On the holomorphically projective correspondence between Kählerian spaces preserving complex structure

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Introduction.

Let (M^n, g) and $(M'^n, g')(n>2)$ be n-dimensional Riemannian spaces with positive definite metric tensors g and g', respectively. The celebrated Beltrami's theorem (see [1][1] §40) of classical differential geometry states that if (M^n, g) is of constant curvature then the space which is in projective correspondence with it is necessarily of constant curvature. N. S. Sinyukov [5] has generalized the above theorem to the non-affine projective correspondence of (M^n, g) with (M'^n, g') which is locally symmetric. Further W. Roter [4] and J. Robinson and J. D. Zund [3] have generalized N. S. Sinyukov's theorem to the case when (M'^n, g') is recurrent.

On the other hand T. Ōtsuki and Y. Tashiro [2] have studied holomorphically projective correspondences between Kählerian spaces. Let (M^n, g, F) and (M'^n, g', F') be Kählerian spaces of real dimension $n(=2m>2)$ with the structures (g, F) and (g', F'), respectively. Then they have proved the generalization of Beltrami's theorem, that is, if (M^n, g, F) is of constant holomorphic sectional curvature then the space which is in holomorphically projective correspondence with it preserving complex structure is necessarily of constant holomorphic sectional curvature.

The purpose of the present paper is to investigate what restrictions the requirement that (M^n, g, F) and (M'^n, g', F') be in holomorphically projective correspondence preserving complex structure imposes on these spaces. An example of such a restriction is given by the above result of T. Ōtsuki and Y. Tashiro.

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§1. Preliminaries.

We shall first give preliminary formulas on the Kählerian space and the holomorphically projective correspondence between Kählerian spaces.

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1) Numbers in brackets refer to the references at the end of the paper.

Let us consider an $n(=2m>2)$ real dimensional Kählerian space with local coordinates $\{x^i\}$. Then the positive definite Riemannian metric g_{ji} and the complex structure $F_i{}^h$ satisfy the following equations

$$F_j{}^a F_a{}^i = -\delta_j{}^i, \qquad g_{ab} F_j{}^a F_i{}^b = g_{ji},$$

$$\nabla_k F_j{}^h = 0, \qquad \nabla_k g_{ji} = 0,$$

where ∇_k denotes the operator of the covariant differentiation with respect to the Christoffel symbols $\begin{Bmatrix} h \\ ji \end{Bmatrix}$ constructed from g_{ji}.

The Riemannian curvature tensor of g_{ji} is defined by

$$R_{kji}{}^h = \partial_k \begin{Bmatrix} h \\ ji \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ ki \end{Bmatrix} + \begin{Bmatrix} h \\ ka \end{Bmatrix} \begin{Bmatrix} a \\ ji \end{Bmatrix} - \begin{Bmatrix} h \\ ja \end{Bmatrix} \begin{Bmatrix} a \\ ki \end{Bmatrix}, \quad \partial_j = \partial/\partial x^j.$$

We put $R_{kjih} = R_{kji}{}^a g_{ah}$, $R_{ji} = R_{aji}{}^a = R_{ajib} g^{ab}$, $R = R_{ab} g^{ab}$ and

$$S_{ji} = F_j{}^a R_{ai},$$

then the following identities are valid

(1. 1)
$$R_{aji}{}^h F_k{}^a = -R_{kai}{}^h F_j{}^a, \qquad R_{kja}{}^h F_i{}^a = R_{kji}{}^a F_a{}^h,$$
$$R_{kjih} = R_{kjab} F_i{}^a F_h{}^b,$$

(1. 2) $\quad R_{ji} = R_{ab} F_j{}^a F_i{}^b, \quad F_j{}^a R_{ai} = -R_{ja} F_i{}^a, \quad F_j{}^a R_a{}^h = R_j{}^a F_a{}^h,$

