A remark on conformal Killing tensors of a Riemannian manifold of constant curvature

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Introduction. Recently S. Tachibana [2][1] and T. Kashiwada [3] has introduced a notion of conformal Killing tensor field of degree p ($p \geqq 2$) in a Riemannian manifold. They discussed such the tensor fields and obtained many interesting results.

In the previous paper [4] the author proved by the mathematical induction that a Riemannian manifold of constant curvature admitting a conformal Killing vector field admits necessarily a conformal Killing tensor field of general degree. But, the author is pointed out by Prof. Y. Katsurada that a part of the proof by the mathematical induction in the previous paper is imperfect. Accordingly, the purpose of the present paper is to give its complete proof.

In this paper we shall denote by R^n an n-dimensional Riemannian manifold of constant curvature. In § 1 we give the definition of a conformal Killing tensor field of degree $p \geqq 2$, and prove that if R^n admits a conformal Killing vector, then R^n admits conformal Killing tensor fields of degree 2 and 3. Making use of these results, in § 2 we shall show that R^n admits a conformal Killing tensor field of degree $p (p \leqq n)$.

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§ 1. Conformal Killing tensor fields of degree 2 and 3. Let $R^n (n>2)$ be an n-dimensional Riemannian manifold of constant curvature whose metric tensor is given by g_{ij}.

Let ξ^i be a vector field in R^n such that

$$\underset{\xi}{\pounds} g_{ij} = \xi_{i;j} + \xi_{j;i} = 2\phi g_{ij} \tag{1.1}$$

where ϕ is a scalar field in R^n and the symbol $\underset{\xi}{\pounds}$ and " ; " denote the operator of Lie derivation with respect to ξ^i and of covariant differentiation with respect to the Riemann connection determined by g_{ij} respectively. Then ξ^i is called a conformal Killing vector field. If ϕ vanishes identically

1) Numbers in brackets refer to the references at the end of the paper.

in (1.1), then ξ^i is called a Killing vector field.

A skew symmetric tensor field $T_{i_1\cdots i_p}$ is called a conformal Killing tensor field of degree p in a Riemannian manifold, if there exists a skew symmetric tensor field $\rho_{i_1\cdots i_{p-1}}$ such that

$$\begin{aligned} T_{i_1\cdots i_p;i}+T_{ii_2\cdots i_p;i_1} &= 2\rho_{i_2\cdots i_p}g_{i_1 i} \\ &-\sum_{h=2}^{p}(-1)^h\{\rho_{i_1\cdots \hat{i}_h\cdots i_p}g_{ii_h}+\rho_{ii_2\cdots \hat{i}_h\cdots i_p}g_{i_1 i_h}\} \end{aligned} \tag{1.2}$$

where $\hat{i}_h$ means that i_h is omitted. We call $\rho_{i_1\cdots i_{p-1}}$ the associated tensor field of $T_{i_1\cdots i_p}$. If $\rho_{i_1\cdots i_{p-1}}$ vanishes identically in (1.2), then $T_{i_1\cdots i_p}$ is called a Killing tensor field of degree p, [2], [3].

At the first, we prove that R^n admitting a conformal Killing vector field ξ^i admits a conformal Killing tensor field of degree 2.

In 1957, K. Yano and T. Nagano [1] proved the following lemma:

LEMMA (K. Yano and T. Nagano). *If R^n admits a conformal Killing vector field ξ^i, then R^n admits a non-zero scalar function ϕ such that*

$$\phi_{;i;j}=-k\phi g_{ij}\,,\qquad k=\frac{R}{n(n-1)}\,, \tag{1.3}$$

where R is the scalar curvature.

Now, we put

$$\rho_i=\xi_i+\frac{1}{k}\phi_i\,,\qquad\qquad (\phi_i=\phi_{;i})\,.$$

Differentiating this covariantly, by means of (1.1) and (1.3) we get

$$\rho_{i;j}+\rho_{j;i}=0\,. \tag{1.4}$$

Differentiating (1.4) covariantly, we have

$$\rho_{i;j;k}+\rho_{j;i;k}+\rho_{i;k;j}+\rho_{k;i;j}-(\rho_{j;k;i}+\rho_{k;j;i})=0$$

