Some characterizations of vanishing Bochner curvature tensor

By Toyoko KASHIWADA

Introduction. In Riemannian manifolds, it is well known that the Weyl conformal curvature tensor vanishes if and only if $R_{abcd}=0$ for indices a, b, c, d, which differ from one another. In this paper, we get the analogous property to it for the Bochner curvature tensor, and by this, we have some necessary and sufficient conditions in terms of sectional curvatures in order that Kähler manifolds have vanishing Bochner curvature tensor. These theorems are analogous to results of J. Haantjes and W. Wrona [3] and R. S. Kulkarni [4].

The author wishes to express her hearty gratitude to Prof. S. Tachibana for his valuable criticisms.

§1. Relations of R_{abcd} and B_{abcd}. Let (M^n, g) be an $n(=2m)$ dimensional Kähler manifold and φ_{ab}, $R_{abc}{}^d$, $R_{ab}(=R_{eab}{}^e)$ be its complex structure, the Riemannian curvature tensor and the Ricci tensor respectively. Let $S_{ab}=\varphi_a{}^e R_{eb}$. With respect to a φ-base $\{e_1, \cdots, e_m, \varphi e_1, \cdots, \varphi e_m\}$ the components of these tensors have relations as follows[1]:

$$g_{ab}=\delta_{ab}\,,$$

$$\varphi_{ii^*}=-\varphi_{i^*i}=1\,,\qquad \varphi_{ia}=0\quad (a\neq i^*)\,,$$

$$R_{abkl^*}=-R_{abk^*l}\,,\qquad R_{ij}=R_{i^*j^*}\,,\qquad R_{ij^*}=-R_{i^*j}\,,$$

$$S_{ij}=S_{i^*j^*}=R_{i^*j}\,,\qquad S_{ij^*}=-S_{i^*j}=R_{ij}\,.$$

Let B be the Bochner curvature tensor, *i.e.*

$$B_{abcd}=R_{abcd}+\frac{1}{n+4}U_{abcd}$$

where we put

$$\begin{aligned}U_{abcd}=&R_{ac}g_{bd}-R_{bc}g_{ad}+R_{bd}g_{ac}-R_{ad}g_{bc}\\&+S_{ac}\varphi_{bd}-S_{bc}\varphi_{ad}+S_{bd}\varphi_{ac}-S_{ad}\varphi_{bc}+2S_{ab}\varphi_{cd}+2S_{cd}\varphi_{ab}\\&-\frac{R}{n+2}(g_{ac}g_{bd}-g_{bc}g_{ad}+\varphi_{ac}\varphi_{bd}-\varphi_{bc}\varphi_{ad}+2\varphi_{ab}\varphi_{cd})\,.\end{aligned}$$

It follows that

---

1) $a, b, \cdots = 1, \cdots, m, 1^*, \cdots, m^*$; $\quad i^*=i+m$; $\quad i, j, \cdots = 1, \cdots, m$.

$$(1.1)\qquad \begin{aligned} U_{abba} &= -\left(R_{aa}+R_{bb}-\frac{R}{n+2}\right) \qquad (|a| \neq |b|)^{1)}, \\ U_{ii^*i^*i} &= -\left(8R_{ii}-\frac{4R}{n+2}\right). \end{aligned}$$

We get an analogous property of conformally flat manifold as follows.

PROPOSITION[2]. *Let M, $n=2m \geqq 8$, be a Kähler manifold. If*

$$R_{abcd}=0 \qquad (|a|, |b|, |c|, |d| \neq)^{3)}$$

holds good for every φ-base $\{e_1, \cdots, e_m, \varphi e_1, \cdots, \varphi e_m\}$, then the Bochner curvature vanishes. The converse is true.

To prove this proposition, at first, we remark the following property, (Bishop & Goldberg [1]).

LEMMA. *Let L be a semi-curvature-like tensor, i.e. tensor field of type* (1. 3) *such that*

(1) $L_{abcd}=-L_{bacd}$

(2) $L_{abcd}=L_{cdab}$

(3) *1-st Bianchi's identity is satisfied.*

Then $L=0$ if and only if $L_{abba}=0$ for every base.

Especially, in the case of a Kähler manifold, for $L=0$, it suffices that $L_{abba}=0$ for every φ-base.

