A characterization of A_7 and M_{11}, I

Dedicated to Professor Yataro Matsushima on his 60th birthday

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1. Introduction

In this paper we shall prove the following theorem.

THEOREM. *Let G be a doubly transitive group on the set $\Omega=\{1, 2, \cdots, n\}$ containing no regular normal subgroup. If the stabilizer $G_{1,2}$ of points 1 and 2 is isomosphic to a simple group $PSL(2, 2^m)$, then one of the following holds:*

(1) *$n=7$ and G is the alternating group A_7 of degree seven,*

(2) *$n=12$ and G is the Mathieu group M_{11} of degree eleven.*

In [12] Yamaki proved Theorem in the case $m=2$. Therefore we may assume $m>2$. A proof of Theorem is similar to that of [7].

Let X be a subset of a permutation group. Let $F(X)$ denote the set of all fixed points of X and $\alpha(X)$ be the number of points in $F(X)$. $N_G(X)$ acts on $F(X)$. Let $\chi_1(X)$ and $\chi(X)$ be the kernel of this representation and its image, respectively. The other notation is standard.

2. Preliminaries

Let $G_{1,2}$ be $PSL(2, 2^m)$ with $m>2$. Let K be a Sylow 2-subgroup of $G_{1,2}$. Then $N_{G_{1,2}}(K)$ is a complete Frobenius group with complement H. Let I be an involution of G with the cycle structure $(1, 2)\cdots$. Then I normalizes $G_{1,2}$.

LEMMA 1. *It may be assumed that the action of I on $G_{1,2}$ is trivial or the field automorphism.*

PROOF. Let ϕ be a homomorphism of $<I, G_{1,2}>$ into Aut $PSL(2, 2^m)$. If ker $\phi\neq1$ and $\phi(I)\neq1$, then we can replace I by an element $(\neq1)$ of ker ϕ. If ker $\phi=1$, then I induces an outer automorphism. Since $<I, G_{1,2}>$ has two classes of involution, I is conjugate to the field automorphism.

By Lemma 1 I is contained in $N_G(H)\cap N_G(K)$. Let τ be an involution of $C_K(I)$. Let τ fix i points of Ω, say $1, 2, \cdots, i$. By a theorem of Witt [11, Th. 9.4] $\chi(\tau)$ is doubly transitive on $F(\tau)$.

LEMMA 2. $n=i(\beta i-\beta+\gamma)/\gamma$, *where β is the number of involutions with* *the cycle structures* $(1, 2)\cdots$ *which are conjugate to* τ *and* $\gamma=[G_{1,2}: C_G(\tau)\cap G_{1,2}]$ $=2^{2m}-1$.

PROOF. See [4], [5] or [7].

LEMMA 3. (1) $|C_K(I)|=|K|$ *or* $\sqrt{|K|}$ *and every involution of* $C_K(I)$ *is* $C_H(I)$*-conjugate to* τ.

(2) *Every involution of* G *is conjuate to* I *or* $I\tau$.

PROOF. The property (1) is trivial from Lemma 1. Every involution of G is conjugate to an involution of $<K, I>-K$. By (1) every involution of $<I, K>-K$ is $C_H(I)$-conjugate to I or $I\tau$. This proves the lemma.

LEMMA 4. *If* G *has one class of involutions, then* $\beta=[G_{1,2}: C_{G_{1,2}}(I)]$ *ro* 2^{2m}. *If* G *has two classes of involutions, then* $\beta=1$ *and* $\alpha(I)=i$ *or* $\beta=2^{2m}-1$ *and* $\alpha(I\tau)=i$, *and* I *contralizes* $G_{1,2}$.

PROOF. If $C_K(I)\neq K$, then I is conjugate to $I\tau$ by Lemma 1. Therefore if G has two classes of involutions, $C_K(I)=K$, and hence I centralizes $G_{1,2}$ and $|C_{G_{1,2}}(I\tau)|=|\mathrm{K}|$. This proves the lemma.

LEMMA 5. $\chi(\tau)$ *contains a regular normal subgroup, or the following hold*:

$$\chi(\tau)=PSL(3,2),\ i=7,\ |K|=16,\ |C_K(I)|=4,$$
$$\alpha(HK)=\alpha(K)=3 \text{ and } <I, K> \text{ is indecomposable.}$$

PROOF. See [7, Lem. 4].

LEMMA 6. $C_K(I)\neq K$ *if every involution is conjugate to* τ.

PROOF. See [7, Lem. 5].

LEMMA 7. *If* $C_K(I)\neq K$, *then* K *has no orbit of length* 2.

PROOF. See [7, Lem. 6].

3. The case n is odd

If $\chi(\tau)$ contains a regular normal subgroup, then let i be a power of a prime p.

Let $g_1^*(2)$ be the number of involutions in G_1 which fix only the point 1.

LEMMA 8. $g_1^*(2)$ *is the number of involutions with the cycle structure* $(1, 2)\cdots$ which are not conjugate to τ.

PROOF. See [7, Lem. 1].

LEMMA 9. $\alpha(HK)$ *is odd if* G *has two classes of involutions.*

PROOF. See [7, Lem. 8].