Then by virtue of Ricci and Bianchi identity, the above equation reduces to

$$\rho_{i;j;k}+\rho_h R^h{}_{kji}=0$$

where $R^h{}_{kji}$ is the curvature tensor.
Since R^n is a space of constant curvature, the last equation turns to

$$\rho_{i;j;k}=k(\rho_j g_{ki}-\rho_i g_{jk})\,.$$

We put $T_{ij}=\rho_{i;j}$, then the above equation is rewritten as follows:

$$T_{ij;k}=k(\rho_j g_{ki}-\rho_i g_{jk})\,, \tag{1.5}$$

and hence we obtain

$$T_{ij;k}+T_{kj;i}=k(2\rho_j g_{ki}-\rho_i g_{jk}-\rho_k g_{ij}).$$

Therefore we have

LEMMA 1.1. *If R^n admits a conformal Killing vector field ξ^i, then R^n admits a conformal Killing tensor field of degree* 2, [4].

Next, we shall show that a conformal Killing tensor field of degree 3 can be constructed by a conformal Killing tensor field of degree 2 and the vector ϕ_i.

We put

$$T_{ijk}=T_{ij}\phi_k+T_{jk}\phi_i+T_{ki}\phi_j. \tag{1.6}$$

Then it is clear that T_{ijk} is skew symmetric with respect to all indices.

Differentiating (1.6) covariantly, by means of (1.5) and (1.3) we have

$$T_{ijk;l}=k\Big\{(\rho_j\phi_k-\rho_k\phi_j-\phi T_{jk})g_{il}+(\rho_k\phi_i-\rho_i\phi_k-\phi T_{ki})g_{jl}$$
$$+(\rho_i\phi_j-\rho_j\phi_i-\phi T_{ij})g_{kl}\Big\}$$

Put $\rho_{jk}=\rho_j\phi_k-\rho_k\phi_j-\phi T_{jk}$, then the last equation turns to

$$T_{ijk;l}=k(\rho_{jk}g_{il}+\rho_{ki}g_{jl}+\rho_{ij}g_{kl}), \tag{1.7}$$

and hence we get

$$T_{ijk;l}+T_{ljk;i}=k(2\rho_{jk}g_{il}-\rho_{ik}g_{jl}-\rho_{lk}g_{ji}+\rho_{ij}g_{kl}+\rho_{lj}g_{ki}).$$

Therefore we have

LEMMA 1.2. *If R^n admits a conformal Killing vector field ξ^i, then R^n admits a conformal Killing tensor field of degree* 3, [4].

§2. Conformal Killing tensor field of general degree. At the first, we prove that R^n admits a conformal Killing tensor field of degree $2p$ $(2p\leqq n)$, under the assumption that R^n admits a conformal Killing tensor field of degree $2p-2\geqq 2$.

We assume that R^n admits a skew symmetric tensor field $T_{i_1\cdots i_{2p-2}}$ such that

$$T_{i_1\cdots i_{2p-2};i}=-k\sum_{h=1}^{2p-2}(-1)^h\rho_{i_1\cdots\hat{i}_h\cdots i_{2p-2}}g_{i_h i}, \tag{2.1}$$

where $\rho_{i_1\cdots\hat{i}_h\cdots i_{2p-2}}$ denotes the associated tensor field of $T_{i_1\cdots i_{2p-2}}$. We put

$$T_{i_1\cdots i_{2p}}=-\sum_{\substack{h,k=1\\(h<k)}}^{2p}(-1)^{h+k}T_{i_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_{2p}}T_{i_h i_k},$$

where T_{ij} means $\rho_{i;j}$ defined in § 1.
Then it is clear that $T_{i_1\cdots i_{2p}}$ is skew symmetric with respect to all indices.

Differentiating (2. 1) covariantly, we have

$$T_{i_1\cdots i_{2p};i}=-k\sum_{\substack{h,k=1\\(h<k)}}^{2p}(-1)^{h+k}(T_{i_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_{2p};i}T_{i_hi_k}+T_{i_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_{2p}}T_{i_hi_k;i})\,.$$

