A Characterization of Conway's Group C_3

By Tomoyuki YOSHIDA

§ 1. Introduction

In this paper we characterize the Conway's simple group C_3 of order $2^{10}\cdot 3^7\cdot 5^3\cdot 7\cdot 11\cdot 23$ by the structure of the centralizer of a noncentral involution.

Main theorem. *Let G be a finite group satisfying the following properties*:

(*i*) *G has an involution e with $C_G(e)\cong Z_2\times M_{12}$,*

(*ii*) $e\in O^2(G)$.

Then $G\cong C_3$.

The centralizer of a central involution of the Conway's group C_3 is isomorphic to the perfect central extention of $S_p(6,2)$ by a group of order 2. The main difficulty in proving the main theorem is in the determination of the structure of a S_2-subgroup of G. If this is established, we can easily know that G has the same involution fusion pattern and the centralizer of a central involution as the Conway's group C_3. Thus the characterization theorem of C_3 by D. Fendel [1] implies that $G\cong C_3$.

Throughout, all group considered are finite. Most of our notations are standard (see [2]) and we use the "bar" convention for homomorphic images. Furthermore we use the following notations:

$x\sim y$	x is conjugate to y,
$a:\ x\longrightarrow y$	$y=x^a=a^{-1}xa$,
x^H	$=\{x^h \mid h\in H\}$,
$\langle x^H\cap K\rangle$	$=\langle y \mid y\in K,\ x\sim y \text{ in } H\rangle$,
$A*B$	the central product,
$A\wr B$	the wreathed product.

§ 2. Preliminalies

A. Mathieu group $\mathbf{M}_{12}$. We list some properties of Mathieu group $M_{12}=M$. Let c be an involution of the center of a S_2-subgroup of M.

(1) *Generators and relations of the centralizer of c.*

$C_M(c)=\langle a_1, a_2, b_1, b_2, s, t\rangle$,

$a_1^2=a_2^2=b_1^2=b_2^2=[a_1, a_2]=[b_1, b_2]=c$,

$$[a_1, b_1] = [a_1, b_2] = [a_2, b_1] = [a_2, b_2] = 1\,,$$
$$s^3 = t^2 = (st)^2 = 1\,,$$
$$s:\ a_1 \longrightarrow a_2 \longrightarrow a_1a_2,\ b_1 \longrightarrow b_2 \longrightarrow b_1b_2\,,$$
$$t:\ a_1 \longrightarrow a_1c,\ a_2 \longrightarrow a_1a_2c,\ b_1 \longrightarrow b_1c,\ b_2 \longrightarrow b_1b_2c\,.$$

(2) *The fusion pattern of involutions.* $T_0 = \langle a_1, a_2, b_1, b_2, c, t\rangle$ is the S_2-subgroup of M and every involution of M is conjugate to c or $a_2b_1b_2$. Furthermore the following hold:

$$c \sim a_1b_1 \sim a_2b_2 \sim t\,,$$
$$a_2b_1b_2 \sim a_1b_2 \sim a_2b_1\,.$$

(3) *New generators of T_0.*
Set as follows:

$$a = a_2b_2t,\ b = a_2b_1b_2ct,\ u = a_2b_1b_2,\ r = a_1b_2\,.$$

Then a, b, u and r generate T_0, and

$$a^4 = b^4 = u^2 = r^2 = [a, b] = [u, r] = 1\,,$$
$$u:\ a \longrightarrow a^{-1},\ b \longrightarrow b^{-1},\ r:\ a \longleftrightarrow b\,,$$
$$a_1 = au,\ b_1 = ab^{-1},\ a_1b_1 = a^2,\ c = a^2b^2,$$
$$a_2 = a^2ur,\ b_2 = a^{-1}b^{-1}r,\ a_2b_2 = a^{-1}bu,$$
$$t = a^2bu\,.$$

(4) *Another 2-local subgroup.*

$$N_M(\langle a^2, b^2\rangle) = \langle a, b, u, s', r\rangle,$$
$$s'^3 = (s'r)^2 = 1,$$
$$s':\ a \longrightarrow b \longrightarrow a^{-1}b^{-1},\ u \longrightarrow u\,.$$

For the original generators, s' normalizes $\langle c, a_1, b_1, a_2b_2, t\rangle$ and

$$s':\ c \longrightarrow a_1b_1 \longrightarrow a_1b_1c,\ t \longrightarrow a_2b_2 \longrightarrow a_1t\,,$$
$$a_1 \longrightarrow a_1a_2b_1b_2t,\ b_1 \longrightarrow ca_1a_2b_1b_2t\,.$$

(5) $C_M(c)$ and $N_M(\langle a^2, b^2\rangle)$ are maximal 2-local subgroups of M. In particular, $N_M(\langle c, a_1b_1, a_2b_2\rangle) = C_M(c)$.

