On infinitesimal projective transformations

By Kazunari YAMAUCHI

§ 1. Introduction

For the infinitesimal conformal transformations, the following results are well known.

THEOREM A. *Let M be a complete Riemannian manifold with parallel Ricci tensor. If M admits nonisometric infinitesimal conformal transformations, then M is isometric to a sphere.*

THEOREM B. *Let M be a compact Riemannian manifold with constant scalar curvature. If the scalar curvature is nonpositive, then an infinitesimal conformal transformation is a motion.*

THEOREM C. *Let M be a compact Riemannian manifold with positive constant scalar curvature. If M admits nonisometric infinitesimal conformal transformations, then M is isometric to a sphere.*

And for the infinitesimal projective transformations, the following results are known.

THEOREM D. *Let M be a complete Riemannian manifold with parallel Ricci tensor. If M admits nonaffine infinitesimal projective transformations, then M is a space of positive constant curvature.* [1]

THEOREM E. *Let M be a complete analytic Riemannian manifold. If M admits nonaffine infinitesimal projective transformations, then M is a space of positive constant curvature.* [2]

The purpose of this paper is to prove the following theorems:

THEOREM 1. *Let M be a compact Riemannian manifold with constant scalar curvature K. If the scalar curvature is nonpositive, then an infinitesimal projective transformation is a motion.*

THEOREM 2. *Let M be a compact Riemannian manifold satisfying a condition $\nabla_k K_{ji} - \nabla_j K_{ki} = 0$, $(K \neq 0)$, where ∇_k, K_{ji} denote a covariant derivative and Ricci tensor respectively. The projective killing vector v^h can be decomposed uniquely as follows,*

$$v^h = w^h + q^h,$$

where w^h and q^h are killing vector and gradient projective killing vector respectively.

THEOREM 3. *Let M be a compact Riemannian manifold satisfying a condition* $\nabla_k K_{ji} - \nabla_j K_{ki} = 0$, $(K \neq 0)$. *If M admits a nonisometric infinitesimal projective transformation, then M is a space of positive constant curvature.*

COROLLALY. *Let M be a compact conformally flat Riemannian manifold with positive constant scalar curvature. If M admits a nonisometric infinitesimal projective transformation, then M is a space of positive constant curvature.*

For this Corollaly, see [3].

A vector field v^h is called an infinitesimal projective transformation or a projective killing vector if it satisfies

$$\mathcal{L}\left\{ {h \atop ji} \right\} = \nabla_j \nabla_i v^h + K_{kji}{}^h v^k = \delta_j^h \phi_i + \delta_i^h \phi_j ,$$

where $\mathcal{L}$, $\left\{ {h \atop ji} \right\}$, $K_{kji}{}^h$, ϕ_i denote a Lie derivation with respect to v^h, Christoffel's symbol, curvature tensor, and associated vector respectively. From this equation, we get following results

$$\mathcal{L} K_{kji}{}^h = -\delta_k^h \nabla_j \phi_i + \delta_j^h \nabla_k \phi_i ,$$
$$\mathcal{L} K_{ji} = -(n-1) \nabla_j \phi_i ,$$
$$\nabla^i \nabla_i v_j + K_{ji} v^i = 2\phi_j ,$$
$$\nabla_j (\Delta_i v^i) = (n+1) \phi_j .$$

If we put $\frac{1}{n+1} \nabla_i v^i = f$, then we have $f_j = \phi_j$, where f_j means $\nabla_j f$. Therefore ϕ_j is a gradient vector and in the following discussions we use f_j instead of ϕ_j.

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§ 2. Proof of Theorem 1

In this section we assume Riemannian manifold M is compact and the scalar curvature is constant.

LEMMA 1. *There exists the following equation*

$$(n-1) \Delta^2 f + 2K \Delta f + 2K_{ji} \nabla^j f^i = 0 ,$$

where Δ *means* $g^{ji} \nabla_j \nabla_i$.