(1. 3)
$$S_{ji} + S_{ij} = 0, \quad S_{ji} = S_{ab} F_j{}^a F_i{}^b, \quad F_j{}^a S_{ai} = -S_{ja} F_i{}^a = -R_{ji},$$
$$S_{ji} = -\frac{1}{2} F^{ab} R_{jiab} = F^{ab} R_{ajib}.$$

The holomorphically projective curvature tensor $P_{kji}{}^h$ is given by

(1. 4) $\quad P_{kji}{}^h = R_{kji}{}^h + \dfrac{1}{n+2} (R_{ki} \delta_j{}^h - R_{ji} \delta_k{}^h + S_{ki} F_j{}^h - S_{ji} F_k{}^h + 2 S_{kj} F_i{}^h).$

We can obtain easily the following identities

(1. 5)
$$P_{kji}{}^h + P_{jki}{}^h = 0, \qquad P_{kji}{}^h + P_{jik}{}^h + P_{ikj}{}^h = 0, \qquad P_{aji}{}^a = 0,$$
$$P_{kji}{}^a F_a{}^h = P_{kja}{}^h F_i{}^a, \qquad P_{aji}{}^h F_k{}^a = P_{aki}{}^h F_j{}^a,$$

from which we get

(1. 6) $\qquad P_{kja}{}^a = 0, \qquad P_{aji}{}^b F_b{}^a = 0, \qquad P_{kja}{}^b F_b{}^a = 0.$

A necessary and sufficient condition for $P_{kji}{}^h = 0$ is that the space is of constant holomorphic sectional curvature, i. e., a space whose curvature tensor $R_{kji}{}^h$ takes the form

$$R_{kji}{}^h = k(g_{ki} \delta_j{}^h - g_{ji} \delta_k{}^h + F_{ki} F_j{}^h - F_{ji} F_k{}^h + 2 F_{kj} F_i{}^h),$$

where we put

$$k = -\frac{R}{n(n+2)} \cdot$$

Let (M^n, g, F) and (M'^n, g', F') be Kählerian spaces of real dimension $n (=2m>2)$ with the structures (g, F) and (g', F'), respectively. A necessary and sufficient condition for a correspondence f from (M^n, g, F) to (M'^n, g', F') to be holomorphically projective is that there exists a vector field ψ_i defined on (M^n, g, F) such that

$$(1.7) \qquad \left\{{\bar{h}\atop ji}\right\} = \left\{{h\atop ji}\right\} + \psi_j \delta_i{}^h + \psi_i \delta_j{}^h - \bar{\psi}_j F_i{}^h - \bar{\psi}_i F_j{}^h,$$

where $\left\{{\bar{h}\atop ji}\right\}$ is the Christoffel symbols constructed from $\bar{g} = f^* g'$ and $\bar{\psi}_i = \psi_a F_i{}^a$.

From (1.7) we know that ψ_i is necessarily gradient. Especially if $\psi_i = 0$, then the correspondence is said to be affine.

For convenience' sake we put "—" over the quantities induced from the quantities on (M'^n, g', F') by f. From (1.7) we have

$$(1.8) \quad \bar{R}_{kji}{}^h = R_{kji}{}^h + \psi_{ki} \delta_j{}^h - \psi_{ji} \delta_k{}^h - (\psi_{ka} F_j{}^h - \psi_{ja} F_k{}^h) F_i{}^a - (\psi_{ka} F_j{}^a - \psi_{ja} F_k{}^a) F_i{}^h,$$

where ψ_{ji} is a symmetric tensor defined by

$$\psi_{ji} = \nabla_j \psi_i - \psi_j \psi_i + \bar{\psi}_j \bar{\psi}_i .$$

Contracting h and k in (1.8), we get

$$(1.9) \qquad \bar{R}_{ji} = R_{ji} - n \psi_{ji} - 2 F_j{}^a F_i{}^b \psi_{ab} .$$

We assume henceforth that the holomorphically projective correspondence preserves the complex structure, i. e., $f^* F' = \bar{F} = F$. Then we have

$$(1.10) \qquad \bar{F}_i{}^j = F_i{}^j, \qquad \bar{\nabla}_k F_i{}^j = 0,$$

where $\bar{\nabla}_k$ denotes the operator of the covariant differentiation with respect to $\left\{{\bar{h}\atop ji}\right\}$.