(6) $R_0 = \langle a_1, a_2, b_1, b_2, c\rangle$ is the unique subgroup of T_0 isomorphic to $Q_8 * Q_8$. Quaternion subgroups of R_0 are only $\langle a_1, a_2, c\rangle$ and $\langle b_1, b_2, c\rangle$. Furthermore *Aut* $Q_8 * Q_8 \cong S_4 \wr Z_2$.

$B_0 = \langle a, b, u\rangle = \langle a_1, b_1, a_2b_2, c, t\rangle$ is the unique subgroup of T_0 isomorphic to B_0.

PROOF. We recall the definition of M_{12} by Witt [4], and also a set of generators and relations for $C_M(c)$ by Wong [5]. Let α be a primitive element of $GF(9)$ satisfying $\alpha^2+\alpha=1$. As a permutation group on the projective line $L=GF(9)\cup\{\infty\}$, we define

$$M_{10}=\langle PSL(2,9),\ s_1\rangle,$$

where $s_1: x\longrightarrow \alpha x^3$. If new points v and w are adjoining to L, M_{11} and M_{12} are defined as the transitive extensions in succession as follows.

$$M_{11}=\langle M_{10}, s_2\rangle,\ M_{12}=\langle M_{11}, S_3\rangle,$$

where

$$s_2:\ x\longrightarrow \alpha^2 x+\alpha x^3,\ \infty\longleftrightarrow v,$$
$$s_3:\ x\longrightarrow x^3,\ v\longleftrightarrow w.$$

Let

$$\begin{aligned}
&\pi:\ x\longrightarrow -x,\\
&\beta:\ x\longrightarrow \alpha^{-1}x^3,\\
&\gamma:\ x\longrightarrow \alpha x^3,\\
&\tau:\ x\longrightarrow \alpha x^{-1},\\
&\varepsilon:\ x\longrightarrow -\alpha x^3-\alpha^3x^{-1}(x\neq 0,\infty),\ 0\longleftrightarrow v,\ \infty\longleftrightarrow w,\\
&\lambda:\ x\longrightarrow \alpha^2x^{-1}+\alpha x^{-3}(x\neq 0,\infty),\ 0\longrightarrow v\longrightarrow\infty\longrightarrow 0,\\
&\mu:\ x\longrightarrow x^{-1}.
\end{aligned}$$

Then π is in the center of a S_2-subgroup of M and $C(\pi)$ is generated by $\pi, \beta, \gamma, \tau, \varepsilon, \lambda, \mu$. If we put

$$c=\pi,\ a_1=\beta\gamma,\ a_2=\gamma,\ b_1=\beta\tau,\ b_2=\pi\gamma\varepsilon,\ s=\lambda,\ t=\mu,$$

then we can check easily that a_1, a_2, b_1, b_2, c, s and t satisfy the relations in (1) using relations (1) in [5].

Since M_{11} has only one class of involutions, we have the fusion pattern of involutions in (2).

If we set

$$s'=(0,\alpha^2,\alpha^3)(\infty,-\alpha^2,-\alpha^3)(v,1,-\alpha)(w,-1,\alpha),$$

then $s'\in M_{12}$ and s' satisfies the relations in (4). The proof of (3), (5) and (6) are easy.

B. The order of a S_2-subgroup.

LEMMA. *Let H be a subgroup of a group G and e an involution of $H\cap O^2(G)$.*

(1) *If χ is a character of G, then $\chi(1)\equiv\chi(e)$ (mod 4).*

(2) *Assume that* $e^G \cap H = e_1^H + \cdots + e_n^H$. *Let* α *be a character of* H. *Then*

$$\alpha^G(e) = \frac{|C^G(e)|}{|H|} \sum_i |e_i^H| \alpha(e_i) \equiv |G:H| \alpha(1) \qquad (mod\ 4).$$

(3) *Let* I *be the principal character of* H. *If* $I^G(e)$ *is odd, then* $|G|_2 = |H|_2$. *If* $I^G(e) \equiv 2$ $(mod\ 4)$, *then* $|G|_2 = 2|H|_2$.

