (Sc), put $S=X$, then we see that (Sc) implies (D). To prove that (BN) implies (Sc), by Lemma 1.4, it is sufficient to show that (BN) implies (D); we assume that $\mathcal{E}$ satisfies (BN). If $f \notin B$, then there exists $\mu \in b(B^{\perp})$ such tha$\rceil$ $\mu(f) \neq 0$. By the assumption, there exist $E \in \mathcal{E}$ and $\nu \in b(B^{\perp})$ such that

$$|\nu(f)| \geq |\mu(f)| \neq 0 , \qquad \operatorname{supp}(\nu) \subset \operatorname{supp}(\mu) \cap E .$$

Then $\nu \in b(B_E^{\perp})$ and $\nu(f) \neq 0$, so we have $f|E \notin B_E$, therefore (D) hold. Finally, we show that (GA) implies (BN). Under the assumption that $\mathcal{E}$ satisfies (GA), take any $\mu \in b(B^{\perp})$, $f \in C(X)$ such that $\mu(f) \neq 0$. Using Lemma 1.4

when $S=\operatorname{supp}(\mu)$, $\mathcal{E}|S$ has the property (GA) for B_S. The function $\nu\mapsto|\nu(f)|$ attains the maximum on $b(B_S^\perp)$ at $\nu_0\in b(B_S^\perp)^e$. Therefore, there is a set $S\cap E\in\mathcal{E}|S$ such that $\operatorname{supp}(\nu_0)\subset E\cap S$, and since $\mu\in b(B_S^\perp)$, we have

$$|\nu_0(f)|\geq|\mu(f)|\,,\quad \text{and} \quad \operatorname{supp}(\nu_0)\subset\operatorname{supp}(\mu)\cap E\,.$$

This shows that $\mathcal{E}$ has the property (BN). That completes the proof.

Next, concerning the properties (GA) and (BN), we shall show that there exists the finest decomposition for each property (c. f. Arenson [1], Nishizawa [7]).

For the convenience of the notations, we agree to use the following: Let $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_\alpha$, and $\mathcal{F}$ denote families of subsets of X. If, for any $E_1\in\mathcal{E}_1$, there exists $E_2\in\mathcal{E}_2$ such that $E_1\subset E_2$, then we shall say $\mathcal{E}_1$ is finer than $\mathcal{E}_2$, and we denote $\mathcal{E}_1\prec\mathcal{E}_2$. For $\mathcal{E}_\alpha(\alpha\in\mathfrak{A})$, we define

$$\bigwedge_{\alpha\in\mathfrak{A}}\mathcal{E}_\alpha=\{\bigcap_{\alpha\in\mathfrak{A}}E_\alpha:\ E_\alpha\in\mathcal{E}_\alpha\}\,.$$

Let S be a subset of X which satisfies the following;

$$\text{if} \quad F\in\mathcal{F} \quad \text{and} \quad S\cap F\neq\emptyset, \quad \text{then} \quad F\subset S\,.$$

We shall say such a set S is saturated with $\mathcal{F}$, and if all the elements of $\mathcal{E}$ is saturated with $\mathcal{F}$, then we shall also say $\mathcal{E}$ is saturated with $\mathcal{F}$. The following facts are easy to verify:

(1. 1) *If all $\mathcal{E}_\alpha$ are partitions of X, then $\bigwedge_\alpha\mathcal{E}_\alpha$ is a partition of X.*

(1. 2) *If $\mathcal{F}$ is finer than all $\mathcal{E}_\alpha$, then $\mathcal{F}\prec\bigwedge_\alpha\mathcal{E}_\alpha$.*

(1. 3) *If all $\mathcal{E}_\alpha$ are saturated with $\mathcal{F}$, then $\bigwedge_\alpha\mathcal{E}_\alpha$ is saturated with $\mathcal{F}$.*

(1. 4) *Let $\mathcal{E}$ be a partition of X. Then $\mathcal{E}$ is saturated with $\mathcal{F}$ if and only if $\mathcal{E}\succ\mathcal{F}$.*

THEOREM 1. 5. *Let $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_\alpha$ be families of closed subsets of X, and B be a closed subspace of $C(X)$.*

(i) *If $\mathcal{E}_1\prec\mathcal{E}_2$, and $\mathcal{E}_1$ has the property* (C), *then $\mathcal{E}_2$ has also the property* (C). *Here* (C) *denotes* (D), (Sc), (BN), *and* (GA).

(ii) *Let $\mathcal{E}_1$ have the property* (Sc) *and $\mathcal{E}_2$ have the property* (D), *then $\mathcal{E}_1\wedge\mathcal{E}_2$ has the property* (D). *Moreover, if $\mathcal{E}_2$ also has the property* (Sc), *then $\mathcal{E}_1\wedge\mathcal{E}_2$ has the property* (Sc).

(iii) *If all $\mathcal{E}_\alpha$ have the property* (GA), *then $\bigwedge_\alpha\mathcal{E}_\alpha$ has the property* (GA).

(iv) *If all $\mathcal{E}_\alpha$ have the property* (BN), *then $\bigwedge_\alpha\mathcal{E}_\alpha$ has the property* (BN).

PROOF: (i) follows from the definitions.

(ii) Let $f\in C(X)$, and we assume that $f|E_1\cap E_2\in B_{E_1\cap E_2}$ for all $E_1\in\mathcal{E}_1$, $E_2\in\mathcal{E}_2$. We let $E_2\in\mathcal{E}_2$ to be fixed. Then $\mathcal{E}_1|E_2$ has the property (D) for B_{E_2}. Thus, by the assumption, we have $f|E_2\in B_{E_2}$. Since this holds for any $E_2\in\mathcal{E}_2$, we have $f\in B$. Now we assume that $\mathcal{E}_2$ has the property (Sc). Then, for any closed subset S of X, both $\mathcal{E}_1|S$ and $\mathcal{E}_2|S$ have the property (Sc) for B_S. Since $(\mathcal{E}_1\wedge\mathcal{E}_2)|S=(\mathcal{E}_1|S)\wedge(\mathcal{E}_2|S)$, the first half implies $(\mathcal{E}_1\wedge\mathcal{E}_2)|S$ has the property (D).

(iii) Clearly, $\mathcal{E}_\alpha$ is a (GA)-family if and only if $\mathcal{E}_\alpha\succ\{\mathrm{supp}\,(\mu):\ \mu\in b(B^\perp)^e\}$. Thus (iii) follows from (1.2).

(iv) To execute the proof, we may assume that $\{\mathcal{E}_\alpha\}$ is well-ordered: $\mathcal{E}_0$, $\mathcal{E}_1,\cdots$, $\mathcal{E}_\alpha,\cdots$ $(\alpha<\omega)$, where $\mathcal{E}_0=\{X\}$. Set $\mathscr{F}_\alpha=\bigwedge_{\beta<\alpha}\mathcal{E}_\beta$. We note that at the end ω of the transfinite seriese, $\mathscr{F}_\omega$ coinsides $\bigwedge_{\alpha\in\mathfrak{A}}\mathcal{E}_\alpha$; in the following, we shall prove that all $\mathscr{F}_\alpha$ have the property (BN), simultaneously. Let $f\in C(X)$ and $\mu\in b(B^\perp)$ such that $\mu(f)\neq 0$. For each α, we wish to choose the measures $\mu_\alpha\in b(B^\perp)$ (for $\alpha\leq\omega$), and the sets $E_\alpha\in\mathcal{E}_\alpha$ (for $\alpha<\omega$) which satisfy the following:

$$(\star)\quad\begin{cases}\text{(a)}\quad \mu_0=\mu\\ \text{(b)}\quad \mathrm{supp}\,(\mu_\alpha)\subset\mathrm{supp}\,(\mu_\beta)\cap E_\beta,\quad |\mu_\alpha(f)|\geq|\mu_\beta(f)|\qquad\text{for any }\beta<\alpha.\end{cases}$$

We shall constract, by transfinite induction for $\delta(\leq\omega)$, the measures $\mu_\alpha\in b(B^\perp)$ $(\alpha\leq\delta)$ and the sets $E_\alpha\in\mathcal{E}_\alpha$ $(\alpha<\delta)$ satisfying $(\star)$. When $\delta=0$, $(\star)$ holds for the measure $\mu_\delta=\mu$. Now we assume that the measures μ_α $(\alpha\leq\tau)$ and the sets E_α $(\alpha<\tau)$ have been constracted for all $\tau<\delta$. If δ has the immediately before element γ, then, let $S=\bigcap_{\alpha<\gamma}E_\alpha$, we have $\mathrm{supp}\,(\mu_\gamma)\subset S$, or $\mu_\gamma\in b(B_S^\perp)$. Since, by Lemma 1.4, $\mathcal{E}_\gamma|S$ has the property (BN) for B_S, there exist $\mu_\delta\in b(B_S^\perp)$ and $E_\gamma\in\mathcal{E}_\gamma$ such that $|\mu_\delta(f)|\geq|\mu_\gamma(f)|$ and $\mathrm{supp}\,(\mu_\delta)\subset E_\gamma\cap\mathrm{supp}\,(\mu_\gamma)$. Then μ_α $(\alpha\leq\delta)$ and E_α $(\alpha<\delta)$ satisfy $(\star)$. If δ has not the immediatly before element, then, let μ_δ be a weak* cluster point of $\{\mu_\alpha\}_{\alpha<\delta}$ in $b(B^\perp)$, it is easy to verify $\mathrm{supp}\,(\mu_\delta)\subset\mathrm{supp}\,(\mu_\alpha)$ and $|\mu_\delta(f)|\geq|\mu_\alpha(f)|$ for $\alpha<\delta$. Hence, the measures μ_α $(\alpha\leq\delta)$ and the sets E_α $(\alpha<\delta)$ satisfy $(\star)$. This completes the construction. Now we let $\nu_\alpha=\mu_\alpha$ and $F_\alpha=\bigcap_{\beta<\alpha}E_\beta$, then

$$|\nu_\alpha(f)|\geq|\mu(f)|,\quad\text{and}\quad \mathrm{supp}\,(\nu_\alpha)\subset\mathrm{supp}\,(\mu)\cap F_\alpha.$$

This completes the proof.