LEMMA 10. $\alpha(G_{1,2})$ *is odd if* G *has two classes of involutions.*

PROOF. By Lemma 4 $C_G(I)$ contains $G_{1,2}$. By Lemma 9 $F(<I,HK>)$ contains unique point a. If a is a point of $F(G_{1,2})$, then $\alpha(G_{1,2})$ is odd. Assume a is not a point of $F(G_{1,2})$. Let Δ be an orbit of $G_{1,2}$ containing a. Since I centralizes $G_{1,2}$, $F(I)$ contains Δ. Since HK is a maximal subgroup of $G_{1,2}$, $G_{1,2,a}=HK$ and H fixes two point of Δ. Thus $\alpha(<I, H>)\geq 2$ and $<I, H>$ is isomorphic to a subgroup of $G_{1,2}$. This is a contradiction.

LEMMA 11. $g_1^*(2)=0$.

PROOF. The proof is similar to that of [7, Lem. 9]. Assume $g_1^*(2)\neq 0$. By Lemma 4 I centralizes $G_{1,2}$. By Lemma 10 $\alpha(G_{1,2})$ is odd. Let a be the point of $F(<I, G_{1,2}>)$. Every involution of $<I, G_{1,2}>$ fixes the point a and by Lemma 8 $<I, G_{1,2}>$ contains every involution which fixes only the point a. If $\alpha(I)=1$, then $g_1^*(2)=1$ and G has a regular normal subgroup by Z^*-theorem [3]. Thus $\alpha(I)=i$ and $\alpha(I\tau)=1$. The subgroup generated by all involutions which fix only a is a characteristic subgroup of G_a and it is $<G_{1,2}, I>$. Thus it is half-transitive on $\Omega-\{a\}$. Since $\{1, 2\}$ is an orbit of $<I, G_{1,2}>$, $G_{1,2}$ must be a 2-group. This is a contradiction.

By this lemma it may be assumed that every involution is conjugate to τ. Thus a Sylow 2-subgroup of $C_G(\tau)$ is also that of G.

LEMMA 12. $\chi(\tau)$ contains a regular normal subgroup, $\alpha(\tau)>\alpha(K)$ and K has an orbit of length 2.

PROOF. See [7, Lem. 10~Lem. 12].

Since $C_K(I)\neq K$ by Lemma 6, Lemma 12 contradicts Lemma 7.

4. The case n is even

By Lemma 5 $\chi(\tau)$ contains a regular normal subgroup. By [1] $\chi(\tau)$ is either a group of semi-linear transformations over $GF(q)$, q even, or $PSL(2, q)V$, where V is a 2-dimensional vector space over $GF(q)$.

Case (I). $\alpha(\tau)=\alpha(K)$. Sylow 2-subgroups of G_1 are independent. By [9] G_1 contains a normal subgroup G_1' of odd index such that $G_1'/0(G_1)$ is isomorphic to $PSL(2, 2^m)\cong G_{1,2}$ and $0(G_1)$ is contained in $Z(G_1)$. Thus $G_1'=0(G_1)G_{1,2}$ and $G_{1,2}$ is normal in G_1, which is a contradiction.

Case (II). $\alpha(\tau)>\alpha(K)$.

LEMMA 13. $\chi(\tau)=PSL(2, q)V$.

PROOF. See [7, Lem. 23].

LEMMA 14. $|K|\neq 8$.

PROOF. Assume $|K|=8$. By Lemma 3 I centralizes HK and hence $G_{1,2}$. By Lemma 4 and 6 G has two classes of involutions and $\beta=1$ or

63. Since $\chi(\tau)_1=PSL(2, 4)$, $i=|V|=16$. Since $n=i(\beta(i-1+)\gamma)/\gamma$, $\beta=63$, and $n=16^2$. Thus H is a Sylow 7-subgroup of G. Since $\alpha(I)=0$, $j=\alpha(H)$ is even. By the theorem of Witt $|N_G(H)|=2j(j-1)|H|$. Since $|\chi(H)_{1.2}|=1$ or 2, j is a factor of 16^2 by [4]. Since $j-1$ is a factor of $9(n-1)=3^3\cdot5\cdot17$ and $n-j$ is divisible by 7, $j=4$. Let P be a subgroup of $G_{1.2}$ of order 3. Since I centralizes P, $\alpha(P)=j'$ is even. By the theorem of Witt $|N_G(P)|=2\cdot9j'(j'-1)$ and $j'-1$ is divisible by 3 since a Sylow 3-subgroup of $G_{1.2}$ is cyclic. $|\chi(P)_{1.2}|=1,2$ or 6. By [4] and [6] $j'=6$, 28 or j' is a power of 2. Since $j'-1$ is a factor of $15\cdot17\cdot7$ and $n-j'$ is divisible by 3, $j'=4$ or 16. Let Q be a Sylow 17-subgroup of G_1. If $N_{G_1}(Q)=C_{G_1}(Q)$, it may be assumed by the Frattini argument that Q normalizes K. Since $|N_G(K)|=|KH|\alpha(K)(\alpha(K)-1)$ and $\alpha(K)\leq i$, this is a contradiction. Thus $|N_{G_1}(Q)|$ is even and $|C_{G_1}(Q)|$ is odd. $[G_1: N_{G_1}(Q)]$ is a multiple of $4\cdot7\cdot9$ and a factor of $\cdot4\cdot7\cdot9\cdot15$. This contradicts the theorem of Sylow. This completes the proof.

Since $\chi(\tau)_1=PLS(2, q)$, $C_{G_1}(\tau)$ is nonsolvable. Since G_1 has one class of involutions, so is $G_1/0(G_1)$. By [10] G_1 has a normal subgroup G_1' of odd index such that $G_1'/0(G_1)$ is isomorphic to $PSL(2, 2^m)$. Thus $C_{G_1}(\tau)$ is solvable, which is a contradiction.

Thus the proof of Theorem is complete.

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