On the other hand we know that R_{ji} and $\bar{R}_{ji}$ are hybrid by (1.10), i. e.,

$$R_{ji} = R_{ab} F_j{}^a F_i{}^b, \qquad \bar{R}_{ji} = \bar{R}_{ab} F_j{}^a F_i{}^b.$$

Taking use of this fact, we obtain by means of (1.9)

$$(n-2)(\psi_{ji} - \psi_{ab} F_j{}^a F_i{}^b) = 0,$$

which implies because of $n>2$ that ψ_{ji} is hybrid, i. e.,

$$(1.11) \qquad \psi_{ji} = \psi_{ab} F_j{}^a F_i{}^b .$$

Substituting (1.11) into (1.8) and (1.9) we find

(1. 12) $\bar{R}_{kji}{}^h = R_{kji}{}^h + \phi_{ki}\delta_j{}^h - \phi_{ji}\delta_k{}^h - \tilde{\phi}_{ki}F_j{}^h + \tilde{\phi}_{ji}F_k{}^h - 2\tilde{\phi}_{kj}F_i{}^h ,$

(1. 13) $\bar{R}_{ji} = R_{ji} - (n+2)\phi_{ji} ,$

where $\tilde{\phi}_{ji}$ is a skew-symmetric tensor defined by

$$\tilde{\phi}_{ji} = \phi_{ja}F_i{}^a = \nabla_j\tilde{\phi}_i - \phi_j\tilde{\phi}_i - \tilde{\phi}_j\phi_i .$$

From (1. 11) we have that $\tilde{\phi}_{ji}$ is hybrid, i.e.,

(1. 14) $\tilde{\phi}_{ji} = \tilde{\phi}_{ab}F_j{}^a F_i{}^b .$

By transvecting (1. 13) with $F_k{}^j$ we obtain

(1. 15) $\bar{S}_{ki} = S_{ki} + (n+2)\tilde{\phi}_{ki} ,$

where $\bar{S}_{ki}$ is a skew-symmetric tensor defined by

$$\bar{S}_{ki} = \bar{F}_k{}^a \bar{R}_{ai} = F_k{}^a \bar{R}_{ai} .$$

Kählerian space is said to be recurrent if there exists a vector field K_l satisfying the following identity

$$\nabla_l R_{kji}{}^h = K_l R_{kji}{}^h .$$

In this case, we shall call K_l the recurrent vector. We know that the recurrent vector K_l is locally a gradient one. If $K_l = 0$, then a recurrent Kählerian space is said to be locally symmetric.

§2. A holomorphically projective correspondence preserving complex structure.

In this section we shall prove the following

THEOREM. *Let* $(M^n, g, F)(n>2)$ *be an arbitrary Kählerian space and* (M'^n, g', F') *a recurrent Kählerian space. If there exists a non-affine holomorphically projective correspondence between* (M^n, g, F) *and* (M'^n, g', F') *preserving complex structure,* (M^n, g, F) *and* (M'^n, g', F') *are of constant holomorphic sectional curvature.*

We shall first prove the following

LEMMA. *Let* (M^n, g, F) *and* $(M'^n, g', F')(n>2)$ *be two Kählerian spaces such that there exists a holomorphically projective correspondence between* (M^n, g, F) *and* (M'^n, g', F') *preserving complex structure. If* (M'^n, g', F') *satisfies*

$$\nabla'_m\nabla'_l R'_{ijk}{}^h - \nabla'_l\nabla'_m R'_{ijk}{}^h = 0 ,$$

where ∇'_k *denotes the operator of the covariant differentiation with respect to* $\begin{Bmatrix} h \\ ji \end{Bmatrix}'$ *constructed from* g' *and* $R'_{ijk}{}^h$ *the Riemannian curvature tensor of*

g'_{ji}, *then one of the following two conditions must be satisfied.*

$$R_{ij} - \frac{1}{n} R g_{ij} = 0 .$$

$$\Delta g_{ij} - n \psi_{ij} = 0 ,$$

where $\Delta = g^{ab} \psi_{ab}$.