REMARK. In the proof of (iv), the condition $\mathrm{supp}\,(\nu)\subset\mathrm{supp}\,(\mu)\cap E$ is an essential fact. Nishizawa ([7]) has attended that Bishop had proved not only the fact $\mathrm{supp}\,(\nu)\subset E$ but also the fact $\mathrm{supp}\,(\nu)\subset\mathrm{supp}\,(\mu)\cap E$, and showed that there exists the finest (BN)-partition of p-sets for function algebras. So the idea of this proof entirely due to her.

In virtue of this theorem,

$$\mathscr{E}_{GA} = \wedge \{\mathcal{E} : \mathcal{E} \text{ has the property (GA)}\}$$

is the finest closed set's family which satisfies the property (GA), and

$$\mathscr{E}_{BN} = \wedge \{\mathcal{E} : \mathcal{E} \text{ has the property (BN)}\}$$

is the finest closed set's family which satisfies the property (BN). In other words the following holds.

COROLLARY 1. 6. *Let $\mathcal{E}$ be a family of closed sets.*

(i) *$\mathcal{E}$ is a (GA)-family if and only if $\mathcal{E} \succ \mathscr{E}_{GA}$.*

(ii) *$\mathcal{E}$ is a (BN)-family if and only if $\mathcal{E} \succ \mathscr{E}_{BN}$.*

However, in the case (GA), we have only to consider $\{\mathrm{supp}(\mu) : \mu \in b(B^{\perp})^e\}$, and yet, $\mathscr{E}_{GA}$ contains many redundant sets which are all closed subsets of $\mathrm{supp}(\mu)$ for $\mu \in b(B^{\perp})^e$. Similarly, $\mathscr{E}_{BN}$ also contains redundant sets; but these sets can not be pointed out distinctly.

If we only consider the partitions of X as the family of closed sets, then

$$\mathscr{E}^*_{GA} = \wedge \{\mathcal{E} : \mathcal{E} \text{ is a (GA)-partition}\}$$

$$\mathscr{E}^*_{BN} = \wedge \{\mathcal{E} : \mathcal{E} \text{ is a (BN)-partition}\}$$

are the finest partitions of X for each property.

By the way, we consider families of p-sets for B; p-set is a closed set E of X such that $\mu_E \in B^{\perp}$ for any $\mu \in B^{\perp}$. If E_1 and E_2 are p-sets, then $E_1 \cap E_2$ is a p-set, and if E_ι is a family of p-sets, then $\bigcap_\iota E_\iota$ is a p-set. Thus our arguments up to this time hold for p-set's families without change, and we can also define $\mathscr{P}_{GA}$, $\mathscr{P}_{BN}$, $\mathscr{P}^*_{GA}$, $\mathscr{P}^*_{BN}$, which correspond with closed set's families. By Glicksberg ([4], Th. 3. 3), the following relation holds for $\mathscr{E}^*_{GA}$ and $\mathscr{P}^*_{GA}$.

COROLLARY 1. 7. *If a set $E \in \mathscr{E}^*_{GA}$ is a G_δ-set, then E is a p-set for B. In particular, if X is a metrizable space, then $\mathscr{E}^*_{GA} = \mathscr{P}^*_{GA}$.*

For another finest decomposition in some sense, we can consider the following; let $\mathscr{F}$ be a certain family of subsets of X, and we consider the families of closed sets, or of p-sets of X which are saturated with $\mathscr{F}$. In fact, Arenson has studied the finest closed set's families which are saturated with the weakly analytic sets (see [1]). As one more example, we can consider the famiy $\mathcal{E}$ of p-sets such that if E_1 and E_2 are distinct elements of $\mathcal{E}$, then $E_1 \cap E_2$ is a interpolation set, i. e., $B|E_1 \cap E_2 = C(E_1 \cap E_2)$.

However, we don't know any notable properties for these families.

2. Decompositions of function algebras. Let A be a function algebra on X, i. e., uniform closed subalgebra of $C(X)$ which contains the constant functions and separates the points of X. A closed subset E of X is a peak set for A (frequently, we will also say "on X") if there is a function $f \in A$ such that $f(x)=1$ for $x \in E$, and $|f(y)|<1$ for $y \in X \backslash E$, and then the function f is said to be a peaking function for E. For function algebras, E is a p-set if and only if E is a intersection of peak sets ([6], Th. 4. 8). Let $\mathcal{M}(A)$ be the maximal ideal space of A, and $\hat{f}$ denotes the Gelfand transform of $f \in A$. Then the total $\hat{A}$ of $\hat{f}$ is regarded as a function algebra on $\mathcal{M}(A)$. For closed subset E of X,

$$\tilde{E} = \left\{a \in \mathcal{M}(A) : |\hat{f}(a)| \leq \|f\|_E \quad \text{for all } f \in A\right\}$$

is said to be the A-convex hull of E, where $\|f\|_E = \sup_{x \in E} |f(x)|$. We need the following well known facts which are easily seen.

(2. 1) *$a \in \tilde{E}$ if and only if there exists a representing measure for a supported on E, i. e., there exists a positive measure μ on E such that $\int f d\mu = \hat{f}(a)$ for all $f \in A$. Then, necessarily, $\|\mu\| = \mu(X) = 1$.*

(2. 2) *$\tilde{E}$ is identified with the maximal ideal space of A_E, and then $\widehat{f|E} = \hat{f}|E$ for any $f \in A$.*

A subset S of X is said to be antisymmetric for A if all real functions of $A|S$ are constants. Then (2. 1) and (2. 2) imply

(2. 3) *E is antisymmetric for A if and only if $\tilde{E}$ is for $\hat{A}$.*

Let $\mathcal{K} = \{K\}$ be the family of maximal antisymmetric sets for A, then it is easy to show that $\mathcal{K}$ is a partition of X by closed subsets. In our terminology, Bishop ([2]) has proved that $\mathcal{K}$ is a family of p-sets and has the property (BN), and Glicksberg ([4]) has give the simple proof of the fact: $\mathcal{K}$ has the property (GA). Furthermore, he showed that $\tilde{\mathcal{K}} = \{\tilde{K} : K \in \mathcal{K}\}$ (we will use this notation without notice, in general) is the family of maximal antisymmetric sets for $\hat{A}$. Thus we obtain the following theorem.

THEOREM. *Let $\mathcal{K}$ be the family of maximal antisymmetric sets for function algebra A. Then $\mathcal{K}$ has the following properties.*

(a) *$\mathcal{K}$ is a family of p-sets on X and has the property* (GA).

(b) $\tilde{\mathscr{K}}$ *is the family of maximal antisymmetric sets for* $\hat{A}$.

In vatue of Lemma 1. 4, we obtain a generalization of a Glicksberg's result ([4], Corollary 3. 4).

COROLLARY 2. 1. *Let S be any closed subset of X. Then* $\mathscr{K}|S=\{S\cap K: K\in\mathscr{K}\}$ *is a family of p-sets and a* (GA)-*partition for A. Moreover* $(\widetilde{\mathscr{K}|S})=\tilde{\mathscr{K}}|\tilde{S}$, *thus* $(\widetilde{\mathscr{K}|S})$ *is also a family of p-sets and a* (GA)-*partition for* $\hat{A}_{\tilde{S}}$.

The last statement will be made clear by Corollary 2. 4. Also, the following corollary is not difficult to prove.