PROOF. Let ρ be a matrix representation of G with the character χ. Then the characteristic roots of the matrix $\rho(e)$ are 1 and -1. Let their multiplicities be a and b, where $a, b \geqq 0$. Since $e \in O^2(G)$, $det\ \rho(e) = 1$. Thus we have that b is even. Since $\chi(1) = a+b$ and $\chi(e) = a-b$, $\chi(1) - \chi(e) = 2b \equiv 0$ $(mod\ 4)$, proving (1). The rest of the lemma is easily proved.

§ 3. The proof of the main theorem

Throughout this section G denotes a simple group satisfying the hypothesis of the main theorem, and let e be an involution of G such that $C_G(e) = \langle e \rangle \times M$, where $M = M_{12}$. Furthermore a_1, a_2, b_1, b_2, c, s, t, a, b, u, r and s' denote the same elements of M as those in § 2.

LEMMA 1. *e is not a central involution of G.*

PROOF. Assume false, in which case $T = \langle e \rangle \times \langle a_1, a_2, b_1, b_2, c, t \rangle \in Syl_2 G$ and $T \cap M \in Syl_2 M$. By § 2 (1), c is the square of an element of G and e is not. Thus $e \not\sim c$. Since $Z(T) = \langle e, c \rangle$, Burnside's theorem ([1], Theorem 7. 1. 1) implies that c, e and ce are not conjugate in G each other. However M possesses exactly two conjugate classes of involutions, and so it follows from Thompson's fusion theorem that two of c, e and ce are conjugate in G each other, a contradiction.

Set $T = \langle e \rangle \times \langle a_1, a_2, b_1, b_2, c, t \rangle$ and $B = \langle e \rangle \times \langle a, b, u \rangle$. Then we note that B is weakly closed in T.

LEMMA 2. $|N_G(T) : T| = 2$ *and* $N_G(B)/B = N_G Z(B)/B \cong S_4$. *In particular* $|G|_2 \geqq 2^9$.

PROOF. Since $Z(T) = \langle c, e \rangle \cong Z_2^2$ and $N_G(T) \cap C_G(e) = T$, we have that $|N_G(T) : T| = 2$ by Lemma 1. Since $B = C_T \Omega_1 Z_2(T)$ *char* T, if we set $Z = Z(B)$ and $N = N_G(Z)$, then $N_G(T) \subseteq N$, and so $e \sim ec$ in N. Since $ea^2 \sim eb^2 \sim ea^2b^2$ in N by § 2(3), we have that N acts transitively by conjugation on the set $e^G \cap Z = e\langle a^2, b^2 \rangle$. Since $C_N(Z) = B$ and $C_N(e)/B \cong S_3$, we conclude that $B \lhd N$ and $N/B \cong S_4$.

LEMMA 3. *There exists an element* $d \in N_G(T) - T$ such that

$$d^2 = 1,\ [d, s] = 1,\ [d, t] = c^\alpha, \text{ where } \alpha = 0 \text{ or } 1,$$
$$[a_1, d] = [a_2, d] = [b_1, d] = [b_2, d] = 1 .$$

PROOF. Set $R=\langle e\rangle\times\langle a_1, a_2, b_1, b_2, c\rangle\cong \boldsymbol{Z}_2\times(Q_8{}^*Q_8)$. Then $R/Z(T)\cong \boldsymbol{Z}_2^4$ is the unique abelian maximal subgroup of $T/Z(T)\cong \boldsymbol{Z}_2^2 \wr \boldsymbol{Z}_2$, and so R char T. Thus $T^*=N_G(T)\subseteq N_G(R)=N$. Since $Z(T)=Z(R)=\langle c, e\rangle \lhd N$, $|N:C_N(e)|=2$. By Frattini argument, $N=N_N(\langle s\rangle)R$, and so $|C_N(s)|=24$. Since $C_N(s)\not\subseteq C_G(e)$, $O_2C_N(s)\cong D_8$. Let d be an involution of $C_N(s)-T$. Then we have that $T^*=\langle T, d\rangle$, $C_N(s)=\langle s\rangle\times\langle c, d, e,\rangle$ and $[d, e]=c$. t normalizes $\langle c, d, e\rangle$ and $[d, t]=c^\alpha$ for $\alpha=0$ or 1 (because otherwise $(dt)^2=[d, t]\sim e$).