PROOF. Since the scalar curvature K is constant, we have

$$
\begin{aligned}
0 &= \mathcal{L}(g^{ja}\nabla_j K_{ia}) \\
&= (\mathcal{L}g^{ja})\nabla_j K_{ia} + g^{ja}\mathcal{L}\nabla_j K_{ia} \\
&= (\mathcal{L}g^{ja})\nabla_j K_{ia} + g^{ja}\left\{\nabla_j \mathcal{L}K_{ia} - \left(\mathcal{L}\left\{{b \atop ji}\right\}\right)K_{ba} - \left(\mathcal{L}\left\{{b \atop ja}\right\}\right)K_{ib}\right\} \\
&= (\mathcal{L}g^{ja})\nabla_j K_{ia} - (n-1)\nabla^j\nabla_j f_i - (\delta_j^b f_i + \delta_i^b f_j)K_b^j \\
&\quad -(\delta_j^b f_a + \delta_a^b f_j)g^{ja}K_{ib} \\
&= (\mathcal{L}g^{ja})\nabla_j K_{ia} - (n-1)\nabla_i(\Delta f) - (n-1)K_{ai}f^a \\
&\quad -Kf_i - 3K_{ai}f^a \\
&= (\mathcal{L}g^{ja})\nabla_j K_{ia} - (n-1)\nabla_i(\Delta f) - Kf_i \\
&\quad -(n+2)K_{ai}f^a.
\end{aligned} \tag{1.1}
$$

And operate ∇_i for (1. 1), we obtain the following equation

$$
\begin{aligned}
0 &= (\nabla_i \mathcal{L}g^{ja})\nabla_j K_a^i + (\mathcal{L}g^{ja})\nabla_i\nabla_j K_a^i - (n-1)\Delta^2 f \\
&\quad -K\Delta f - (n+2)K_{ai}\nabla^i f^a.
\end{aligned} \tag{1.2}
$$

On theother hand, we have the following equentions

$$
\begin{aligned}
(\nabla_i \mathcal{L}g^{ja})\nabla_j K_a^i &= \left\{\mathcal{L}(\nabla_i g^{ja}) - \left(\mathcal{L}\left\{{j \atop ib}\right\}\right)g^{ba} - \left(\mathcal{L}\left\{{a \atop ib}\right\}\right)g^{jb}\right\}\nabla_j K_a^i \\
&= -\left\{(\delta_i^j f_b + \delta_b^j f_i)g^{ba}\right. \\
&\quad \left. +(\delta_i^a f_b + \delta_b^a f_i)g^{jb}\right\}\nabla_j K_a^i \\
&= 0
\end{aligned} \tag{1.3}
$$

$$
\begin{aligned}
(\mathcal{L}g^{ja})\nabla_i\nabla_j K_a^i &= (\mathcal{L}g^{ja})(\nabla_j\nabla_i K_a^i + K_{ijb}{}^i K_a^b - K_{ija}{}^b K_b^i) \\
&= (\mathcal{L}g^{ja})(K_{jb}K_a^b - K_{ija}{}^b K_b^i) \\
&= \mathcal{L}\left\{g^{ja}(K_{jb}K_a^b - K_{ija}{}^b K_b^i)\right\} \\
&\quad -g^{ja}(\mathcal{L}K_{jb}K_a^b - K_{ija}{}^b K_b^i) \\
&= -g^{ja}\left\{(\mathcal{L}K_{jb})K_a^b + K_{jb}\mathcal{L}K_a^b - (\mathcal{L}K_{ija}{}^b)K_b^i\right. \\
&\quad \left. -K_{ija}{}^b\mathcal{L}K_b^i\right\} \\
&= -(\mathcal{L}K_{jb})K^{jb} + (\mathcal{L}K_{ija}{}^b)g^{ja}K_b^i \\
&= (n-1)\nabla_j f_b K^{jb} + (-\delta_i^b\nabla_j f_a + \delta_j^b\nabla_i f_a)g^{ja}K_b^i \\
&= nK_{jb}\nabla^j f^i - K\Delta f.
\end{aligned} \tag{1.4}
$$

Substituting (1. 3) and (1. 4) into (1. 2), we have Lemma 1.

LEMMA 2. *There is the following relation*

$$
\int_M f\Delta^2 f\, d\sigma = \int_M (\Delta f)^2 d\sigma,
$$

where $d\sigma$ is the volume element of M.