PROOF. From our assumption we have

(2. 1) $$\bar{V}_m \bar{V}_l \bar{R}_{ijk}{}^h - \bar{V}_l \bar{V}_m \bar{R}_{ijk}{}^h = 0 .$$

By the Ricci's identity we find

$$\bar{R}_{ajk}{}^h \bar{R}_{mli}{}^a + \bar{R}_{iak}{}^h \bar{R}_{mlj}{}^a + \bar{R}_{ija}{}^h \bar{R}_{mlk}{}^a - \bar{R}_{ijk}{}^a \bar{R}_{mla}{}^h = 0 .$$

Using (1. 12) we now replace $\bar{R}_{kji}{}^h$ by $R_{kji}{}^h$, transvect the result with g^{ii} and lower the index h. The symmetric part in the indices m, j and k, h of its result can be written as

(2. 2) $$Q_{hjkm} + Q_{kjhm} = 0 ,$$

where

$$Q_{hjkm} = g_{hj} \psi_{ka} R_m{}^a + g_{km} \psi_{ha} R_j{}^a - R_{hj} \psi_{km}$$
$$- R_{km} \psi_{hj} - F_{hj} \psi_{ka} S_m{}^a - F_{km} \psi_{ha} S_j{}^a + S_{hj} \bar{\psi}_{km} + S_{km} \bar{\psi}_{hj} ,$$

after some complicated calculation.

Contracting i and h in (2. 1), we have

(2. 3) $$\bar{V}_m \bar{V}_l \bar{R}_{jk} - \bar{V}_l \bar{V}_m \bar{R}_{jk} = 0 .$$

By the Ricci's identity we find

$$\bar{R}_{ak} \bar{R}_{mlj}{}^a + \bar{R}_{ja} \bar{R}_{mlk}{}^a = 0 .$$

Using (1. 12) and (1. 13) we replace $\bar{R}_{kji}{}^h$ and $\bar{R}_{kj}$ by $R_{kji}{}^h$ and R_{kj} and transvect the result with g^{jk}. Then we have

(2. 4) $$\psi_{ma} R_l{}^a = \psi_{la} R_m{}^a ,$$

after some calculation.
Transvecting (2. 4) with $F_k{}^l$, we obtain

(2. 5) $$\psi_{ma} S_k{}^a = - \psi_{ka} S_m{}^a .$$

We put $Q_{hjkm} = A_{hjkm} + B_{hjkm}$,
where

$$A_{hjkm} = g_{hj} \psi_{ka} R_m{}^a + g_{km} \psi_{ha} R_j{}^a - R_{hj} \psi_{km} - R_{km} \psi_{hj} ,$$

$$B_{hjkm} = - F_{hj} \psi_{ka} S_m{}^a - F_{km} \psi_{ha} S_j{}^a + S_{hj} \bar{\psi}_{km} + S_{km} \bar{\psi}_{hj} .$$

By (2. 4) and (2. 5) we have

(2. 6)
$$A_{hjkm} = A_{jhkm} = A_{kmhj},$$
$$B_{hjkm} = -B_{jhkm} = B_{kmhj}.$$

By a straight computation, we obtain

$$F_h{}^a F_j{}^b Q_{abkm} = Q_{hjkm}, \qquad F_j{}^a F_k{}^b Q_{habm} = -Q_{hjkm}.$$