Now d normalizes $[R, s]=Q_1Q_2$ where $Q_1=\langle a_1, a_2, c\rangle$ and $Q_2=\langle b_1, b_2, c\rangle$. Since $[R, s]\cong Q_8{}^*Q_8$ possesses exactly two quaternion subgroups, which are Q_1 and Q_2, we have that either d: $Q_1\longrightarrow Q_1$, $Q_2\longrightarrow Q_2$ or d: $Q_1\longleftrightarrow Q_2$. If d: $Q_1\longrightarrow Q_1$, $Q_2\longrightarrow Q_2$, then since $Aut\ Q_8\cong S_4$ and d commutes with the element of order 3, d centralizes both Q_1 and Q_2. Thus in this case the lemma holds.

Next we assume that d: $Q_1\longleftrightarrow Q_2$. Since $[d, t]=c^\alpha$, we have that d: $a_1\longrightarrow b_1c^i$ for $i=0$ or 1. Thus d: $a_2=a_1^s\longrightarrow a_1^{sd}=a_1^{ds}=b_1^sc^i=b_2c^i$. Similarly d: $a_2^s\longrightarrow b_2^sc^i=b_1b_2c^i$. On the other hand d: $a_2^s=a_1a_2\longrightarrow b_1c^ib_2c^i=b_1b_2$. Hence we have that $i=0$, and so

$$d:\ a_1\longleftrightarrow b_1,\ a_2\longleftrightarrow b_2,\ e\longrightarrow ec.$$

Thus we have that $T_1=C_{T^*}(ea_1b_1)=\langle c, a_1, b_1, a_2b_2, e, t, a_2d\rangle$ and $cl\ T_1=4$, and so T_1 is not isomorphic to T. On the other hand, it follows from Lemma 2 that $x=d^{s'}\in N_G(T^*)$ and x: $e\longrightarrow a_1b_1e$. Thus $T_1=T_1^x\cong T$, a contradiction. The lemma is proved.

LEMMA 4. *Set* $Z=Z(B)=\langle e, a^2, b^2\rangle$ *and set* $d'=d^{s'}$. *Then* $N_G(Z)=\langle B, d, d', s', r\rangle$ and the following relations holds:

(i) d: $a\longrightarrow ac^\alpha$, $b\longrightarrow bc^\alpha$, $u\longrightarrow u$, $e\longrightarrow ec$

(ii) d': $a\longrightarrow a$, $b\longrightarrow ba^{2\alpha}$, $u\longrightarrow u$, $e\longrightarrow ea^2$,

(iii) s': $d\longrightarrow d'\longrightarrow dd'u^\alpha c^\beta e^\beta$,

(iv) r: $d\longrightarrow d$, $d'\longrightarrow dd'u^\alpha c^\beta e^\beta$,

(v) $[d, d']=(dd')^2=b^{2\beta}$,

where $\beta=0$ or 1.

PROOF. By §2 (4) and Lemma 3, (i) and (ii) hold. Set $N=N_G(Z)$ and let s': $d'\longrightarrow d''$. Then $N/B\cong S_4$ by Lemma 2 and d centralizes $B'=\langle a^2, b^2\rangle$, and so $\langle B, d, d'\rangle=C_N(B')=O_2(N)$. Thus $[d, d']$ and $dd'd''\in B$. Set $x=d'dd''$. Then

$$x:\ a\longrightarrow a^{1+2\alpha},\ b\longrightarrow b^{1+2\alpha},\ u\longrightarrow u,\ e\longrightarrow e.$$

Thus we can write

$$x=u^{\alpha}a^{2i}b^{2j}e^{\beta},\text{ where } i,j,\beta=0 \text{ or } 1\,.$$

Since $s'\colon d\longrightarrow d'\longrightarrow d''\longrightarrow d$, we have that $s'\colon x=d'dd''\longrightarrow d''d'd=dd'xd'd=xb^{2\beta}$. Thus $b^2=[x,s']=[u^{\alpha}a^{2i}b^{2j}e^{\beta},s']=a^{2i+2j}b^{2i}$, and so $i=j=\beta$, proving (iii). Since $r\colon s'\longrightarrow s'^{-1}$, (iv) follows easily from (iii). Finally since $d''^2=(dd'x)^2=(dd')^2x^{dd'}x=(dd')^2b^{2\beta}=1$, (v) also holds. The lemma is proved.