COROLLARY 2. 2. *Let $\mathcal{E}$ be a* (C)-*family of closed sets. Then there exists a* (C)-*family $\mathcal{E}'$ which consists of antisymmetric closed sets of X and is finer than $\mathcal{E}$. Moreover, if $\mathcal{E}$ consists of p-sets, then $\mathcal{E}'$ also consists of p-sets, and if $\tilde{\mathcal{E}}$ is a covering of $\mathscr{M}(A)$, then $\tilde{\mathcal{E}}'$ also is a covering of $\mathscr{M}(A)$. Here,* (C) *denotes* (D), (Sc), (BN), *and* (GA).

Now we shall study the relations between the decompositions on X and the decompositions on $\mathscr{M}(A)$ by p-set's families.

THEOREM 2. 3. *Let E be a p-set for A.*

(i) *If $a\in E$, then any representing measure for a is supported on E.*

(ii) *For any closed subset S of X,*

$$\widetilde{E\cap S}=\tilde{E}\cap\tilde{S}.$$

In particular, if $E\cap S=\emptyset$, then $\tilde{E}\cap\tilde{S}=\emptyset$.

(iii) *E is saturated with Gleason parts in $\mathscr{M}(A)$.*

PROOF: Let $a\in\tilde{E}$. Then there exists a representing measure μ for a supported on E. Let ν be any representing measure for a. Then $(\mu-\nu)_E=\mu-\nu_E\in A^{\perp}$. Thus ν_E is also representing measure for a. The norms of representing measures are always equal to 1, and hence ν must be supported on E. Therefore (i) holds. Let $a\in\tilde{E}\cap\tilde{S}$, and take the representing μ for a supported on S. Then μ must be supported on E by (i). So we have $\tilde{E}\cap\tilde{S}\subset\widetilde{E\cap S}$. Clearly, the converse inclusion holds. This shows (ii). If a and b belong to a same Gleason part, then there exist mutually absolutely continuous representing measures for a and for b. Therefore (iii) follows from (i).

COROLLARY 2. 4. *Let $\mathcal{E}$ be a family of p-sets on X.*

(i) $(\widetilde{\mathcal{E}|S})=\tilde{\mathcal{E}}|\tilde{S}$ *for any closed subset S of X.*

(ii) *$\tilde{\mathcal{E}}$ is a covering of $\mathscr{M}(A)$ if and only if* $\mathcal{E}\succ\{\mathrm{supp}(\mu)$: *$\mu$ is a representing measure for A*$\}$.

THEOREM 2.5. *There is a one-to-one correspondence between p-sets E on X and p-sets F on $\mathcal{M}(A)$ such that $\tilde{E}=F$, and $F\cap X=E$. For this correspondence, it follows that*:

(i) *E is a peak set on X if and only if $\tilde{E}$ is a peak set on $\mathcal{M}(A)$.*

(ii) $\widetilde{\bigcap_\iota E_\iota}=\bigcap_\iota \tilde{E}_\iota$, *where E_ι is a p-set on X.*

PROOF: Let E be a peak set on X and $f\in A$ a peaking for E. Then $\hat{f}$ is a peaking function for $\tilde{E}$. For, if $|\hat{f}(a)|=1$, then we let μ be a representing measure for a and we have

$$1=|\hat{f}(a)|=\left|\int f\,d\mu\right|\leq\int|f|\,d\mu\leq 1\,.$$

Since $\|f\|\leq 1$ and f is a continuous function, $|f|=1$ on the support of μ. For f is a peaking function for E, $|f(x)|=1$ implies $x\in E$, and we have $\operatorname{supp}(\mu)\subset E$. This shows $a\in\tilde{E}$. Conversely, if $a\in\tilde{E}$, then $\hat{f}(a)=1$. So, we have

$$\tilde{E}=\left\{a\in\mathcal{M}(A):\ \hat{f}(a)=1\right\},\tag{1}$$

and $|\hat{f}(a)|<1$ for $a\in\mathcal{M}(A)\backslash\tilde{E}$. Hence $\tilde{E}$ is a peak set on $\mathcal{M}(A)$. On the other hand, let F be a peak set on $\mathcal{M}(A)$ and $\hat{f}\in\hat{A}$ a peaking function for F. Then

$$F\cap X=\left\{x\in X:\ f(x)=1\right\}.\tag{2}$$

Therefore f is a peaking function for $F\cap X$ and $F\cap X$ is a peak set on X. From (1) and (2), we obtain a one-to-one correspondence for peak sets: $\tilde{E}\cap X=E$, $\widetilde{F\cap X}=F$. Next, we prove (ii). Clearly, $\widetilde{\bigcap_\iota E_\iota}\subset\bigcap_\iota\tilde{E}_\iota$. To see the equality, let $a\in\bigcap_\iota\tilde{E}_\iota$. Let μ be a representing measure for a. Since $a\in\tilde{E}_\iota$ and E_ι is a p-set on X, μ must be supported on any E_ι. Thus $\operatorname{supp}(\mu)\subset\bigcap_\iota E_\iota$, and we have $a\in\widetilde{\bigcap_\iota E_\iota}$. Therefore (ii) holds. Finally, let E be a p-set on X and F a p-set on $\mathcal{M}(A)$. We can write E and F as intersections of peak sets, i.e., $E=\bigcap_\iota E_\iota$, $F=\bigcap_\iota F_\iota$, where E_ι is a peak set on X and F_ι is a peak set on $\mathcal{M}(A)$. Then, by (ii), we have

$$\tilde{E}\cap X=(\bigcap_\iota\tilde{E}_\iota)\cap X=\bigcap_\iota(\tilde{E}_\iota\cap X)=\bigcap_\iota E_\iota=E\,,$$

$$\widetilde{F\cap X}=\bigcap_\iota\widetilde{(F_\iota\cap X)}=\bigcap_\iota\widetilde{(F_\iota\cap X)}=\bigcap_\iota F_\iota=F\,.$$

This completes the proof.

We will denote the essential set of A by E_A. The definition of E_A

and the following properties are found in [6], § 4. 4.

(2. 4) *If* $f\in C(X)$ *and* $f|E_A\in A|E_A$, *then* $f\in A$.

(2. 5) E_A *is a p-set and* $\tilde{E}_A=E_{\hat{A}}$.

(2. 6) *If* F *a set satisfies the property* (2. 4), *then* $E_A\subset\bar{F}$.

THEOREM 2. 6. *Let* $\mathcal{E}$ *be a family of p-sets on* X.

(i) *If* $\mathcal{E}$ *has the property* (D) *on* X, *then* $E_A\subset\overline{\cup\mathcal{E}}$, *where* $\cup\mathcal{E}$ *denotes* $\bigcup_{E\in\mathcal{E}}E$.

(ii) *Let* $\mathcal{E}$ *has the property* (D) *on* X. *Then* $\tilde{\mathcal{E}}$ *has the property* (D) *on* $\mathcal{M}(A)$ *if and only if* $E_{\hat{A}}\subset\overline{\cup\tilde{\mathcal{E}}}$.

(iii) *Let* $\tilde{\mathcal{E}}$ *has the property* (D) *on* $\mathcal{M}(A)$, *and if* $\tilde{\mathcal{E}}$ *is a covering of* $\mathcal{M}(A)$, *then* $\mathcal{E}$ *has the property* (D) *on* X.

(iv) *Let* $\mathcal{E}$ *has the property* (D) *on* X. *Let* a *be in* $\mathcal{M}(A)\backslash\cup\tilde{\mathcal{E}}$. *If a function* $f\in A$ *does not vanish on* X *and* $\hat{f}(a)=0$, *then* $\hat{f}$ *must vanish at some points on* $\cup\tilde{\mathcal{E}}$.

(v) *If* $\tilde{\mathcal{E}}$ *has the property* (Sc) *or* (BN), (GA) *on* $\mathcal{M}(A)$, *then* $\mathcal{E}$ *has the same property on* X.

PROOF: (i) Let $f\in C(X)$, $f|\overline{\cup\mathcal{E}}\in A|\overline{\cup\mathcal{E}}$. Then $f|E\in A|E$ for any $E\in\mathcal{E}$. Thus $f\in A$. Now (i) follows from (2. 6).

(ii) Suppose $\overline{\cup\tilde{\mathcal{E}}}\supset E_{\hat{A}}$. Let $f\in C(\mathcal{M}(A))$ and $f|\tilde{E}\in\hat{A}|\tilde{E}$ for all $E\in\mathcal{E}$. Restrict f to X, we have $g=f|X\in C(X)$ and $g|E\in A|E$ for all $E\in\mathcal{E}$. This implies $g\in A$. Since $\hat{g}|\tilde{E}=\widehat{g|E}=\widehat{f|E}=f|\tilde{E}$ for $E\in\mathcal{E}$, we see that g and f agree on $\cup\tilde{\mathcal{E}}$, and hence, on $\overline{\cup\tilde{\mathcal{E}}}$. By $E_{\hat{A}}\subset\overline{\cup\tilde{\mathcal{E}}}$, $f|E_{\hat{A}}=\hat{g}|E_{\hat{A}}\in\hat{A}|E_{\hat{A}}$. Thus $f\in\hat{A}$ by (2. 4). This shows that $\tilde{\mathcal{E}}$ has the property (D). The converse follows from (i).