From (2. 2) and these equations we get

$$0 = F_h{}^a F_j{}^b (Q_{abkm} + Q_{kbam}) = Q_{hjkm} - Q_{kjhm}.$$

Hence, from (2. 2) we obtain

$$Q_{hjkm} = 0.$$

From (2. 6) we have

$$0 = Q_{hjkm} A^{hjkm} = A_{hjkm} A^{hjkm}.$$

Hence we get

(2. 7) $$A_{hjkm} = g_{hj}\phi_{ka}R_m{}^a + g_{km}\phi_{ha}R_j{}^a - R_{hj}\phi_{km} - R_{km}\phi_{hj} = 0.$$

Transvecting (2. 7) with g^{hj}, we obtain

$$n\phi_{ka}R_m{}^a + g_{km}\phi_{ab}R^{ab} - R\phi_{km} - \Delta R_{km} = 0,$$

which upon further transvection with g^{km} yields

$$n\phi_{ab}R^{ab} = \Delta R.$$

The last two equations enable us to write (2. 7) in the following convenient form

$$\left(R_{hj} - \frac{R}{n}g_{hj}\right)(\Delta g_{km} - n\phi_{km}) + \left(R_{km} - \frac{R}{n}g_{km}\right)(\Delta g_{hj} - n\phi_{hj}) = 0,$$

which clearly means that either

$$R_{ij} - \frac{R}{n}g_{ij} = 0,$$

or

$$\Delta g_{ij} - n\phi_{ij} = 0. \qquad\qquad \text{Q. E. D.}$$

PROOF OF THE THEOREM.

Since (M'^n, g', F') is recurrent, we have by definition

(2. 8) $$\nabla'_l R'_{ijk}{}^h = K'_l R'_{ijk}{}^h.$$

Let us consider the integrability conditions of (2. 8):

$$\nabla'_m \nabla'_l R'_{ijk}{}^h - \nabla'_l \nabla'_m R'_{ijk}{}^h = (\nabla'_m K'_l - \nabla'_l K'_m) R'_{ijk}{}^h + K'_l \nabla'_m R'_{ijk}{}^h - K'_m \nabla'_l R'_{ijk}{}^h.$$

Since K'_i is locally a gradient vector and by (2.8), the right-hand of the last equation vanishes. Hence, by the lemma we have one of the following two cases.

Case A: $\qquad R_{ij} - \dfrac{R}{n} g_{ij} = 0$.

Case B: $\qquad \Delta g_{ij} - n \psi_{ij} = 0$.

Let us consider first the case A. By (2.8) we have

$$\bar{V}_l \bar{R}_{ijk}{}^h = \bar{K}_l \bar{R}_{ijk}{}^h, \qquad \bar{V}_l \bar{R}_{jk} = \bar{K}_l \bar{R}_{jk}, \qquad \bar{V}_l \bar{S}_{ij} = \bar{K}_l \bar{S}_{ij}$$

and hence

$$\bar{V}_l \bar{P}_{ijk}{}^h = \bar{K}_l \bar{P}_{ijk}{}^h.$$

By construction the holomorphically projective curvature tensor is invariant under a holomorphically projective correspondence and hence in the last equation we can replace $\bar{P}_{ijk}{}^h$ by $P_{ijk}{}^h$, i.e.,

$$\bar{V}_l P_{ijk}{}^h = \bar{K}_l P_{ijk}{}^h.$$

This equation may be written out explicity giving

$$V_l P_{ijk}{}^h + \phi_a P_{ijk}{}^a \delta_l{}^h - \bar{\phi}_a P_{ijk}{}^a F_l{}^h - 2\phi_l P_{ijk}{}^h - \phi_i P_{ljk}{}^h - \phi_j P_{ilk}{}^h$$
$$- \phi_k P_{ijl}{}^h + (\bar{\phi}_i P_{ajk}{}^h + \bar{\phi}_j P_{iak}{}^h + \bar{\phi}_k P_{ija}{}^h) F_l{}^a = \bar{K}_l P_{ijk}{}^h.$$