LEMMA 5. *The following hold*:

(1) *Set* $V=\langle c,v_1,v_2,e\rangle$, $E=\langle a_1,a_2,b_1,b_2,c,d,e\rangle$ *and* $N=N_G(V)$, *where* $v_1=a_1b_1$, $v_2=a_2b_2$. *Then* N *normalizes* $\langle c\rangle$, $\langle c,v_1,v_2\rangle$ *and* E. $N/E\cong S_4$. $|N|=2^{10}\cdot 3$. *Furthermore with respect to the basis* $\{c,v_1,v_2,e\}$ *of* $V\cong Z_2^4$, N/V *is represented as the subgroup of* $GL(4.2)(\cong Aut\ V)$ *of matrices*

$$\begin{bmatrix}1&0&0&0\\ *&*&*&0\\ *&*&*&0\\ *&*&*&1\end{bmatrix}$$

(2) $\alpha=\beta$

(3) *If we set* $w=d'(ta_1a_2b_2de^{\alpha})^{\alpha}c^{\gamma}$ *and* $w'=w^s$ *for a suitable* $\gamma=0$ *or* 1, *then*

$$\begin{aligned}w\colon\ & v_1\longrightarrow v_1,\ v_2\longrightarrow v_2c^{\alpha},\ e\longrightarrow v_1ec^{\alpha},\\ & d\longrightarrow(v_1c)^{\alpha}d,\ a_1\longrightarrow a_1v_1^{\alpha},\ b_1\longrightarrow b_1v_1^{\alpha},\\ & a_2\longrightarrow a_2(v_1v_2c)^{\alpha}de^{\alpha},\\ & b_2\longrightarrow b_2(v_1v_2)^{\alpha}de^{\alpha},\\ w'\colon\ & v_1\longrightarrow v_1c^{\alpha},\ v_2\longrightarrow v_2,\ e\longrightarrow v_2ec^{\alpha},\\ & d\longrightarrow(v_2c)^{\alpha}d,\ a_2\longrightarrow a_2v_2^{\alpha},\ b_2\longrightarrow b_2v_2^{\alpha},\\ & a_1\longrightarrow a_1(v_1v_2)^{\alpha}de^{\alpha},\\ & b_1\longrightarrow b_1(v_1v_2c)^{\alpha}de^{\alpha},\end{aligned}$$

(4) $\langle w,w'\,s,t\rangle\cong S_4$ *and* $N=E\cdot\langle w,w',s,t\rangle$.
Furthermore

$$\begin{aligned}&w^2=w'^2=[w,w']=1\,,\\ &s\colon\ w\longrightarrow w'\longrightarrow ww',\\ &t\colon\ w\longrightarrow w,\ w'\longrightarrow ww'.\end{aligned}$$

PROOF. By Lemma 4 (ii) and §2 (4), we see that d' normaizes $V=\langle c,v_1,v_2,e\rangle=\langle a^2,b^2,a^{-1}bu,e\rangle$ and $d'\colon e\longrightarrow v_1e$. Thus $e^G\cap V=\langle c,v_1,v_2\rangle e=e^N\cap V$. Set $\bar{N}=N/V$. Then since $C_N(e)=\langle e\rangle\times C_M(c)$ *by Lemma* §2 (5), we have that $C_{\bar{N}}(e)\cong S_4$. Thus it follows that $|\bar{N}:C_{\bar{N}}(e)|=|e^N\cap V|=8$, and so

$|\bar{N}|=2^6\cdot 3$. Now we can regard $\bar{N}$ as a subgroup of $GL(4,2)\cong Aut\ V$. With respected to the basis $\{c, v_1, v_2, e\}$, we have that

$$s\longrightarrow\begin{pmatrix}1&&&\\&&1&\\&1&1&\\&&&1\end{pmatrix},\quad t\longrightarrow\begin{pmatrix}1&&&\\&1&&\\&1&1&\\&&&1\end{pmatrix},$$

$$a_1\longrightarrow\begin{pmatrix}1&&&\\&1&&\\1&&1&\\&&&1\end{pmatrix},\quad a_2\longrightarrow\begin{pmatrix}1&&&\\1&1&&\\&&1&\\&&&1\end{pmatrix},$$

$$d'\,t^{\alpha}\longrightarrow\begin{pmatrix}1&&&\\&1&&\\&&1&\\&1&&1\end{pmatrix},\quad (d'\,t^{\alpha})^{s}\longrightarrow\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&1&1\end{pmatrix},$$

$$d\longrightarrow\begin{pmatrix}1&&&\\&1&&\\&&1&\\1&&&1\end{pmatrix}.$$

Hence it follows from a comparison between the orders that

$$\bar{N}\longrightarrow\left\{\begin{pmatrix}1&0&0&0\\ *&*&*&0\\ *&*&*&0\\ *&*&*&1\end{pmatrix}\in GL(4,2)\right\},$$

and so (1) follows at once.