The Bochner curvature tensor B is a semi-curvature-like tensor. So, by virtue of this lemma, it suffices to prove that $B_{abba}=0$ for every φ-base.

PROOF OF PROPOSITION: For a φ-base $\{e_1, \cdots, e_m, \varphi e_1, \cdots, \varphi e_m\}$,

$$(1.2)\qquad R_{abcd}=0 \qquad (|a|, |b|, |c|, |d| \neq).$$

We take another φ-base

$$(*)\qquad \begin{aligned} e'_i &= ce_i+se_j \\ e'_j &= -se_i+ce_j \\ e'_a &= e_a \qquad (|a| \neq i, j) \end{aligned}$$

where c and s are real numbers such that $c^2+s^2=1$ and $cs \neq 0$.
As (1. 2) is true for this base, we have

$$\begin{aligned} 0 &= g\left(R(e'_i, e_a)e'_j, e_b\right) \\ &= -cs(R_{iaib}-R_{jajb}), \end{aligned}$$

---

1) $|i|=i, \quad |i^*|=i.$

2) Cf. Eisenhart [2], p. 124.

3) This means that $|a|, |b|, |c|, |d|$ differ from one another.

i. e. $$R_{aiib}=R_{ajjb} \qquad (|a|, |b|, i, j \neq).$$

By replacing e_i with e_{i*}, we have

$$R_{ai*i*b}=R_{ajjb}.$$

So we get

$$(1.3) \qquad R_{aiib}=R_{ai*i*b} \qquad (|a|, |b|, i \neq).$$

Since (1. 3) is true for every φ-base, for φ-base (*) we know

$$g\left(R(e'_i, e_{k*})e_{k*}, e'_j\right)=-g\left(R(e'_i, e_k)e_k, e'_j\right)$$

which implies

$$(1.4) \qquad R_{ik*k*i}-R_{jk*k*j}=R_{ikki}-R_{jkkj}.$$

Replacing e_j with e_{j*} and adding it to (1. 4) we have

$$(1.5) \qquad R_{ik*k*i}=R_{ikki} \qquad (i \neq k)$$

Since (1. 5) is true for every φ-base, computing (1. 5) with respect to φ-base (*),

$$g\left(R(e'_i, e'_{j*})e'_{j*}, e'_i\right)=g\left(R(e'_i, e'_j)e'_j, e'_i\right),$$

we obtain after all,

$$(1.6) \qquad R_{ii*i*i}+R_{jj*j*j}=8R_{ijji} \qquad (i \neq j).$$

Then we have

$$\sum_{j(\neq i)}^{m}(R_{ii*i*i}+R_{jj*j*j})=8\sum_{j=1}^{m}R_{ijji}$$

$$(m-2)R_{ii*i*i}+\mu=4\left(\sum_{j=1}^{m}R_{ijji}+\sum_{j(\neq i)}^{m}R_{ij*j*i}\right)$$

$$=4(R_{ii}-R_{ii*i*i}),$$

i. e.

$$(1.7) \qquad R_{ii*i*i}=\frac{1}{m+2}(4R_{ii}-\mu)$$

where we put $\mu=\sum_{j=1}^{m}R_{jj*j*j}$ and take account of $R_{ijji}=R_{ij*j*i}$. Taking sum of (1. 7) from $i=1$ to $i=m$, we have

$$(1.8) \qquad R=(m+1)\mu.$$

So from (1. 7) and (1. 8) we get

$$(1.9) \qquad R_{ii*i*i}=\frac{1}{n+4}\left(8R_{ii}-\frac{4R}{n+2}\right).$$

On account of (1. 6), it follows

$$(1.10)\qquad R_{ijji}=\frac{1}{n+4}\left(R_{ii}+R_{jj}-\frac{R}{n+2}\right).$$

On the other hand,

$$(1.11)\qquad R_{i*j*j*i*}=R_{ijji}.$$

Then, from (1. 1), (1. 5) and (1. 9)~(1. 11) we obtain

$$(1.12)\qquad \begin{aligned} B_{abba}&=R_{abba}+\frac{1}{n+4}U_{abba}=0 \qquad (|a|\neq|b|),\\ B_{ii*i*i}&=R_{ii*i*i}+\frac{1}{n+4}U_{ii*i*i}=0. \end{aligned}$$

So by lemma, we get

$$B=0.$$

The converse is trivial since $U_{abcd}=0$ for $|a|, |b|, |c|, |d|\neq$. Q. E. D.