(iii) For $a\in\mathcal{M}(A)$, we denote the total of the representing measures for a by M_a. Let $M_R=\bigcup_{a\in\mathcal{M}(A)}M_a$. Then M_R can be expressed as follows;

$$M_R=\bigcap_{f,g\in A}\left\{\mu\in\Sigma:\ \mu(f)\mu(g)=\mu(fg)\right\},$$

where
$$\Sigma=\left\{\mu\in M(X):\ \|\mu\|=\mu(X)=1\right\}.$$

Therefore M_R is a weak* compact subset of $M(X)$. Now we can naturally regard $\mathcal{M}(A)$ as a quotient space of M_R. Let $f\in C(X)$, and suppose $f|E\in A|E$ for all $E\in\mathcal{E}$. Define the function $\tilde{f}$ on M_R by $\tilde{f}(\mu)=\int f\,d\mu$, then $\tilde{f}$ is a continuous function on M_R. For any fixed $a\in\mathcal{M}(A)$, there is a p-set $E\in\mathcal{E}$ such that $a\in\tilde{E}$. Since μ is supported on E for $\mu\in M_a$, $\widehat{f|E}(a)=\int f\,d\mu$. This shows that the value $\int f\,d\mu$ is independent of the choice $\mu\in M_a$. Thus we can regard $\tilde{f}$ as a continuous function g on the quotient space $\mathscr{M}(A)$. Moreover, $g|\tilde{E}=\widehat{f|E}\in\hat{A}|\tilde{E}$ for all $E\in\mathcal{E}$. Since $\tilde{\mathcal{E}}$ has the property (D) on $\mathscr{M}(A)$, we have $g\in\hat{A}$, and $g|X=f\in A$.

(iv) Since f does not vanish on X, $1/f\in C(X)$. If $\hat{f}$ has no zero on $\cup\tilde{\mathcal{E}}$, then $\hat{f}|\tilde{E}$ is invertible in $\hat{A}|\tilde{E}$ for any $E\in\mathcal{E}$, so we have $1/(f|E)=(1/f)|E\in A|E$ for any $E\in\mathcal{E}$. Hence, we must have $1/f\in A$. This shows that f has no zero on $\mathscr{M}(A)$, and contradicts the assumption $\hat{f}(a)=0$.

(v) It is clear by Lemma 1.4. This completes the proof.

In the proof of (iii), $\cup\tilde{\mathcal{E}}$ may well lack of the point $a\in\mathscr{M}(A)$ which has a unique representing measure. So, if any $a\in\mathscr{M}(A)$ has a unique representing measure on X, then $\mathcal{E}$ has the property (D) on X whenever $\tilde{\mathcal{E}}$ has the property (D) on $\mathscr{M}(A)$. More generally we have the following:

COROLLARY 2.7. *If any $a\in\mathscr{M}(A)$ has a unique Jensen measure, then $\mathcal{E}$ has the property* (D) *on X whenever $\tilde{\mathcal{E}}$ has the property* (D) *on $\mathscr{M}(A)$.*

PROOF: Let J_R be the total of the Jensen measures for all $a\in\mathscr{M}(A)$. In this time, we can write

$$J_R=\bigcap_{f\in A,\varepsilon>0}\left\{\mu\in M_R:\ |\hat{f}(a)|\leq\exp\left(\int\log(|f|+\varepsilon)\,d\mu\right)\right\}.$$

Therefore J_R is a weak* compact subset of $M(X)$, and we find that J_R is homeomorphic to $\mathscr{M}(A)$. The remains of the proof is the same of (iii)

REMARK. In (ii), even if p-set's family $\mathcal{E}$ is a (D)-partition on X, we can construct an example such that $\tilde{\mathcal{E}}$ has not the property (D) on $\mathscr{M}(A)$ (Example 3). In (iii), if $\tilde{\mathcal{E}}$ fails to cover $\mathscr{M}(A)$, then there exists a counter example (Example 2). For the converse of (v), even the following is unknown.

QUESTION. *If a* (GA)-*family $\mathcal{E}$ of p-sets is a partition on X, then are $\tilde{\mathcal{E}}$ a covering of $\mathscr{M}(A)$?*

However, the following partial results hold.

THEOREM 2.8. *Let $a\in\mathscr{M}(A)\backslash X$.*

(i) *Suppose, for any neighborhood U of a in $\mathscr{M}(A)$, there exists $f\in A$ such that $\hat{f}(a)=0$ and $\{b\in\mathscr{M}(A):\ \hat{f}(b)=0\}\subset U$. Then $a\in\overline{\cup\tilde{\mathcal{E}}}$ for any* (D)-*family $\mathcal{E}$ of p-sets on X.*

(ii) *Suppose a has a unique representing measure, and the Gleason part which contains a, contains at least two point. Then $a\in\cup\tilde{\mathcal{E}}$ for any* (GA)-*family $\mathcal{E}$ of p-sets on X.*

PROOF: (i) Let $a\notin\overline{\cup\tilde{\mathcal{E}}}$. There is a neighborhood of a such that

$U\cap((\overline{\cup\tilde{\mathcal{E}}})\cup X)=\emptyset$, and then there exists a function $f\in A$ such that $\hat{f}(a)=0$ and $\{b\in\mathcal{M}(A):\ \hat{f}(b)=0\}\subset U$. Thus we obtain the function $f\in A$ which does not vanish on X and $\hat{f}(a)=0$. This contradicts Theorem 2.6, (iv).

(ii) Let m be a representing measure for a and $H^\infty(m)$ the weak* closure of A in $L^\infty(m)$. Note that $H_0^1(m)=\{f\in L^1(m):\ \int fg\,dm=0$ for all $g\in H^\infty(m)\}$ ([8], Th. 2.3.8). Since $H_0^1(m)$ is a simply invariant space, there exists a function $F\in H_0^\infty(m)$ such that $H_0^1(m)=FH^1(m)$ and $|F|=1$ a.e. $[m]$ (c.f. [3], Chap. V, Th. 6.2, and Th. 7.2). Now we want to show that $\mu=Fm\in b(A^\perp)^e$. Clearly, $\mu\in b(A^\perp)$. Thus we let $\mu=t\nu_1+(1-t)\nu_2$, where $0<t<1$, and ν_1, $\nu_2\in b(A^\perp)$. Let $\nu_i=h_im+\nu_i'$ be the Lebesgue decomposition of measure ν_i for $i=1,2$. Then we have

$$\mu=th_1m+(1-t)h_2m\,.$$

Thus we must have $\nu_i=h_im$ $(i=1,2)$ in a manner similar to the proof of Lemma 1.3, (iii). Now $h_i\perp H^\infty(m)$, for $h_im\in b(A^\perp)$. Therefore, $h_i\in H_0^1(m)$ $=FH^1(m)$. Thus we can write $h_i=Fg_i$, where $g_i\in H^1(m)$, and we have

$$F=tg_1F+(1-t)g_2F\,.$$

Since $F\bar{F}=1$, we obtain

$$1=tg_1+(1-t)g_2\,.$$

On the other hand,

$$\|g_i\|_1=\int|g_i|\,dm=\int|g_iF|\,dm=\|h_im\|=1\,.$$

Since 1 is an extremal function of $H^1(m)$ (c.f. [3], Chap. V, Th. 9.5, and Lemma 9.1), we must have $g_1=g_2=1$, i.e., $\nu_1=\nu_2=\mu$, and hence, we obtain $\mu\in b(A^\perp)^e$. Now there is a set $E\in\mathcal{E}$ such that $\operatorname{supp}(\mu)\subset E$, and since $\operatorname{supp}(m)$ $=\operatorname{supp}(\mu)$, we have $a\in\tilde{E}$. That completes the proof.

In the proof of (ii), our purpose was to find the element of $b(A^\perp)^e$ which is absolutly continuous to a representing measure for a. Along these line one can ask the following: If a is in $\mathcal{M}(A)\backslash X$ and the Gleason part which contains a, contains at least two points, then are there an element of $b(A^\perp)^e$ which is absolutly continuous to a representing measure for a?

For the representing measures for A and the elements of $b(A^\perp)^e$, we note the following property.