Next set $x=d'(ta_1a_2b_2de^{\beta})^{\alpha}$, $y=x^s$ and $z=y^s$ we see that $\langle E, x, y\rangle=O_2(N)$ by the matrix representation of N. By Lemma 4 (ii) and §2 (1),

$$\begin{aligned}x:\ & c\longrightarrow c,\ v_1\longrightarrow v_1,\ v_2\longrightarrow v_2,\ e\longrightarrow c^{\alpha}v_1e,\\ & d\longrightarrow c^{\alpha+\beta+\alpha\beta}v_1^{\beta}d,\ a_1\longrightarrow a_1v_1^{\alpha},\\ & a_2\longrightarrow a_2(v_1c)^{\beta}v_2^{\alpha}de^{\beta},\ t\longrightarrow t\\ (sx)^3:\ & c\longrightarrow c,\ v_1\longrightarrow v_1,\ v_2\longrightarrow v_2,\ e\longrightarrow e,\\ & d\longrightarrow c^{\alpha+\beta}d,\ a_1\longrightarrow a_1c^{\alpha+\alpha\beta}v_1^{\alpha+\beta},\\ & a_2\longrightarrow a_2(v_2c)^{\alpha+\beta}.\end{aligned}$$

Thus $(sx)^3 \in C_G(V)=V$. Since $[V, a_1]=E'=\langle c\rangle$, we have $\alpha=\beta$, proving (2). Thus $(sx)^3$ centralizes E, and so $(sx)^3 \in C_G(E)=\langle c\rangle$. Let $(sx)^3=c^\gamma$, $\gamma=0$ or 1. Then $zyx=c^\gamma$.

Since $|x|=2$ by Lemma 4 (ii) and §2 (1), we have that $|y|=|z|=2$, and so $1=z^2=(c^\gamma xy)^2=[x, y]$. Thus if we set $w=xc^\gamma$, then (3) and (4) follow from (2).

LEMMA 6. $\alpha=1$.

PROOF. Assume that $\alpha=0$. Then $N=N_G(V)$ possesses fourteen conjugate classes of involutions, representatives of which together with their cardinalities are given by

c	d	v_1	$v_1 d$	e	u	ue
1	2	6	6	8	24	24
w	cw	$v_2 w$	uw	t	dt	et
12	12	24	24	48	48	48.

Here $u=a_2 b_1 b_2=b_1 v_2$ and $w=d' c^\gamma$. By Lemma 4, Lemma 5 and §2 (4), we have that s' normalizes $\langle c, a_1, b_1, v_2, d, e, t, w\rangle$ and

$$\begin{aligned} s':\ & c \longrightarrow v_1 \longrightarrow v_1 c,\ t \longrightarrow v_2 \longrightarrow a_1 t, \\ & a_1 \longrightarrow v_1 v_2 t,\ b_1 \longrightarrow c v_1 v_2 t,\ e \longrightarrow e, \\ & d \longrightarrow d' = wc^\gamma,\ w \longrightarrow dwc^\gamma v_1^\gamma . \end{aligned}$$

Thus it follows that

$$\begin{aligned} & c \sim v_1 \sim t \sim dt \sim v_2 w \sim uw \sim v_1 d \sim w \sim cw \sim d, \\ & e \sim et \sim ue,\ u \sim uw \end{aligned}$$

in $Ns'N$. Furthermore, we have that $C_S(u)=\langle c, u, v_1, b_1 b_2, d, e, w\rangle$, and so $C_S(u)'=\langle c, v_1, d\rangle \cong Z_2^3$. On the other hand, since $T'=\langle c, a_1, b_1\rangle \cong Z_2 \times Z_4$, $e \not\sim u$. Hence $e^G=e^N+(et)^N+(ue)^N$.

By Lemma (2) of §2,

$$I_N^G(e) \doteq \frac{|C_G(e)|}{|N|}\{8+24+48\} \equiv 2 \qquad (\text{mod } 4)$$

Thus Lemma (3) of §2 implies that $|G|_2=2^{11}$.

Next we shall prove that E is characteristic in S. Let D be a normal subgroup of S isomorphic to E. Then since $D \cong E = D_8 D_8 * D_8 *$ and $S/E \cong D_8$, we have that $Z(D)=D'=\langle c\rangle$ and if $D \neq E$, then $w \in DE$. Set $\bar{S}=S/\langle c\rangle$. Then $|\bar{D} \cap \bar{E}| \geqq 16$. Since $C_{\bar{D}}(\bar{w})=\langle \bar{v}_1, \bar{v}_2, \bar{a}_1, \bar{d}\rangle \subseteq \bar{D} \cap \bar{E}$, we have that $\bar{D} \cap \bar{E}=\langle \bar{v}_1, \bar{v}_2, \bar{a}_1, \bar{d}\rangle$. Thus $\bar{D} \subseteq C_{\bar{S}}(\bar{v}_1, \bar{v}_2, \bar{a}_1, \bar{d})=\langle \bar{E}, \bar{w}\rangle$, and so $|\bar{D}|=32$, a contradiction. This means that E is the unique normal subgroup of S isomorphic to $D_8 * D_8 * D_8$. Hence E char S.