PROOF. This is obious from the following equation,

$$\nabla^i\left\{(\Delta f)f_i - f\nabla_i(\Delta f)\right\} = (\Delta f)^2 - f\Delta^2 f.$$

LEMMA 3. *There exists the following relation*

$$\int_M f\Delta f d\sigma = -\int_M f_i f^i d\sigma .$$

This proof is trivial.

LEMMA 4. *We have the following equations*

$$\int_M fK_{ji}\nabla^j f^i d\sigma = -\int_M K_{ji} f^j f^i d\sigma$$
$$= \int_M\left\{(\nabla_j f_i)(\nabla^j f^i) - (\Delta f)^2\right\} d\sigma .$$

PROOF. These are immediate consequence from the following equations,

$$\nabla^j(K_{ji}ff^i) = K_{ji}f^j f^i + fK_{ji}\nabla^j f^i,$$
$$\tfrac{1}{2}\Delta(f_i f^i) = (\nabla^j\nabla_j f_i)f^i + (\nabla_j f_i)(\nabla^j f^i)$$
$$= \left\{\nabla_i(\Delta f) + K_{ji}f^j\right\}f^i + (\nabla_j f_i)(\nabla^j f^i)$$
$$= \nabla_i(f^i\Delta f) - (\Delta f)^2 + K_{ji}f^j f^i$$
$$+ (\nabla_j f_i)(\nabla^j f^i) .$$

We have the following equation by means of Lemma 1,

$$(n-1)f\Delta^2 f + 2Kf\Delta f + 2fK_{ji}\nabla^j f^i = 0 .$$

We apply Lemma 2, Lemma 3, and Lemma 4 for the avove equation, and we obtain,

$$\int_M\left\{(n-3)(\Delta f)^2 + 2(\nabla_j f_i)(\nabla^j f^i) - 2Kf_i f^i)\right\} d\sigma = 0 .$$

This complets the proof of Theorem 1.

§ 3. Proof of Theorem 2 and Theorem 3.

In this section we assume M is compact and Ricci curvature satisfies $\nabla_k K_{ji} - \nabla_j K_{ki} = 0$.

LEMMA 5. *K is constant.*

This proof is trivial.

LEMMA 6. *There exist the following equations*

$$K_{kji}{}^{l} f_l = \frac{1}{n-1}(f_k K_{ji} - f_j K_{ki}),$$

$$K_{jl} f^l = \frac{K}{n} f_j .$$

PROOF. From $\nabla_k K_{ji} - \nabla_j K_{ki} = 0$, we obtain,

$$\begin{aligned}
0 &= \mathcal{L}(\nabla_k K_{ji} - \nabla_j K_{ki}) \\
&= \nabla_k \mathcal{L} K_{ji} - \left(\mathcal{L}\left\{ {a \atop kj} \right\}\right) K_{ai} - \left(\mathcal{L}\left\{ {a \atop ki} \right\}\right) K_{ja} - \nabla_j \mathcal{L} K_{ki} \\
&\quad + \left(\mathcal{L}\left\{ {a \atop jk} \right\}\right) K_{ai} + \left(\mathcal{L}\left\{ {a \atop ji} \right\}\right) K_{ak} \\
&= -(n-1)(\nabla_k \nabla_j f_i - \nabla_j \nabla_k f_i) - (\delta_k^a f_i + \delta_i^a f_k) K_{ja} \\
&\quad + (\delta_j^a f_i + \delta_i^a f_j) K_{ak} \\
&= (n-1) K_{kji}{}^{l} f_l - (f_k K_{ji} - f_j K_{ki}) .
\end{aligned}$$

This completes the proof of the first equation, and proof of the second equation is easy from the first equation.

LEMMA 7. *We have the following equation*

$$\Delta f = -\frac{2(n+1)}{n(n-1)} K f .$$

PROOF. By means of $K = \text{const.}$ and $K_{ji} f^j = \frac{K}{n} f_i$, we have the following equation

$$\begin{aligned}
K_{ji} \nabla^j f^i &= \nabla^j (K_{ji} f^i) \\
&= \frac{K}{n} \Delta f .
\end{aligned}$$

From Lemma 1, we obtain,

$$\begin{aligned}
0 &= (n-1)\Delta^2 f + 2K\Delta f + \frac{2K}{n}\Delta f \\
&= (n-1)\Delta\left(\Delta f + \frac{2(n+1)}{n(n-1)} K f\right).
\end{aligned}$$

Therefore we have

$$\Delta f + \frac{2(n+1)}{n(n-1)} K f = \text{constant} .$$

On theother hand,

$$\int_M f \, d\sigma = \int_M \Delta f \, d\sigma = 0 .$$

Thus we have Lemma 7.