THEOREM 2.9. *Let measure μ be a representing measure for A or an element of $b(A^\perp)^e$. Then, for any p-set E on X,*

$$\operatorname{supp}(\mu)\subset E \quad \textit{or} \quad |\mu|(E)=0\,.$$

PROOF: Let μ be a representing measure for $a\in\mathcal{M}(A)$. Suppose

$\text{supp}(\mu)\not\subset E$. Then $a \notin E$. We may regard μ as a representing measure on $\mathcal{M}(A)$, and we have $\mu-\delta_a\in\hat{A}$. Here δ_a denote the unit point mass at a. It follows $(\mu-\delta_a)_E=\mu_E\in\hat{A}$. Since μ_E is a positive measure and anihilates 1, we must have $\mu_E=0$. Now let $\mu\in b(A^\perp)^e$. Suppose $\mu_E\neq 0$ and $\mu_{X\setminus E}\neq 0$. Then we have

$$\mu=\|\mu_E\|\frac{\mu_E}{\|\mu_E\|}+\|\mu_{X\setminus E}\|\frac{\mu_{X\setminus E}}{\|\mu_{X\setminus E}\|},$$

$$\text{and,}\qquad \|\mu_E\|+\|\mu_{X\setminus E}\|=1,\quad 0<\|\mu_E\|,\quad \|\mu_{X\setminus E}\|<1\,.$$

Since E is a p-set, it follows $\mu/_E\|\mu_E\|\in b(A^\perp)$ and $\mu_{X\setminus E}/\|\mu_{X\setminus E}\|\in b(A^\perp)$. Thus we must have $\mu=\mu_E/\|\mu_E\|=\mu_{X\setminus E}/\|\mu_{X\setminus E}\|$. This contains self-contradiction.

REMARK. If there exists a (GA)-parition $\mathcal{E}$ of p-sets on X such that $\tilde{\mathcal{E}}$ does not cover $\mathcal{M}(A)$, then we let $a\in\mathcal{M}(A)\setminus\cup\tilde{\mathcal{E}}$ and μ be a representing measure for a with minimal suppot, and we will find that $A_{\text{supp}(\mu)}$ has some interesting property by Lemma 1.4 and Theorem 2.9.

3. Partitions of $\mathcal{M}(A)$ by p-sets and decompositions of $R(X)$.

DEFINITION 3.1. *Let A be a function algera on X. Define $\hat{\mathcal{P}}_*=\{\hat{L}_a\}$ be the finest p-set's partition of $\mathcal{M}(A)$, where $\hat{L}_a$ indicates the element of $\hat{\mathcal{P}}_*$ which contains a. Let $\hat{L}_a\cap X=L_a$ and define the p-set's partition $\mathcal{P}_*$ of X by $\{L_a\}$.*

3.2. PROPERTIES OF $\{L_a\}$:

(i) $\tilde{L}_a=\hat{L}_a$.

(ii) *The sets L_a are antisymmetric.*

(iii) *$\mathcal{P}_*$ is characterized as the finest p-set's partition which is saturated with all supports of representing measures for A.*

COROLLARY 3.3. *Let $\mathcal{E}$ be a p-set's partition on X. The following are equivalent.*

(i) *$\tilde{\mathcal{E}}$ is a partition of $\mathcal{M}(A)$.*

(ii) $\mathcal{E}\succ\mathcal{P}_*$.

(iii) *$\mathcal{E}$ is saturated with all the supports of representing measures on X.*

In general, $\{L_a\}$ may not define the decomposition of function algebra. For an example, we propose the Cole's example. However, we have the following:

THEOREM 3.4. *Let A be a function algebra such that the anihilating measures for A which are singular to all representing measures are only zero. Then $\mathcal{P}_*$ is a* (GA)*-partition for A.*

PROOF: Let $\mu \in b(A^{\perp})^{e}$. Then, by hypothesis, there is a representing measure λ for a such that $\mu \ll \lambda$ ([5], Cor. 1. 3). By the property of $\mathscr{P}_{*}$, we have $\mathrm{supp}(\mu) \subset \mathrm{supp}(\lambda) \subset L_{a}$.

Let X be a compact plane set and $R(X)$ the uniform closure in $C(X)$ of all rational functions with poles off X. Since $R(X)$ satisfies the hypothesis in Theorem 3. 4, $\mathscr{P}_{*}$ is a (GA)-partition for $R(X)$. Moreover, for $R(X)$, the following holds.

THEOREM 3. 5. *Let* $\mathscr{E}_{G}=\{\bar{P}$: P *is a Gleason part in* $X\}$. *Then* $\mathscr{E}_{G}$ *is a* (GA)-*family of closed sets.*

PROOF: Let $\mu \in b(R(X)^{\perp})^{e}$. Then there is a representing measure m for $a \in X$ such that $\mu \ll m$. Then, by the Wilken's theorem ([9], Th. 3. 3), we have $\mathrm{supp}(\mu) \subset \mathrm{supp}(m) \subset \bar{P}_{a}$. Here, P_{a} is the Gleason part which contains a.

In § 5, we shall give an example of a compact plane set such that $R(X)$ is antisymmetric and $\mathscr{P}_{*}$ is a proper partition of X (see Example 1).

4. Topological characterization. We have seen that $\mathscr{E}^{*}_{GA}$, $\mathscr{E}^{*}_{BN}$, etc. generally are finer than the family $\mathscr{K}$ of maximal antisymmetric sets. In this section we shall show that if we decompose a function algebra A to $\{A_{E}\}_{E \in \mathscr{E}}$ in some methods such that $\mathscr{E}$ is finer than $\mathscr{K}$, then the family $\mathscr{K}$ of maximal antisymmetric sets determined by the condition that the sets of $\mathscr{E}$ topologically interwine in X, where we only assume that $\mathscr{E}$ has the following property.

(S) *For any p-set S which is saturated with* $\mathscr{E}$, $\mathscr{E}|S$ *has the property* (D) *for* A_{S}. (Note: (Sc)$\Rightarrow$(S)$\Rightarrow$(D).)

To state the theorem, we begin with the following definition; let Δ be a general topological space. $C_{R}(\Delta)$ denotes all continuous real functions on X. For two points δ_1, $\delta_2 \in \Delta$, if $f(\delta_1)=f(\delta_2)$ for all $f \in C_{R}(\Delta)$, then we shall say that δ_1 and δ_2 are H-equivalent. The total of the H-equivalent class will be denoted by $\mathscr{H}(\Delta)=\{\Delta_{h}\}$, and we define the partition $\mathscr{H}_{\alpha}=\{\Delta_{a}\}$ of Δ for each ordinal number α, by transfinite induction, as follows.

(i) If $\alpha=0$, then $\mathscr{H}_{0}=\{\Delta\}$.

(ii) If α has not the immediatly be fore element, then $\mathscr{H}_{\alpha}=\bigwedge_{\beta<\alpha}\mathscr{H}_{\beta}$.

(iii) If α has the immediatly before element β, then $\mathscr{H}_{\alpha}=\bigcup_{\Delta_{b}\in\mathscr{H}_{\beta}}\mathscr{H}(\Delta_{b})$.

We let $\sigma(\Delta)$ denote the minimum ordinal number α such that $\mathscr{H}_{\alpha}=\mathscr{H}_{\alpha+1}$. We shall call $\mathscr{H}_{\sigma(\Delta)}$ by Šilov decomposition of Δ, and if $\mathscr{H}_{\sigma(\Delta)}$ consists of one point sets $\{\delta\}$ for all $\delta \in \Delta$, then Δ will be said to be Hausdorff decomposable.

THEOREM 4.1. *Let A be a function algebra on X. Let $\mathcal{E}$ be a* (S)-*partition for A which is finer than the family $\mathscr{K}$ of maximal antisymmetric sets. Let Δ be the quotient space of X which is obtained from $\mathcal{E}$. Then the partition of X which is defined by the Šilov decomposition $\mathscr{H}_{\sigma(\Delta)}$ of Δ, coinsides with $\mathscr{K}$.*

PROOF: Let q be the natural quotient mapping on X onto Δ. We let $\tilde{\mathscr{H}}_\alpha$ denote the partition of X which is defined by $\mathscr{H}_\alpha$, i.e., $\tilde{\mathscr{H}}_\alpha$ consists of all the set $\tilde{\Delta}_a=q^{-1}(\Delta_a)$ for all $\Delta_a \in \mathscr{H}_\alpha$. Then it follows that;

(a) $\tilde{\Delta}_a$ is a p-set which is saturated with $\mathcal{E}$.

(b) $\mathscr{H}_\alpha \succ \mathscr{K}$.