Set $L=N_G(E)$. Then $L \supseteq \langle N, N_G(S) \rangle$ and clearly $|e^L|=|L:C_L(e)|=|L:C_G(c,e)|=|L|/2^7 \cdot 3$. If $e^L=e^N$, then $L \subseteq N_G(\langle e^N \rangle)=N_G(V)=N$, and so $L=N$. Thus $|G|_2=|S|=2^{10}$, which contra dicts to the fact that $|G|_2=2^{11}$. Hence $e^L=e^N+(ue)^N$. Thus we have that $|L|=2^7 \cdot 3|e^L|=2^7 \cdot 3(8+24)=2^{12} \cdot 3$, a contradiction. The lemma is proved.

LEMMA 7. *The following hold*:

(1) *Every involution of N is conjugate to c or e.*
Furthermore,

$$c \sim v_1 \sim t \sim d \sim w,$$
$$e \sim et \sim det \sim a_1 we \sim u \sim ue,$$

where $u=a_2 b_1 b_2 = v_1 v_2 a_1$.

(2) $S=\langle E, w, w', t \rangle \in Syl_2 G$.

(3) $L/E \cong GL(3,2)$ *and* $L'=L$, *where* $L=N_G(E)$.

PROOF. N has eleven conjugate classes of involutions. Their representatives and cardinalities are given by

c	v_1	d	e	u	ue	w	a_1we	t	det	et
1	6	8	8	24	24	24	48	48	48	48.

By Lemma 4, Lemma 5 and §2 (3), (4), we have that s' normalizes $\langle c, u_1, v_2, a_1, t, e, d, w \rangle$ and

$$s': \; c \longrightarrow v_1 \longrightarrow cv_1, \; t \longrightarrow v_2 \longrightarrow a_1 t,$$
$$a_1 \longrightarrow v_1 v_2 t, \; b_1 \longrightarrow cv_1 v_2 t, \; e \longrightarrow e,$$
$$d \longrightarrow d' = c^\tau v_1 v_2 a_1 dewt,$$
$$w \longrightarrow v_1^\tau v_2 cd.$$

Thus it follows that

$$c \sim v_1 \sim t \sim d \sim w,$$
$$e \sim et \sim det \sim a_1 we \sim u \sim ue$$

in $Ns'N$, proving (1).

By Lemma of §2, we have that

$$I_N^G(e) = \frac{|C_G(e)|}{|N|} \{8+48+48+48+24+24\}$$
$$\equiv 1 \qquad (\text{mod } 2).$$

Thus N contains a S_2-subgroup of G, proving (2).

Let D be a subgroup of S isomorphic to E. Then since $S/E\cong D_8$ and $D\cong E\cong D_8*D_8*D_8$, we have that $|D\cap E|\geqq 16$, and so $c\in D\cap E$. Thus $Z(D)=\langle c\rangle$. It follows easily from Lemma 5 (3) that $E/\langle c\rangle$ is the unique elementary abelian subgroup of $S/\langle c\rangle$ of order 64. Hence $D=E$. This means that E is weakly closed in S.

Let x be an involution of E conjugate to e in G. Then $x\sim e$, u or ue in N. Thus $C_E(x)\cong Z_2\times(Q_8*Q_8)$. Since $C_E(e)=\langle a_1, a_2, b_1, b_2, c, e\rangle$ is weakly closed in $T=C_E(e)\langle t\rangle\in Syl_2 C_G(e)$, x and e are conjugate in $C_G(c)$ each other. Thus it follows from Sylow's theorem that x and e are conjugate in $N_G(E)=L$. Thus $|e^L|=|e^G\cap E|=|e^N|+|u^N|+|(ue)^N|=8+24+24=2^3\cdot 7$. Since $|C_L(e)|=|C_G(c, e)|=2^7\cdot 3$, we have that $|L|=2^{10}\cdot 3\cdot 7$, and so $|L/E|=2^3\cdot 3\cdot 7$. Set $A=\langle c^G\cap E\rangle$. Then it follows from (1) and Lemma 5 (3) that $A=\langle c, v_1, v_2, d\rangle\cong Z_2^4$ and $C_N(A)=A<S$. Thus $C_G(A)=A\times K$, where $K=OC_G(A)$. Clearly $C_K(e)=1$. Since L acts on K and $e\sim u\sim eu$ in L, we have that $K=1$, and so $C_G(A)=A$. $N/E\cong S_4$ acts faithfully on $A/\langle c\rangle$. Thus we see that $C_L(A/\langle c\rangle)=E$, and so L/E acts faithfully on $A/\langle c\rangle\cong Z_2^3$. Hence $L/E\cong GL(3, 2)$. Furthermore, since $\langle e^G\cap E\rangle=\langle e^L\rangle=E$, it follows that $L'\supseteq E$, proving (3).