LEMMA 8. *There is the following equation*:

$$2\nabla_k f_i \nabla_j K^{ki} = \frac{1}{n-1} f_j\left(K_{ki}K^{ki} - \frac{K^2}{n}\right).$$

PROOF. From Lemma 6, we obtain

$$\begin{aligned}
0 &= \nabla^j \nabla^i \left\{K_{kji}{}^l f_l - \frac{1}{n-1}(f_k K_{ji} - f_j K_{ki})\right\} \\
&= \nabla^j \left\{(\nabla^i K_{kji}{}^l) f_l + K_{kji}{}^l \nabla^i f_l - \frac{1}{n-1}(K_{ji}\nabla^i f_k - K_{ki}\nabla^i f_j)\right\} \\
&= -\frac{1}{n-1}\nabla^j (K_{ji}\nabla^i f_k - K_{ki}\nabla^i f_j) \\
&= -\frac{1}{n-1}\left\{K_{ji}\nabla^j \nabla^i f_k - (\nabla^j K_{ki})\nabla^i f_j - K_{ki}\nabla^j \nabla^i f_j\right\} \\
&= -\frac{1}{n-1}\left\{K_{ji}\nabla^j \nabla_k f^i - (\nabla^j K_{ki})\nabla^i f_j \right. \\
&\qquad \left. - K_{ki}(\nabla^i \nabla^j f_j - K^{ji}{}_j{}^l f_l)\right\} \\
&= -\frac{1}{n-1}\left\{K_{ji}(\nabla_k \nabla^j f^i + K^j{}_{kl}{}^i f^l) - (\nabla^j K_{ki})\nabla^i f_j \right. \\
&\qquad \left. - K_{ki}\nabla^i (\Delta f) - K_{ki}K^{il} f_l\right\} \\
&= -\frac{1}{n-1}\left\{K_{ji}\nabla_k \nabla^j f^i - K^{ji}K_{jki}{}^l f_l - (\nabla^j K_{ki})\nabla^i f_j \right. \\
&\qquad \left. + \frac{2(n+1)}{n(n-1)} K f^i K_{ki} - \frac{K^2}{n^2} f_k\right\} \\
&= -\frac{1}{n-1}\left\{K_{ji}\nabla_k \nabla^j f^i - K^{ji}\frac{1}{n-1}(f_j K_{ki} - f_k K_{ji}) \right. \\
&\qquad \left. - (\nabla^j K_{ki})\nabla^i f_j + \frac{2(n+1)}{n^2(n-1)} K^2 f_k - \frac{K^2}{n^2} f_k\right\} \\
&= -\frac{1}{n-1}\left\{K_{ji}\nabla_k \nabla^j f^i - \frac{1}{n-1}\left(\frac{K^2}{n^2} f_k - f_k K_{ji}K^{ji}\right) \right. \\
&\qquad \left. - (\nabla^j K_{ki})\nabla^i f_j + \frac{n+3}{n^2(n-1)} K^2 f_k\right\} \\
&= -\frac{1}{n-1}\left\{K_{ji}\nabla_k \nabla^j f^i - (\nabla^j K_{ki})\nabla^i f_j + \frac{1}{n-1} f_k K_{ji}K^{ji} \right. \\
&\qquad \left. + \frac{n+2}{n^2(n-1)} K^2 f_k\right\} \\
&= -\frac{1}{n-1}\left\{\nabla_k (K_{ji}\nabla^j f^i) - (\nabla_k K_{ji})\nabla^j f^i - (\nabla^j K_{ki})\nabla^i f_j \right.
\end{aligned}$$

$$+\frac{1}{n-1}f_k K_{ji}K^{ji}+\frac{n+2}{n^2(n-1)}K^2 f_k\Big\}$$

$$=-\frac{1}{n-1}\Big\{\nabla_k\nabla^j(K_{ji}f^i)-2(\nabla_k K_{ji})\nabla^j f^i+\frac{1}{n-1}f_k K_{ji}K^{ji}$$