In fact, it holds for the case $\alpha=0$, clearly. We assume that (a) and (b) hold for any $\beta<\alpha$. If α has not the immediatly before element, then, by the definition, $\Delta_a=\bigcap_{\beta<\alpha}\{\Delta_b \in \mathscr{H}_\beta:\ \Delta_a \subset \Delta_b\}$. Therefore, $\tilde{\Delta}_a=\bigcap_{\beta<\alpha}\{\tilde{\Delta}_b \in \tilde{\mathscr{H}}_\beta:\ \tilde{\Delta}_a \subset \tilde{\Delta}_b\}$. Thus (1.2) implies (b), and clearly, (a) holds. Now suppose α has the immediatly before element β. Let $K \in \mathscr{K}$. By the assumption, there is a set $\Delta_b \in \mathscr{H}_\beta$ such that $K \subset \tilde{\Delta}_b$. For $f \in C_R(\Delta_b)$, $f \circ q$ is continuous on $\tilde{\Delta}_b$ and constant on each set $E \in \mathcal{E}$ which is contained in $\tilde{\Delta}_b$. The set Δ_b is saturated with the family $\mathcal{E}$ and $\mathcal{E}$ has the property (S), so we have $f \circ q \in A|\tilde{\Delta}_b$. It follows that the function $f \circ q$ is constant on K, for $f \circ q$ is real valued on K. This holds for any $f \in C_R(\Delta_b)$. Thus there is a set $\Delta_a \in \mathscr{H}_\alpha$ such that $q(K) \subset \Delta_a \subset \Delta_b$. Therefore we obtain $K \subset \tilde{\Delta}_a$, and (b) holds for $\mathscr{H}_\alpha$. To prove (a), it suffices to show that $\tilde{\Delta}_a$ is a p-set. Let Δ_b/H be the quotient space of Δ_b which is obtained by H-equivalence and p: $\Delta_b \to \Delta_b/H$ the natural quotient mapping. For $f \in C_R(\Delta_b/H)$, it holds $f \circ p \in C_R(\Delta_b)$ and $f \circ p \circ q \in A|\tilde{\Delta}_b$. Let F be any closed set of Δ_b/H. Since Δ_b/H is a compact Hausdorff space, it is easy to verify that $(p \circ q)^{-1}(F)$ is a p-set for $A|\tilde{\Delta}_b$. Especially, we identify Δ_a with a point of Δ_b/H, and we have $\tilde{\Delta}_a$ is a p-set for $A|\tilde{\Delta}_b$. Since $\tilde{\Delta}_b$ is a p-set for A, (a) follows. Now we see that $\mathscr{H}_{\sigma(\Delta)} \succ \mathscr{K}$. If $\mathscr{K}$ is actually finer than $\mathscr{H}_{\lambda(\Delta)}$, then there is a set $\Delta_s \in \mathscr{H}_{\sigma(\Delta)}$ which is the union of several maximal antisymmetric sets. Certainly, $\tilde{\Delta}_s$ is not antisymmetric, and hence there is a function $f \in A$ such that $f|\tilde{\Delta}_s$ is nonconstant real valued on $\tilde{\Delta}_s$. Since $f|\tilde{\Delta}_s$ is constant on each set K which is contained in $\tilde{\Delta}_s$, $f|\tilde{\Delta}_s$ defines a nonconstant real function on Δ_s. This contradicts the definition of $\{\Delta_s\}$ and completes the theorem.

COROLLARY 4.2. *Let $\mathcal{E}$ be a* (S)-*partition of closed sets which is finer than $\mathscr{K}$. Then $\mathcal{E}$ coinsides with $\mathscr{K}$ if and only if the quotient space obtained from $\mathcal{E}$ is Hausdorff decomposable.*

5. Examples. In Example 1, let X be a compact plane set as Fig. 1, we shall see that $R(X)$ is antisymmetric algebra, while the finest p-set's partition $\mathscr{P}_*$ define a nontrivial decomposition for $R(X)$. Example 2 shows (D)-family $\mathscr{E}$ of p-sets on $\mathscr{M}(A)$ not necessarily define (D)-family $\mathscr{E}|X$ on X. Example 3 shows Theorem 2.6, (ii) does not hold unconditionally. To construct the last example, we are forced by Theorem 2.8, (i) to use the method of several complex variable.

EXAMPLE 1. Let X be a compact set as Fig. 1. For example, we make as follows; take the rectangle Γ with the sides of ratio 2 : 1, and we shall denote its base by $X_{1,0}$. We choose the sequence of rectangles $X_{1,1}, X_{1,2}, \cdots$ contained in Γ, which converges to $X_{1,0}$, and set $X_1=\bigcup_{i=0}^{\infty} X_{1,i}$. Next, we choose the figure X_2 similar to X_1 with the longer side equal to the shoter side of Γ. We attach X_2 to the shoter side of Γ. We continue the same method as in Fig. 1. We may assume $X_1, X_2, \cdots$ converges to 0, and we set $X=\bigcup_{n=0} X_n$, where $X_0=\{0\}$. Then

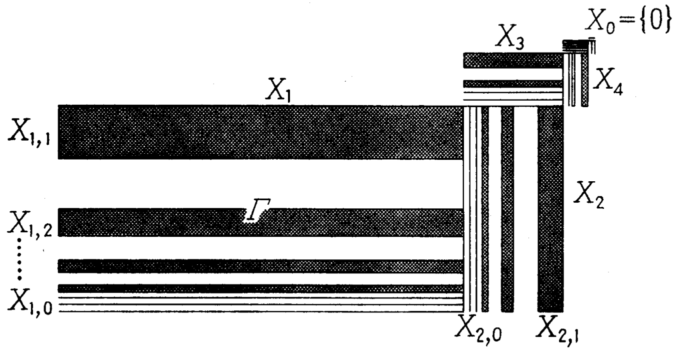

Fig. 1.

(a) *$R(X)$ is an antisymmetric algebra.*

(b) $\mathscr{P}_*=\{X_{n,i}\colon i\neq 0,\ n=1,2,\cdots\}\cup\{x\}_{x\in D}\cup\{0\}$, *where* $D=\bigcup_{n=1}^{\infty}(X_{n,0}\backslash X_{n-1})$.

(a) follows easily. (b) is surely known by intuition. However, we shall prove more exactly. First, we note $\boldsymbol{C}\backslash X$ is connected, so $A(X)=R(X)$ by the Mergelyan's theorem (c. f. [3], Chap. II, Th. 9.1). Here, $A(X)$ denotes the total of continuous functions on X which are analytic in the interior of X. We have only to prove $X_{k+1,0}$ is a peak interpolation set for $R(\bigcup_{n=k+1}^{\infty} X_n\cup X_0)$; indeed, if the assertion holds, then it is easy to see that $\bigcup_{n=0}^{k} X_n$ is a peak set for $R(X)$. Hence, $R(X)\Big|\bigcup_{n=1}^{k} X_n=R(\bigcup_{n=1}^{k} X_n)$, and since $X_{n,i}$ $(i\neq 0,\ 1\leq n\leq k)$ is a peak set for $R(\bigcup_{n=1}^{k} X_n)$, we have $X_{n,i}$ $(i\neq 0,\ n\neq 0)$ is a peak set for $R(X)$. Also, it follows each point of D is a peak point for $R(X)$. Now we should show our assertion. For the condition is the same, it is sufficient to see that $X_{1,0}$ is a peak interpolation set for $A(X)$. We use the following lemma (c.f. [3], Chap. II, Th. 12.5):

LEMMA 5.1. *Let A be a function algebra on X. Let E be a p-set for A, and $f\in A|E$. Then, for any positive continuous function p on X such that $|f(y)|\leq p(y)$ for $y\in E$, there is a function $g\in A$ such that $g|E=f$ and $|g(x)|\leq p(x)$ for all $x\in X$.*

Let f be any continuous function on $X_{1,0}$. We must find the continuous extension g of f such that $|g(x)|<\|f\|_{X_{1,0}}$ for $x\in X\backslash X_{1,0}$, and g is analytic in the interior of X. Since $X_{i,0}$ is a peak interpolation set for $A(X_i)$, first, we extend f to a function g_1 on X_1 which yields to the conditions. Next, we extend $g_1|X_1\cap X_2$ to a function g_2 on X_2 which yields to the conditions, and so on. In above, we can take g_n such that the norm $\|g_n\|_{X_n}$ tends to 0 as $n\to\infty$. Thus we obtain the continuous function g on X which agrees with g_n on X_n and $g(0)=0$. This completes the assertion.