LEMMA 8. $C_G(c)/\langle c\rangle\cong S_p(6, 2)$.

PROOF. Set $C=C_G(c)$ and $\bar{C}=C/\langle c\rangle$. Firstly the S_2-subgroup $\bar{S}$ of $\bar{C}$ is of type A_{12}. Actually the map $S\longrightarrow A_{12}$ given by

$$v_1v_2de\longrightarrow(1\ 2)(3\ 4),\ a_1a_2e\longrightarrow(1\ 3)(2\ 4),$$
$$v_2de\longrightarrow(5\ 6)(7\ 8),\ a_2e\longrightarrow(5\ 7)(6\ 8),$$
$$v_1de\longrightarrow(9\ 10)(11\ 12),\ a_1d\longrightarrow(9\ 11)(10\ 12),$$
$$w\longrightarrow(1\ 2)(5\ 6),\ w'\longrightarrow(1\ 2)(9\ 10),$$
$$t\longrightarrow(1\ 5)(2\ 6)(3\ 7)(4\ 8),$$

defines a homomorphism onto a S_2-subgroup of A_{12} with the kernel $\langle c\rangle$ (See R. Solomon [3], P. 347 and 349), as required.

Next we shall prove that $\bar{C}$ is fusion simple, that is $O(\bar{C})=Z(\bar{C})=1$ and $\bar{C}=O^2(\bar{C})$. Since $L=N_G(E)\subseteq C$, $L'=L$ and $Z(L \text{ mod } O(G))=1$, we have that $\bar{C}=O^2(\bar{C})$ and $Z(\bar{C})=1$. The four group $\langle u, e\rangle$ normalizes $O(C)$, where $u=v_1v_2a_1$, and $u\sim e\sim ue$ in C. Since $O(C)\cap C_G(e)\subseteq OC_G(c, e)=1$, $O(C)=\langle O(C)\cap C_G(x)|x=u, e, eu\rangle=1$. Thus $O(\bar{C})=\overline{O(C)}=1$.

Finally we shall prove that $C\neq L=N_G(E)$. Since $s': c\longrightarrow v_1\longrightarrow v_1c\longrightarrow c$ and $w': v_1\longleftrightarrow v_1c$, we have that $s'w's'\in C$. By §2 (4), Lemma 4 and Lemma 5, we have that

$$s'w's':\ v_2\longrightarrow c^{\tau}v_1^{1-\tau}w\notin E$$

Thus E is not normal in C, as required.

We proved that $\bar{C}$ is fusion simple, $N_{\bar{C}}(\bar{E})/\bar{E} \cong GL(3, 2)$ and $\bar{E}$ is not normal in $\bar{C}$. Hence R. Solomon [3] derives the lemma.

We can now complete the proof of the main theorem. By Lemma 8, $C_G(c)$ is a perfect central extension of $S_p(6, 2)$. Furthremore, since $C_G(e) \not\leq C_G(c)O(G)$, $G \neq C_G(c)O(G)$. Thus we conclude that G is isomorpeic to Conway's group C_3 by D. Fendel [1].

Department of Mathematics
Hokkaido University

References

[1] D. FENDEL: A characterization of Conway's group. 3, J. Alg. 24 (1973), 159–196.

[2] D. GORENSTEIN: "Finite Groups", Harper and Row, New York, 1968.

[3] R. SOLOMON: Finite groups with Sylow 2-subgroups of type A_{12}, J. Alg. 24 (1973), 346–378.

[4] E. WITT: Die 5-fach transitiven Gruppen von Mathieu, Abh. Math. Sem. Univ. Hamburg 12, 256–264 (1938).

[5] W. J. WONG: A characterization of the Mathieu group M_{12}, Math. Zeit. 84 (1964), 378–388.

(Received December 21, 1973)