Substituting (2. 1) and (1. 5) into this equation, we find

$$\begin{aligned}T_{i_1\cdots i_{2p};i}=&-\sum_{\substack{h,k=1\\(h<k)}}^{2p}(-1)^{h+k}\Big\{-k\sum_{\substack{l=1\\(l<h<k,\,h<k<l)}}^{2p}(-1)^{l}\rho_{i_1\cdots\hat{i}_l\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_{2p}}T_{i_hi_k}g_{i_li}\\&-k\sum_{\substack{l=1\\(h<l<k)}}^{2p}(-1)^{l+1}\rho_{i_1\cdots\hat{i}_h\cdots\hat{i}_l\cdots\hat{i}_k\cdots i_{2p}}T_{i_hi_k}g_{i_li}\\&+kT_{i_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_{2p}}(\rho_{i_k}g_{i_hi}-\rho_{i_h}g_{i_ki})\Big\}\\=&\,k\sum_{l=1}^{2p}(-1)^{l}\Big\{\sum_{\substack{h,k=1\\(l<h<k,\,h<k<l)}}^{2p}(-1)^{h+k}\rho_{i_1\cdots\hat{i}_l\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_{2p}}T_{i_hi_k}\\&+\sum_{\substack{h,k=1\\(h<l<k)}}^{2p}(-1)^{h+k+1}\rho_{i_1\cdots\hat{i}_h\cdots\hat{i}_l\cdots\hat{i}_k\cdots i_{2p}}T_{i_hi_k}\\&-\sum_{\substack{k=1\\(l<k)}}^{2p}(-1)^{k}T_{i_1\cdots\hat{i}_l\cdots\hat{i}_k\cdots i_{2p}}\rho_{i_k}+\sum_{\substack{h=1\\(h<l)}}^{2p}(-1)^{h}T_{i_1\cdots\hat{i}_h\cdots\hat{i}_l\cdots i_{2p}}\rho_{i_h}\Big\}g_{i_li}\,.\end{aligned}$$

Hence if we put

$$\begin{aligned}\rho_{i_1\cdots\hat{i}_l\cdots i_{2p}}=&\sum_{\substack{h,k=1\\(l<h<k,\,h<k<l)}}^{2p}(-1)^{h+k}\rho_{i_1\cdots\hat{i}_l\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_{2p}}T_{i_hi_k}\\&+\sum_{\substack{h,k=1\\(h<l<k)}}^{2p}(-1)^{h+k+1}\rho_{i_1\cdots\hat{i}_h\cdots\hat{i}_l\cdots\hat{i}_k\cdots i_{2p}}T_{i_hi_k}\\&-\sum_{\substack{k=1\\(l<k)}}^{2p}(-1)^{k}T_{i_1\cdots\hat{i}_l\cdots\hat{i}_k\cdots i_{2p}}\rho_{i_k}+\sum_{\substack{h=1\\(h<l)}}^{2p}(-1)^{h}T_{i_1\cdots\hat{i}_h\cdots\hat{i}_l\cdots i_{2p}}\rho_{i_h}\,,\end{aligned}$$

then the last equation turns to

$$T_{i_1\cdots i_{2p};i}=k\sum_{l=1}^{2p}(-1)^{l}\rho_{i_1\cdots\hat{i}_l\cdots i_{2p}}g_{i_li}\,,\tag{2. 2}$$

and hence we get

$$\begin{aligned}T_{i_1i_2\cdots i_{2p};i}+T_{ii_2\cdots i_{2p};i_1}&=k\sum_{l=1}^{2p}(-1)^{l}\rho_{i_1\cdots\hat{i}_l\cdots i_{2p}}g_{i_li}+k\sum_{\substack{l=1\\(l\neq1)}}^{2p}(-1)^{l}\rho_{ii_2\cdots\hat{i}_l\cdots i_{2p}}g_{i_li_1}\\&=k\Big\{-\rho_{i_2\cdots i_{2p}}g_{i_1i}+\sum_{l=2}^{2p}(-1)^{l}\rho_{i_1\cdots\hat{i}_l\cdots i_{2p}}g_{i_li}\Big\}\end{aligned}$$

$$+k\left\{-\rho_{i_2\cdots i_{2p}}g_{ii_1}+\sum_{l=2}^{2p}(-1)^l\rho_{ii_2\cdots\hat{i}_l\cdots i_{2p}}g_{i_li_1}\right\}$$

$$=-k\left\{2\rho_{i_2\cdots i_{2p}}g_{ii_1}-\sum_{l=2}^{2p}(-1)^l(\rho_{i_1\cdots\hat{i}_l\cdots i_{2p}}g_{i_li}+\rho_{ii_2\cdots\hat{i}_l\cdots i_{2p}}g_{i_li_1})\right\}.$$

This equation shows that $T_{i_1\cdots i_{2p}}$ is a conformal Killing tensor field of degree $2p$ whose associated tensor field is given by $-k\rho_{i_2\cdots i_{2p}}$.

Next, we shall show that a conformal Killing tensor field of degree $2p+1\geqq 3$ can be constructed by a conformal Killing tensor field of degree $2p$ and the vector ϕ_i. It is given by the method of the previous paper, [4].