EXAMPLE 2. Let $X=\varDelta\times[-1, 1]$, where $\varDelta=\{z\in \boldsymbol{C}: |z|\leq 1\}$ and $[-1, 1]$ denotes the closed interval in the real line. Let

$$A=\left\{\begin{array}{l} f\in C(X): f(z, t) \text{ is analytic in } |z|<1 \\ \qquad\qquad \text{for each fixed } 0\leq t\leq 1. \end{array}\right\}$$

Then the maximal ideal space $\mathcal{M}(A)$ of A is X. The Šilov boundary $\Gamma(A)$ of A is $\{(z, t)\in X: |z|=1 \text{ or } t\leq 0\}$, and the essential set E_A is $\{(z, t)\in X: t\geq 0\}$. We have the following:

(a) *Let $E_t=\{(z, t)\in X: |z|\leq 1\}$. The closed set E_t is a peak set for A, and $\mathcal{E}=\{E_t: t>0\}$ has the property* (D) *on $\mathcal{M}(A)$.*

(b) *The p-set's family $\mathcal{E}|\Gamma(A)$ has not the property* (D) *on $\Gamma(A)$. More precisely, $\bigcup_{t>0} E_t\cap\Gamma(A)$ is not dense in the essential set $E_A\cap\Gamma(A)$ of $A|\Gamma(A)$.*

EXAMPLE 3. First, we take compact sets X_0, X_1, X_2 in $\boldsymbol{C}^2$;

$$X_0=\left\{(z_1, z_2)\in \boldsymbol{C}^2: |z_1|^2+|z_2|^2\leq 1\right\},$$

$$X_1=\left\{(z_1, z_2)\in X_0: |z_2|\geq\frac{1}{2}\right\},$$

$$X_2=\left\{(z_1, z_2)\in X_0: |z_1|\geq\frac{1}{2}\right\}.$$

We denote by $P(X_0)$ the uniform closure on X_0 of all polynomial functions, and by $R(X_i)$ $(i=1,2)$ the uniform closure on X_i of all rational functions which are analytic on X_i. It follows easily:

(5.1) *X_0 is polynomially convex, and X_1, X_2 are rationally convex.*

(5.2) $S_0=\left\{(z_1, z_2)\in \boldsymbol{C}^2:\ |z_1|^2+|z_2|^2=1\right\}$ *is the Šilov boundary of $P(X_0)$.*

(5.3) $S_i=S_0\cap X_i$ $(i=1,2)$ *is the Šilov boundary of $R(X_i)$.*

Let $I=[0,1]$ and $\varDelta=\{z\in \boldsymbol{C}:\ |z|\leq 1\}$, and let S^3 be the 3-sphere. We define function algebras as follows;

$$A_i=C(I)\widehat{\otimes} R(X_i)\quad (i=1,2),\quad \text{and}\quad A_3=C(S^3)\widehat{\otimes} R(\varDelta).$$

Here, $\widehat{\otimes}$ denotes the uniform closure of algebraic tensor product. The secure definition of tensor product and the following facts are compared with [6], §8.4.

(5.4) *For $i=1,2$ and for any closed subset K of I, it follows that*

(i) $\mathscr{M}(A_i)=I\times X_i$, $\Gamma(A_i)=I\times S_i$.

(ii) *$K\times X_i$ is a peak set for A_i.*

(5.5) (i) $\mathscr{M}(A_3)=S^3\times\varDelta$, $\Gamma(A_3)=S^3\times\partial\varDelta$,

where $\partial\varDelta=\{z\in\boldsymbol{C}:\ |z|=1\}$.

(ii) *$S^3\times\{1\}$ is a peak interpolation set for A_3.*

Our example is obtained by pasting A_1, A_2, A_3 to $P(X_0)$. To do this, we need the following lemma; let Q and R be compact Hausdorff spaces. Let E be a closed subset of Q and φ a continuous mapping on E into R. We denote by $Q\# R$ the direct sum of the spaces Q, R, and define the quotient space $Q\#_\varphi R$ of $Q\# R$ by identifying $\{r\}\cup\varphi^{-1}(\{r\})$ to a point for each $r\in\varphi(E)$. Then we can easily verify that $Q\#_\varphi R$ is a compact Hausdorff space.

LEMMA 5.2. *Let X, Y be compact Hausdorff spaces, and A, B function algebras on X, Y, respectively. Suppose the intersection of X and Y, denoted by E, is not empty and a p-set for A, and suppose $A|E\subset B|E$. Then,*

$$\tilde{A}=\left\{f\in C(X\cup Y):\ f|X\in A,\quad f|Y\in B\right\}$$

is a function algebra on $X\cup Y$. For any closed set F of $X\cup Y$,

(5.6) *F is a p-set for $\tilde{A}$ if and only if $F\cap X$ is a p-set for A and $F\cap Y$ is a p-set for B. And then,*

$$\tilde{A}|F=\left\{f\in C(F):\ f|F\cap X\in A|F\cap X,\quad f|F\cap Y\in B|F\cap Y\right\}.$$

Moreover, let $\tilde{E}_A$ be the A-convex hull of E, there exists the natural mapping

φ: $\tilde{E}^A \to \mathcal{M}(B)$ and

(5. 7) *$\mathcal{M}(\tilde{A})$ is homeomorphic to $\mathcal{M}(A)\#_{\varphi}\mathcal{M}(B)$.*

Also,

(5. 8) $\Gamma(\tilde{A}) = (\overline{\Gamma(A)\setminus E}) \cup \Gamma(B)$.

PROOF: (Later, we shall use the case: E is a peak set for A, so we shall prove only in this case for simplicity. The general case follows from the slight modification of this proof). We should note that if $f \in A$ and $g \in B$ are agree on E, then the continuous function h on $X \cup Y$ is defined by $h|X=f$ and $h|Y=g$, and it follows $h \in \tilde{A}$; thus we have $\tilde{A}|Y=B$. In particular, for a function $f \in A$ which is constant on E, to extend f on Y constantly, we define the function, denoted by $\tilde{f}$, of $\tilde{A}$, and we fix a peaking function $e \in A$ for E in the following proof; note that $\tilde{e}$ is a peaking function for Y on $X \cup Y$. To prove that $\tilde{A}$ is a function algebra on $X \cup Y$, it suffice to show that $\tilde{A}$ separates any distinct points x, y of $X \cup Y$. When x, $y \in X \setminus E$, there is a function $f \in A$ such that $f(x)=0$ and $f(y) \neq 0$. Then the function $h=(\widetilde{1-e)f}$ is separates x and y. The other case will be verified more easily, and we have the first assertion. Now, let F be a closed set of $X \cup Y$ such that $F \cap X$ is a p-set for A and $F \cap Y$ is a p-set for B. Let k be any fixed function of $\tilde{A}|F$ and p a positive continuous function on $X \cup Y$ such that $|k(x)| \leq p(x)$ for $x \in F$. We want to prove that there is a function $h \in \tilde{A}$ such that $h|F=k$ and $|h(x)| \leq p(x)$ for $x \in X \cup Y$. Since $F \cap Y$ is a p-set for B, by Lemma 5. 1, there is a function $g \in B$ such that $g|F \cap Y = k|F \cap Y$ and $|g(y)| \leq p(y)$ for $y \in Y$ (when $F \cap Y$ is empty, we let $g=0$ on Y). Let g_1 be the function which agrees with g on E and agrees with k on $F \cap X$. Then g_1 is continuous on $E \cup (F \cap X)$. Since $E \cup (F \cap X)$ is a p-set for A, $g_1 \in A|E \cup (F \cap X)$. Hence there is a function $f \in A$ such that $f|E \cup (F \cap X) = g_1$ and $|f(x)| \leq p(x)$ for $x \in X$. For f and g agree on E, we have the seeking function h from f and g. This implies that F is an intersection of peak set for A. Clearly, the converse holds. Moreover, in above argument, we only use the facts $k|F \cap X \in A|F \cap X$ and $k|F \cap Y \in B|F \cap Y$ to construct the function h. Hence we have $\tilde{A}|F = \{f \in C(F) : f|F \cap X \in A|F \cap X, f|F \cap Y \in B|F \cap Y\}$. Now we shall show (5. 7). Since $a \in \mathcal{M}(A)$ and $b \in \mathcal{M}(B)$ are non-zero multiplicative linear functionals on $\tilde{A}$, we can define the natural mapping τ: $\mathcal{M}(A)\#\mathcal{M}(B) \to \mathcal{M}(\tilde{A})$, i. e., for $h \in \tilde{A}$,

$$\hat{h}\big(\tau(a)\big) = (\widehat{h|X})(a), \quad \hat{h}\big(\tau(b)\big) = (\widehat{h\,|\,Y})(b).$$

Also, we can define the natural mapping φ: $\tilde{E}^A \to \mathcal{M}(B)$, i. e., for $g \in B$ and $a \in \tilde{E}^A$,

$$\hat{g}\big(\varphi(a)\big)=(\widehat{g\,|\,E})(a)\,.$$

We have already noted $\tilde{A}\,|\,Y=B$. Thus τ is injective on $\mathscr{M}(B)$. And, for $a\in\tilde{E}^{A}$ and $h\in\tilde{A}$, we have

$$\hat{h}\big(\tau(\varphi(a))\big)=(\widehat{h\,|\,Y})\big(\varphi(a)\big)=(\widehat{h\,|\,E})(a)=(\widehat{h\,|\,X})(a)=\hat{h}\big(\tau(a)\big).$$