If we lower the index h and symmetrize the result over k and h, we find that

(2.9) $\qquad \phi_a P_{ijk}{}^a g_{lh} + \phi_a P_{ijh}{}^a g_{lk} - \bar{\phi}_a P_{ijk}{}^a F_{lh} - \bar{\phi}_a P_{ijh}{}^a F_{lk} - \phi_k P_{ijlh}$
$$- \phi_h P_{ijlk} + \bar{\phi}_k P_{ijah} F_l{}^a + \bar{\phi}_h P_{ijak} F_l{}^a = 0,$$

since in the case A P_{ijkh} is skew-symmetric with respect to k and h. By transvecting (2.9) with g^{lh} and taking use of (1.5) and (1.6), we find that $\phi_a P_{ijk}{}^a = 0$ and $\bar{\phi}_a P_{ijk}{}^a = 0$ which allow us to reduce (2.9) to

(2.10) $\qquad -\phi_k P_{ijlh} - \phi_h P_{ijlk} + \bar{\phi}_k P_{ijah} F_l{}^a + \bar{\phi}_h P_{ijak} F_l{}^a = 0$.

By transvecting (2.10) with $\phi^k P^{ijlh}$ we find that

$$(\phi_k P_{ijlh})(\phi^k P^{ijlh}) = 0.$$

Since we are considering only non-affine holomorphically projective correspondence, we have clearly $P_{ijlh} = 0$, which implies that (M^n, g, F) is of constant holomorphic sectional curvature. But since the holomorphically projective curvature tensor is invariant under holomorphically projective correspondence, $\bar{P}_{ijk}{}^h = 0$ and hence in the case A, (M'^n, g', F') is also of constant holomorphic sectional curvature.

Consider now the case B, and let

(2.11) $\Pi_{ijk}{}^h = R_{ijk}{}^h + \dfrac{\Delta}{n}(g_{ik}\delta_j{}^h - g_{jk}\delta_i{}^h + F_{ik}F_j{}^h - F_{jk}F_i{}^h + 2F_{ij}F_k{}^h)$.

Using (2.11) we see that

(2.12) $\Pi_{ijk}{}^h = \bar{R}_{ijk}{}^h$, $\bar{V}_l\Pi_{ijk}{}^h = \bar{K}_l\Pi_{ijk}{}^h$.

By a straight computation, we have

$$\Pi_{ijk}{}^a g_{ah} = \Pi_{ijkh} = -\Pi_{jikh} = -\Pi_{ijhk} ,\qquad \Pi_{ijkh} = \Pi_{khij} ,$$

$$\Pi_{ijkh} + \Pi_{jkih} + \Pi_{kijh} = 0 ,$$

$$\Pi_{ajk}{}^a = R_{jk} - (n+2)\phi_{jk} ,\qquad \Pi_{ija}{}^a = 0 ,$$

(2.13)

$$\Pi_{ijk}{}^a F_a{}^h = \Pi_{ija}{}^h F_k{}^a ,\qquad \Pi_{ajk}{}^h F_i{}^a = -\Pi_{iak}{}^h F_j{}^a ,$$

$$\Pi_{ija}{}^b F_b{}^a = 2\{S_{ij} + (n+2)\bar{\phi}_{ij}\} ,\qquad \Pi_{ajk}{}^b F_b{}^a = -\{S_{jk} + (n+2)\bar{\phi}_{jk}\} ,$$

$$F^{ij}\Pi_{ija}{}^b F_b{}^a = R - (n+2)\Delta .$$

From the second equation of (2.12) and (2.13) we get

(2.14)

$$\bar{V}_l\Pi_{ijk}{}^h + \phi_a\Pi_{ijk}{}^a\delta_l{}^h - \bar{\phi}_a\Pi_{ijk}{}^a F_l{}^h - 2\phi_l\Pi_{ijk}{}^h - \phi_i\Pi_{ljk}{}^h - \phi_j\Pi_{ilk}{}^h$$

$$-\phi_k\Pi_{ijl}{}^h + (\bar{\phi}_i\Pi_{ajk}{}^h + \bar{\phi}_j\Pi_{iak}{}^h + \bar{\phi}_k\Pi_{ija}{}^h)F_l{}^a = \bar{K}_l\Pi_{ijk}{}^h .$$