REMARK: In this proof, we know that the property (1. 12) depends only the property (1. 6).

§ 2. Theorems. We have several necessary and sufficient conditions to be $B=0$ in terms of the sectional curvature.

THEOREM 1[1]. *Let M^{2m}, $m\geqq 4$, be a Kähler manifold. Then, the followings are equivalent at every point $p\in M$.*

(1) *The Bochner curvature tensor $B(p)=0$.*

(2)[2] *For every φ-base at p,*

$$\rho(e_i, e_{i*})+\rho(e_j, e_{j*})=8\rho(e_i, e_j)^{3)}.$$

(3) *For each holomorphic 8-plane $W\subseteqq T_p(M)$,*

$$k_p(W, \mathfrak{b})=\rho(e_1, e_2)+\rho(e_3, e_4)$$

is independent of φ-base $\mathfrak{b}=\{e_1, \cdots, e_4, \varphi e_1, \cdots, \varphi e_4\}$ of W.

(4) *For every orthogonal 8 vectors of $T_p(M)$ such that $\{e_1, \cdots, e_4, \varphi e_1, \cdots, \varphi e_4\}$,*

$$\rho(e_1, e_2)+\rho(e_3, e_4)=\rho(e_1, e_4)+\rho(e_2, e_3).$$

Proof. (2)⟹(1) is noted at the last of proof of proposition.

(1)⟹(2) is trivial since (1. 1) and

1) An analogous theorem have been got independently by Ogitsu and Iwasaki, [5].

2) This was remarked by Mr. M. Sekizawa.

3) $\rho(e_a, e_b)$ means Riemannian sectional curvature with respect to tse plane spanned by e_a, e_b.

$$R_{abba}=\frac{-1}{n+4}U_{abba}.$$

(3)$\Longrightarrow$(4) is trivial.

(4)$\Longrightarrow$(1): Let $\{e_1,\cdots,e_m,\varphi e_1,\cdots,\varphi e_m\}$ be arbitrary φ-base of $T_p(M)$. For $\{e_i, e_j, e_k, e_l, \varphi e_i, \varphi e_j, \varphi e_k, \varphi e_l\}$, by assumption,

$$R_{ijji}+R_{klkk}=R_{illi}+R_{jkkj}.$$

We take another orthonormal vectors $\{e_i, e_j', e_k', e_l, \varphi e_i, \varphi e_j', \varphi e_k', \varphi e_l\}$ such that

$$e_j'=ce_j+se_k$$
$$e_k'=-se_j+ce_k, \qquad (c^2+s^2=1, \quad cs \neq 0).$$

Since $\rho(e_i, e_j')+\rho(e_k', e_l)=\rho(e_i, e_l)+\rho(e_j', e_k')$, it follows

(2. 1) $$R_{jiik}=R_{jllk}.$$

Since (2. 1) is true for every φ-base, for the above base,

$$g\left(R(e_ie_j')e_j', e_l\right)=g\left(R(e_ie_k')e_k', e_l\right)$$

which implies

$$R_{ijkl}+R_{ikjl}=0.$$

Then by Bianchi's identity, we get $R_{ijkl}=0$. Replacing $e_i\to e_{i*}$, $e_j\to e_{j*}\cdots$ etc, we obtain $R_{abcd}=0$ ($|a|, |b|, |c|, |d| \neq$). So, by proposition, the Bochner curvature tensor vanishes.

(1)$\Longrightarrow$(3): Let $B=0$. Then, for a φ-base, it follows

$$R_{abba}=\frac{1}{n+4}\left(R_{aa}+R_{bb}-\frac{R}{n+2}\right) \qquad (|a|\neq|b|).$$

Let $\boldsymbol{b}=\{e_1, e_2, e_3, e_4, \varphi e_1, \varphi e_2, \varphi e_3, \varphi e_4\}$, $\boldsymbol{b}'=\{e_1', e_2', e_3', e_4', \varphi e_1', \varphi e_2', \varphi e_3', \varphi e_4'\}$, be basis of $W\subseteq T_p(M)$. We construct two basis of $T_p(M)$ such that

$$\boldsymbol{f}=\{e_1,\cdots,e_m,\varphi e_1,\cdots,\varphi e_m\}$$
$$\boldsymbol{f}'=\{e_1',\cdots,e_4', e_5,\cdots,e_m,\varphi e_1',\cdots,\varphi e_4',\varphi e_5,\cdots,\varphi e_m\}.$$