$$+\frac{n+2}{n^2(n-1)}K^2 f_k\Big\}$$

$$=-\frac{1}{n-1}\Big\{\frac{K}{n}\nabla_k(\Delta f)-2(\nabla_k K_{ji})\nabla^j f^i+\frac{1}{n-1}f_k K_{ji}K^{ji}$$

$$+\frac{n+2}{n^2(n-1)}K^2 f_k\Big\}$$

$$=-\frac{1}{n-1}\Big\{-\frac{2(n+1)}{n^2(n-1)}K^2 f_k-2(\nabla_k K_{ji})\nabla^j f^i+\frac{1}{n-1}f_k K_{ji}K^{ji}$$

$$+\frac{n+2}{n^2(n-1)}K^2 f_k\Big\}$$

$$=-\frac{1}{n-1}\Big\{-2(\nabla_k K_{ji})\nabla^j f^i+\frac{1}{n-1}f_k\Big(K_{ji}K^{ji}-\frac{K^2}{n}\Big)\Big\}.$$

LEMMA 9. *If we put* $w_i=v_i+\frac{n(n-1)}{2K}f_i$, *where* v_i *is a projective killing vector, then* w_i *is a killing vector and* $\frac{n(n-1)}{2K}f_i$ *is a gradient projective killing vector.*

PROOF. We have the following equations

$$\nabla^i w_i=\nabla^i v_i+\frac{n(n-1)}{2K}\Delta f$$
$$=(n+1)f-(n+1)f$$
$$=0,$$

$$\nabla^j\nabla_j w_i+K_{ji}w^j=\nabla^j\nabla_j v_i+\frac{n(n-1)}{2K}\nabla^j\nabla_j f_i+K_{ji}v^j+\frac{n(n-1)}{2K}K_{ji}f^j$$
$$=2f_i+\frac{n(n-1)}{2K}\Big\{\nabla_i(\Delta f)+K_{ji}f^j\Big\}+\frac{n-1}{2}f_i$$
$$=2f_i-(n+1)f_i+\frac{n-1}{2}f_i+\frac{n-1}{2}f_i$$
$$=0.$$

Thus w_i is a killing vector and it is clear that $\frac{n(n-1)}{2K}f_i$ is a gradient projective killing vector.

The proof of uniqueness of this decomposition is as follows. If v_i is decomposed as follows

$$v_i = z_i + q_i$$

where z_i is a killing vector and q_i is a gradient projective killing vector, tnen from Lemma 9, we have

$$\nabla^i w_i - \frac{n(n-1)}{2K} \Delta f = \nabla^i z_i + \nabla^i q_i .$$

Thus we obtain

$$\Delta\left(q + \frac{n(n-1)}{2K} f\right) = 0 ,$$

and consequently we get

$$q + \frac{n(n-1)}{2K} f = \text{constant} .$$

This shows

$$q_i = -\frac{n(n-1)}{2K} f_i .$$

Therefore this completes the proof of Theorem 2.

We have the following equation from Theorem 2

$$\begin{aligned} \mathcal{L} g^{ja} &= -(\nabla^j v^a + \nabla^a v^j) \\ &= \frac{n(n-1)}{K} \nabla^j f^a . \end{aligned}$$

We put this result into (1. 1), and we get

$$\begin{aligned} 0 &= (\mathcal{L} g^{ja}) \nabla_j K_{ia} - (n-1) \nabla_i (\Delta f) - K f_i - (n+2) K_{ji} f^j \\ &= \frac{n(n-1)}{K} \nabla^j f^a \nabla_j K_{ia} + \frac{2(n+1)}{n} K f_i - K f_i - \frac{n+2}{n} K f_i \\ &= \frac{n(n-1)}{K} \nabla^j f^a \nabla_j K_{ia} . \end{aligned}$$

Therefore, from Lemma 8, we have

$$f_j \left(K_{ki} K^{ki} - \frac{K^2}{n} \right) = 0 .$$

Consequently we get

$$K_{ji} K^{ji} = \frac{K^2}{n} .$$

That is, M is an Einstein space. From Theorem D, we arrive at the complete proof of Theorem 3.

Mathematical Institute
College of Liberal Arts
Kagoshima University

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