If we lower the index h and symmetrize the result over k and h, taking use of the first equation of (2.13) we get

(2.15)

$$\phi_a\Pi_{ijk}{}^a g_{lh} + \phi_a\Pi_{ijh}{}^a g_{lk} - \bar{\phi}_a\Pi_{ijk}{}^a F_{lh} - \bar{\phi}_a\Pi_{ijh}{}^a F_{lk}$$

$$-\phi_k\Pi_{ijlh} - \phi_h\Pi_{ijlk} + \bar{\phi}_k\Pi_{ijah}F_l{}^a + \bar{\phi}_h\Pi_{ijak}F_l{}^a = 0 .$$

By transvecting (2.15) with g^{lh} and taking use of (2.13), we obtain

(2.16) $n\phi_a\Pi_{ijk}{}^a + \bar{\phi}_k\Pi_{ija}{}^b F_b{}^a = 0$.

Transvecting (2.16) with $\phi_c\Pi^{ijkc}$ we have

$$n(\phi_a\Pi_{ijk}{}^a)(\phi_c\Pi^{ijkc}) + \phi_c\Pi_{ijk}{}^c \bar{\phi}^k\Pi^{ij}{}_a{}^b F_b{}^a = 0 .$$

If we can show that $\phi_c\Pi_{ijk}{}^c\bar{\phi}^k\Pi^{ij}{}_a{}^b F_b{}^a = 0$, then we have $\phi_a\Pi_{ijk}{}^a = 0$ and $\bar{\phi}_a\Pi_{ijk}{}^a = 0$ which allow us to reduce (2.15) to

$$-\phi_k\Pi_{ijlh} - \phi_h\Pi_{ijlk} + \bar{\phi}_k\Pi_{ijah}F_l{}^a + \bar{\phi}_h\Pi_{ijak}F_l{}^a = 0 .$$

Then we can repeat the argument which was previously used in the case A to deduce that $\Pi_{ijkh} = 0$. Hence, from the first equation of (2.12) this means that (M'^n, g', F') is flat Kählerian space and by (2.11) we see that (M^n, g, F) is of constant holomorphic sectional curvature. By (2.11) in this case the holomorphically projective correspondence preserving complex structure must satisfy $\Delta = $ constant.

Now we shall prove that $\phi_c\Pi_{ijk}{}^c\bar{\phi}^k\Pi^{ij}{}_a{}^b F_b{}^a = 0$. Transvecting (2.16)

with F^{ij} and taking use of (2.13), we obtain

(2.17) $\qquad n\phi_a\left\{S_k{}^a+(n+2)\bar{\phi}_k{}^a\right\}-\bar{\phi}_k\left\{R-(n+2)\varDelta\right\}=0$.

Transvecting (2.17) with $\bar{\phi}^k$ and taking use of $\bar{\phi}^a\bar{\phi}_a=\phi^a\phi_a$, we have

(2.18) $\qquad n\bar{\phi}^k\phi_a\left\{S_k{}^a+(n+2)\bar{\phi}_k{}^a\right\}-\phi^a\phi_a\left\{R-(n+2)\varDelta\right\}=0$.