Then we have

(2. 2) $$\rho(e_1, e_2)+\rho(e_3, e_4)=\frac{1}{n+4}\sum_{\lambda=1}^{4}\left(R_{\lambda\lambda}-\frac{2R}{n+2}\right).$$

Let R_{aa}, R_{aa}' be components of the Ricci tensor with respect to base $\boldsymbol{f}, \boldsymbol{f}'$. So, as $R=\sum R_{aa}=\sum R_{aa}'$ and $R_{ii}'=R_{ii}$ ($i>4$), we have

$$\sum_{\lambda=1}^{4}R_{\lambda\lambda}=\sum_{\lambda=1}^{4}R_{\lambda\lambda}'.$$

Then by virtue of (2.2), we know that $k_p(W, \boldsymbol{b})$ is independent of $\boldsymbol{b}$.

Q.E.D.

By this proof, we know "8-plane" can be changed with arbitrary "$2d$-plane" ($8 \leqq 2d \leqq m$) in this theorem. So, for example,

THEOREM 2. *Let M be a Kähler manifold of dimension $4m \geqq 8$. Then, $B=0$ if and only if, for every φ-base $\boldsymbol{f}$ of $T_p(M)$,*

$$k_p(\boldsymbol{f}) = \rho(e_1, e_2) + \cdots + \rho(e_{2m-1}, e_{2m})$$

is independent of $\boldsymbol{f}$.

§3. Another proof[1] of a part of Kulkarni's result.

Theorem (Kulkarni [4], Theorem 3.2). *Let M^n, $n \geqq 4$, be a Riemennian manifold. Then the Weyl conformal curvature tensor C vanishes if and only if*

$$\rho(e_1, e_2) + \rho(e_3, e_4) = \rho(e_1, e_4) + \rho(e_2, e_3)$$

for every quadruple of (orthogonal) vectors $\{e_1, e_2, e_3, e_4\}$.

Kulkarni proved this by conformal change of metric. Now, as equations in this theorem are all algebraic, we shall give an algebraic proof.

PROOF OF KULKARNI'S THEOREM. Necessity is trivial.

Sufficiency: By assumption,

$$\rho(e_i, e_j) + \rho(e_k, e_l) = \rho(e_i, e_l) + \rho(e_j, e_k) \qquad (i, j, k, l \neq).$$

Taking summation with respect to $l(\neq i, j, k)$ and $k(\neq i, j)$, we have

$$(n-1)(n-2)\rho(e_i, e_j) = (n-1)\left\{\sum_{t=1}^{n} \rho(e_i, e_t) + \sum_{t=1}^{n} \rho(e_j, e_t)\right\} - \sum_{t,r=1}^{n} \rho(e_t, e_r).$$

This equation means

$$R_{ijji} - \frac{1}{n-2}(R_{ii} + R_{jj}) - \frac{R}{(n-1)(n-2)} = 0,$$

which is nothing but

$$C_{ijji} = 0.$$

As the last equation is valid for any base, we know $C=0$ by virtue of lemma.

Q.E.D.

Department of Mathematics,
Ochanomizu University, Tokyo, Japan

1) This proof was ramarked by Prof. S. Tachibana.

Bibliography

[1] R. L. BISHOP and S. I. GOLDBERG: Some implications of the generalized Gauss-Bonnet theorem. Trans. Amer. Math. Soc., 112 (1964), 508-535.

[2] L. P. EISENHART: Riemannian geometry, Princeton Paperbacks.

[3] J. HAANTJES and W. WRONA: Ueber konformeuklidische und Einsteinsche raume gerader dimension.

[4] R. S. KULKARNI: Curvature structures and conformal transformation, J. Differential Geometry, 4 (1969), 425-451.

[5] N. OGITSU and K. IWASAKI: On a characterization of the Bochner curvature tensor=O, Nat. Sci. Rep. of the Ochanomizu Univ., 25 (1974), 1-6.

(Received January 5, 1974)