Therefore, we obtain the natural mapping $\kappa\colon \mathscr{M}(A)\#_{\varphi}\mathscr{M}(B)\to\mathscr{M}(\tilde{A})$ from τ. Now we have to show that τ maps onto $\mathscr{M}(\tilde{A})$ and is injective on $\mathscr{M}(A)\backslash\tilde{E}^{A}$. Let $I_E=\{f\in A\colon f|E=0\}$ and $I_Y=\{h\in\tilde{A}\colon h\,|\,Y=0\}$. Clearly, I_E is isomorphic to I_Y by the correspondence $f\mapsto\tilde{f}$. To prove "onto", we take any point a_0 of $\mathscr{M}(\tilde{A})$. It is clear when $a_0\in\mathscr{M}(B)$, so we assume $a_0\in\mathscr{M}(\tilde{A})\backslash\mathscr{M}(B)$, i.e., $|\hat{e}(a_0)|<1$. Since $1-\tilde{e}\in I_Y$ and $(1-\tilde{e})(a_0)\neq 0$, a_0 is a non-zero multiplicative linear functional on I_Y. Thus a non-zero multiplicative functional ϕ on I_E is obtained from a_0; indeed, ϕ is defined as follows;

$$\phi(f)=\hat{\tilde{f}}(a_0)\quad\text{for}\quad f\in I_E.$$

Moreover, since I_E is an ideal of A, we can extend ϕ uniquely to a multiplicative linear functional a on A, i.e.,

$$\hat{f}(a)=\phi\big((1-e)f\big)\big/\phi(1-e)\qquad\text{for}\quad f\in A\,.$$

Then, for any $h\in\tilde{A}$, we have

$$\begin{aligned}\hat{h}\big(\tau(a)\big)=(\widehat{h\,|\,X})(a)&=\phi\big((1-e)(h\,|\,X)\big)\big/\phi(1-e)\\&=\big(\widehat{(1-e)\widetilde{(h\,|\,X)}}\big)(a_0)/(\widehat{1-\tilde{e}})(a_0)\\&=\big(\widehat{(1-\tilde{e})h}\big)(a_0)/(\widehat{1-\tilde{e}})(a_0)\\&=\hat{h}(a_0)\,.\end{aligned}$$

Thus τ maps onto $\mathscr{M}(\tilde{A})$. And above arguments also prove that τ is a bijection on $\mathscr{M}(A)\backslash\tilde{E}^{A}$ onto $\mathscr{M}(\tilde{A})\backslash\mathscr{M}(B)$. Hence κ is a homeomorphism on $\mathscr{M}(A)\#_{\varphi}\mathscr{M}(B)$ onto $\mathscr{M}(\tilde{A})$, and we have (5.7). To show (5.8), we are sufficient to notice $\Gamma(\tilde{A})=\overline{c(\tilde{A})}$, where $c(\tilde{A})=\{x\in X\cup Y\colon \{x\}\text{ is a }p\text{-set for }\tilde{A}\}$. Then (5.8) follows from (5.6). That completes the proof.

Now, we let X be obtained by pasting together the compact spaces $I\times X_1$, $I\times X_2$, $S^3\times\varDelta$ and X_0 along $\{0\}\times X_1$ at X_1, $\{0\}\times X_2$ at X_2, and $S^3\times\{1\}$ at S_0, respectively. We define the function algebra A on X by

$$A=\left\{\begin{matrix}f\in C(X)\colon f|X_0\in P(X_0)\,,\quad f|(I\times X_i)\in A_i\quad(i=1,2),\\ f\,|\,(S^3\times\varDelta)\in A_3\end{matrix}\right\}$$

Then, by Lemma 5. 2,

(a) $\mathscr{M}(A)=X$,

(b) $\Gamma(A)=S_0\cup(I\times S_1)\cup(I\times S_2)\cup(S^3\times\partial\Delta)$.

Moreover,

(c) *The family* $\mathcal{E}=\{\{t\}\times S_i:\ i=1,2,\ 0<t\leq 1\}\cup\{\{w\}\times\partial\Delta:\ w\in S^3\}$ *is a* (D)*-partition of p-sets on* $\Gamma(A)$.

(d) $\tilde{\mathcal{E}}$ has not the property (D) on $\mathscr{M}(A)$. *Indeed,* $\overline{\cup\tilde{\mathcal{E}}}$ *does not contain* $(0,0)\in X_0$.

$\mathcal{E}$ is a p-set's partition of X follows from Lemma 5. 2. So, to make sure (c), we suppose $f\in C(\Gamma(A))$ and $f|(\{t\}\times S_i)\in A|(\{t\}\times S_i)$ ($i=1,2$, and for all $t\in I$), $f|(\{w\}\times\partial\Delta)\in A|(\{w\}\times\partial\Delta)$ (for all $w\in S^3$). By Lemma 5. 2, $A|(\{w\}\times\partial\Delta)=A_3|(\{w\}\times\partial\Delta)$, and since $\{\{w\}\times\partial\Delta:\ w\in S^3\}$ is the family of maximal anti-symmetric sets for $A_3|(S^3\times\partial\Delta)$ (c. f. [8], § 8. 4, Th. 16), we have $f|(S^3\times\partial\Delta)\in A_3|(S^3\times\partial\Delta)$. For $i=1,2$, it follows $f|(\{0\}\times S_i)\in A_i|(\{0\}\times S_i)$ by uniform convergence. Similary, since $\{t\}\times S_i$ is the family of maximal antisymmetric sets for $A_i|(I\times S_i)$, $f|(I\times S_i)\in A_i|(I\times S_i)$. Therefore, there exist $f_i\in A_i$ for $i=1,2,3$ such that $f_i|(I\times S_i)=f|(I\times S_i)$ ($i=1,2$) and $f_3|(S^3\times\partial\Delta)=f|(S^3\times\partial\Delta)$. If we see that f_1 and f_2 agree on $X_1\cap X_2$, we can define the continuous function g on $(I\times X_1)\cup(I\times X_1)\cup(S^3\times\Delta)$ such that g coincides f_i on each base space. Then g is analytic in the interior of $X_1\cup X_2$. Thus g is uniquely extended to a function on X_0 which is analytic in the interior of X_0 by the well-known theorem in se\eral complex variable. Since X_0 is polynomial convex, we have $g|X_0\in P(X_0)$ by Oka-Weil Aνproximation theorem, and hence, (c) holds. Now, we restric our argument within $\boldsymbol{C}^2$, and show that f_1 and f_2 agree on $X_1\cap X_2$. First, we note that f_1 and f_2 are coincide with f on $S_1\cap S_2$. Let r_1, r_2 be a pair of real numbers such that $r_1>1/2$, $r_2>1/2$, and $r_1^2+r_2^2=1$. Define the function $h(z_1,z_2)$ by

$$h(z_1,z_2)=\frac{1}{(2\pi i)^2}\int_{|\xi_2|=r_2}\int_{|\xi_1|=r_1}\frac{f(\xi_1,\xi_2)}{(\xi_1-z_1)(\xi_2-z_2)}\,d\xi_1 d\xi_2 \qquad \text{for}\quad |z_1|<r_1,\quad |z_2|<r_2.$$

Then h is analytic in $\{(z_1,z_2):\ |z_i|<r_i\}$. The function $f_2(\xi_1,z_2)$ is analytic in $|z_2|<r_2$ for fixed ξ_1 such that $|\xi_1|=r_1$; and $f_2(\xi_1,\xi_2)=f(\xi_1,\xi_2)$ for $|\xi_1|=r_1$, $|\xi_2|=r_2$. Thus we have

$$h(z_1,z_2)=\frac{1}{2\pi i}\int_{|\xi_1|=r_1}\frac{f_2(\xi_1,z_2)}{\xi_1-z_1}\,d\xi_1.$$

Let z_2 tend to ξ_2, then $f_2(\xi_1,z_2)$ uniformly converges to $f_2(\xi_1,\xi_2)$ for $|\xi_1|=r_1$, so we have

$$\lim_{z_2 \to \xi_2} h(z_1, z_2) = \frac{1}{2\pi i} \int_{|\xi_1|=r_1} \frac{f_2(\xi_1, \xi_2)}{\xi_1 - z_1} d\xi_1$$
$$= \frac{1}{2\pi i} \int_{|\xi_1|=r_1} \frac{f(\xi_1, \xi_2)}{\xi_1 - z_1} d\xi_1 = f_1(z_1, \xi_2).$$

Now, let z_1 ($|z_1|<r_1$) to be fixed. Then $h(z_1, z_2)$ is analytic in $|z_2|<r_2$, and extended to $|z_2|\leq r_2$ continuously. This extension agrees with $f_1(z_1, z_2)$ on $|z_2|=r_2$ and $f_1(z_1, z_2)$ is analytic in $r_2<|z_2|<\sqrt{1-|z_1|^2}$. Thus, by Painlevé Theorem, the function $h(z_1, z_2)$ has analytic extension and agrees with $f_1(z_1, z_2)$ on $1/2\leq|z_2|\leq\sqrt{1-|z_1|^2}$. Hence $f_1(z_1, z_2)$ and $h(z_1, z_2)$ agree on $\{(z_1, z_2): |z_i|<r_i\} \cap X_1$. Similary, $f_2(z_1, z_2)$ and $h(z_1, z_2)$ agree on $\{(z_1, z_2): |z_i|<r_i\} \cap X_2$. By arbitariness of r_1 and r_2, and continuity of f_1 and f_2, we see that f_1 and f_2 must agree on $X_1 \cap X_2$. Thus (c) holds. (d) follows clearly. This completes Example 3.

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