Substituting (2.16) into $-\dfrac{n}{2}\phi_a(\Pi_{ijk}{}^a+\Pi_{jki}{}^a+\Pi_{kij}{}^a)=0$ and transvecting the result with $F^{ij}\bar{\phi}^k$, we obtain

(2.19) $\qquad \phi^a\phi_a\left\{R-(n+2)\varDelta\right\}-2\bar{\phi}^k\phi_a\left\{S_k{}^a+(n+2)\bar{\phi}_k{}^a\right\}=0$.

From (2.18) and (2.19) we have

$$(n-2)\bar{\phi}^k\phi_a\left\{S_k{}^a+(n+2)\bar{\phi}_k{}^a\right\}=0 ,$$

which implies because of $n>2$ that

$$\bar{\phi}^k\phi_a\left\{S_k{}^a+(n+2)\bar{\phi}_k{}^a\right\}=0 .$$

Therefore, from the last equation, (2.18) and $\phi_a\neq0$ we obtain

(2.20) $\qquad\qquad\qquad R-(n+2)\varDelta=0$,

which implies the following relations by (2.17)

(2.21) $\quad\begin{aligned}&\phi_a\left\{S_k{}^a+(n+2)\bar{\phi}_k{}^a\right\}=0 , \quad \phi_a\left\{R_j{}^a-(n+2)\phi_j{}^a\right\}=0 ,\\ &\bar{\phi}_a\left\{S_k{}^a+(n+2)\bar{\phi}_k{}^a\right\}=0 , \quad \bar{\phi}_a\left\{R_j{}^a-(n+2)\phi_j{}^a\right\}=0 .\end{aligned}$

Contracting l and h and transvecting with g^{jk} in (2.14) and taking use of (2.20) and (2.21), we obtain

(2.22) $\quad\begin{aligned}&\nabla_a\left\{R_i{}^a-(n+2)\phi_i{}^a\right\} = \bar{K}_a\left\{R_i{}^a-(n+2)\phi_i{}^a\right\} ,\\ &\nabla_a\left\{S_j{}^a+(n+2)\bar{\phi}_j{}^a\right\} = \bar{K}_a\left\{S_j{}^a+(n+2)\bar{\phi}_j{}^a\right\} .\end{aligned}$

On the other hand, operating ∇^i to the third equation of (2.21) and taking use of (2.21) and (2.22), we have

(2.23) $\qquad\qquad \bar{\phi}^{ab}\left\{S_{ab}+(n+2)\bar{\phi}_{ab}\right\}=0$.

From (2.3) we have

$$\nabla_m\nabla_l\bar{S}_{ij}-\nabla_l\nabla_m\bar{S}_{ij}=0 .$$

By the Ricci's identity we get

$$\bar{S}_{aj}\bar{R}_{mli}{}^a+\bar{S}_{ia}\bar{R}_{mlj}{}^a=0 .$$

Using (1.12) and (1.15) we now replace $\bar{R}_{kji}{}^h$ and $\bar{S}_{ji}$ by $R_{kji}{}^h$ and S_{ji} and transvect the result with $g^{ti}\bar{\varphi}^m\varphi^j$, then by (2.20) and (2.21) we have

$$(2.24) \qquad \phi_j\bar{\varphi}^m R_{alm}{}^j\left\{S^{al}+(n+2)\bar{\varphi}^{al}\right\}=0.$$

Hence, by (2.11), (2.21), (2.23) and (2.24) we get

$$\phi_c\Pi_{ijk}{}^c\bar{\varphi}^k\Pi^{ij}{}_a{}^b F_b{}^a=0. \qquad\qquad \text{Q. E. D.}$$

The following corollary is trivial from our theorem.

COROLLARY. *Let* (M^n,g,F) $(n>2)$ *be an arbitrary Kählerian space and* (M'^n,g',F') *a locally symmetric Kählerian space. If there exists a non-affine holomorphically projective correspondence between* (M^n,g,F) *and* (M'^n, g',F') *preserving complex structure,* (M^n,g,F) *and* (M'^n,g',F') *are of constant holomorphic sectional curvature.*

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