We put

$$T_{i_1\cdots i_{2p+1}}=-\sum_{h=1}^{2p+1}(-1)^hT_{i_1\cdots\hat{i}_h\cdots i_{2p+1}}\phi_{i_h}. \tag{2.3}$$

Then it is clear that $T_{i_1\cdots i_{2p+1}}$ is skew symmetric with respect to all indices.

Differentiating (2.3) covariantly we have

$$T_{i_1\cdots i_{2p+1};i}=-\sum_{h=1}^{2p+1}(-1)^hT_{i_1\cdots\hat{i}_h\cdots i_{2p+1};i}\phi_{i_h}-\sum_{h=1}^{2p+1}(-1)^hT_{i_1\cdots\hat{i}_h\cdots i_{2p+1}}\phi_{i_h;i}.$$

Substituting (1.3) and (2.2) into this equation, we find

$$T_{i_1\cdots i_{2p+1};i}=-\sum_{h=1}^{2p+1}(-1)^h\cdot k\sum_{\substack{k=1\\(h\neq k)}}^{2p+1}(-1)^k\rho_{i_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_{2p+1}}\phi_{i_h}g_{i_ki}$$

$$-k\phi\sum_{h=1}^{2p+1}(-1)^hT_{i_1\cdots\hat{i}_h\cdots i_{2p+1}}g_{i_hi}$$

$$=-k\sum_{h=1}^{2p}(-1)^h\left\{\sum_{\substack{k=1\\(h\neq k)}}^{2p+1}(-1)^k\rho_{i_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_{2p+1}}\phi_{i_k}+\phi T_{i_1\cdots\hat{i}_h\cdots i_{2p+1}}\right\}g_{i_hi}.$$

Hence if we put

$$\rho_{i_1\cdots\hat{i}_h\cdots i_{2p+1}}=\sum_{\substack{k=1\\(h\neq k)}}^{2p+1}(-1)^k\rho_{i_1\cdots\hat{i}_h\cdots\hat{i}_k\cdots i_{2p+1}}\phi_{i_k}+\phi T_{i_1\cdots\hat{i}_k\cdots i_{2p+1}},$$

then the last equation turns to

$$T_{i_1\cdots i_{2p+1};i}=-k\sum_{h=1}^{2p+1}(-1)^h\rho_{i_1\cdots\hat{i}_h\cdots i_{2p+1}}g_{i_hi},$$

and hence we get

$$T_{i_1i_2\cdots i_{2p+1};i}+T_{ii_2\cdots i_{2p+1};i_1}=-k\sum_{h=1}^{2p+1}(-1)^h\rho_{i_1\cdots\hat{i}_h\cdots i_{2p+1}}g_{i_hi}-k\sum_{\substack{h=1\\(h\neq 1)}}^{2p+1}(-1)^h\rho_{ii_2\cdots\hat{i}_h\cdots i_{2p+1}}g_{i_hi_1}$$

$$=-k\left\{-\rho_{i_2\cdots i_{2p+1}}g_{i_1i}+\sum_{h=2}^{2p+1}(-1)^h\rho_{i_1\cdots\hat{i}_h\cdots i_{2p+1}}g_{i_hi}\right\}$$

$$-k\left\{-\rho_{i_2\cdots i_{2p+1}}g_{ii_1}+\sum_{h=2}^{2p+1}(-1)^h\rho_{ii_2\cdots\hat{i}_h\cdots i_{2p+1}}g_{i_h i_1}\right\}$$
$$=k\left\{2\rho_{i_2\cdots i_{2p+1}}g_{ii_1}-\sum_{h=2}^{2p+1}(-1)^h(\rho_{i_1\cdots\hat{i}_h\cdots i_{2p+1}}g_{i_h i}+\rho_{ii_2\cdots\hat{i}_h\cdots i_{2p+1}}g_{i_h i_1})\right\}.$$

This equation shows that $T_{i_1\cdots i_{2p+1}}$ is a conformal Killing tensor field of degree $2p+1$ whose associated tensor field is given by $k\rho_{i_2\cdots i_{2p+1}}$. By means of Lemma 1. 1, Lemma 1. 2 and the above calculation, we have

THEOREM 2. 1. *Let R^n be an n-dimensional Riemannian manifold of constant curvature which admits a conformal Killing vector field ξ^i. Then R^n admits a conformal Killing tensor field of degree p ($p\leqq n$).*

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