On a certain subspace of the Riemannian projective recurrent space

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§ 0. Introduction

Riemannian spaces which admit some recurrent tensors have been studied by many authors. Recently, T. Miyazawa and Gorō Chūman [1] have studied the subspaces of a Riemannian recurrent space. In this paper, we would like to further study the subspaces of the Riemannian projective recurrent spaces.

The Riemannian space V_m may be called a projective recurrent space if Weyl's projective curvature tensor

$$P_{kji}{}^h = \bar{R}_{kji}{}^h - \frac{1}{m-1}(\bar{R}_{ji}\delta_k^h - \bar{R}_{ki}\delta_j^h) \tag{0.1}$$

satisfies the relation

$$\nabla_l P_{kji}{}^h = K_l P_{kji}{}^h, \tag{0.2}$$

where ∇_l denotes a covariant differentiation with respect to the metric tensor g_{ij} of the V_m. We will call K_l in (0.2) the vector of recurrence of the space.

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§ 1. Preliminary

Let us consider an n-dimensional subspace V_n, of local coordinate y^a, immersed in an m-dimensional Riemannian space V_m of local coordinate x^i. Let $B_a^i = \partial x^i / \partial y_a$, then the rank of the matrix (B_a^i) is n, where the indices $h, i, j, \cdots,$ take the values $1, \cdots, m$ and the indices $a, b, c. \cdots,$ the values $1, \cdots, n\,(m>n)$. We have the components g_{ab} of the fundamental tensor for V_n given by the relation $g_{ab} = B_a^i B_b^j g_{ij}$, g_{ij} being the components of the fundamental tensor for V_m.

Let $N_P\,(P=n+1, \cdots, m)$ be unit normals to the V_m and mutually orthogonal, then we have the relations

$$g_{ij} N_P^i N_P^j = e_P, \quad g_{ij} N_P^i N_Q^j = 0\,(P \neq Q), \quad g_{ij} B_a^i N_P^j = 0, \tag{1.1}$$

where e_P is an indicator.

The Euler-Schouten's curvature tensor $H_{ab}{}^{i}$ of the V_n is defined by

$$H_{ab}{}^{i} = \nabla_a B_b{}^{i},$$

where ∇_a denotes a covariant differentiation with respect to the fundamental tensor g_{ab} of the V_n. If we put

$$H_{ab}{}^{i} = \sum_P e_P H_{abP} N_P{}^{i}, \tag{1.2}$$

then the second fundamental tensor H_{abP} for $N_P{}^{i}$ is given by

$$H_{abP} = H_{ab}{}^{i} N_{Pi}. \tag{1.3}$$

Therefore (1.2) can be rewritten as

$$H_{ab}{}^{i} = \sum_P e_P H_{ab}{}^{j} N_{Pj} N_P{}^{i}.$$

The Gauss and Codazzi equation for the V_n can be written in the following forms respectively:

$$R_{abcd} = \bar{R}_{ijkl} B_a{}^{i} B_b{}^{j} B_c{}^{k} B_d{}^{l} + \sum_P e_P (H_{bcP} H_{adP} - H_{acP} H_{bdP}), \tag{1.4}$$

$$\bar{R}_{ijkl} B_a{}^{i} N_P{}^{j} B_b{}^{k} B_c{}^{l} = \nabla_b H_{acP} - \nabla_c H_{abP} + \sum_Q e_Q (L_{PQc} H_{abQ} - L_{PQb} H_{acQ}), \tag{1.5}$$

where we put

$$L_{QPa} = \nabla_a N_{Qi} N_P{}^{i} \ (= -L_{PQa}). \tag{1.6}$$

§ 2. Reviews of the known results

We have studied a Riemannian space $V_m (m>2)$ satisfying

$$\nabla_l W_{kji}{}^{h} = K_l W_{kji}{}^{h} \tag{2.1}$$

for a non-zero vector K_l, where $W_{kji}{}^{h}$ is the so-called concircular tensor given by K. Yano [2] as follows:

$$W_{kji}{}^{h} = \bar{R}_{kji}{}^{h} - \frac{1}{m(m-1)} \bar{R} (g_{ji} \delta_k{}^{h} - g_{ki} \delta_j{}^{h}). \tag{2.2}$$

For brevity, we denote by CCK$_m$-space a Riemannian space defined by (2.1).

We shall denote the following results that are necessary to prove our theorems.

LEMMA 1. (T. Miyazawa [3])

A CCK$_m$-space is a projective recurrent space.

LEMMA 2. (T. Miyazawa [3])

A projective recurrent space is a CCK$_m$-space.

§ 3. A totally umbilical surface immersed in a projective recurrent space

From lemma 1 and lemma 2 we find that a CCK_m-space is equal to a projective recurrent space. We assume that a V_m is a Riemannian projective recurrent space, that is, CCK_m^*-space. If $H_{ab}{}^i$ satisfies the following relation:

$$H_{ab}{}^i = g_{ab} H^i, \tag{3.1}$$

where H^i is called the mean curvature vector and satisfies

$$H^i = \frac{1}{n} g^{ab} H_{ab}{}^i, \tag{3.2}$$

then the V_n is called a totally umbilical surface. We assume that the subspace V_n immersed in the V_m is totally umbilical.

Substituting (3.1) into (1.3), we have

$$H_{abP} = g_{ab} H^i N_{Pi}. \tag{3.3}$$

Putting $H^i N_{Pi} = \rho_P$, (3.3), (3.2) and (3.1) can be rewritten respectively as:

$$H_{abP} = \rho_P g_{ab}, \tag{3.4}$$

$$H^i = \sum_P e_P \rho_P N_P{}^i, \tag{3.5}$$

$$H_{ab}{}^i = \sum_P e_P \rho_P N_P{}^i g_{ab}. \tag{3.6}$$

Using (1.1) and (3.5), we have

$$H_i H^i = \sum_P e_P \rho_P^2. \tag{3.7}$$

Hereafter, for brevity, we will put $H^2 = \sum_P e_P \rho_P^2$. Then the mean curvature H is written as $H^2 = |H_i H^i|$.

Substituting (3.4) into (1.4), we have

$$R_{abcd} = \bar{R}_{ijkl} B_a{}^i B_b{}^j B_c{}^k B_d{}^l + H_i H^i (g_{bc} g_{ad} - g_{ac} g_{bd}). \tag{3.8}$$

Differentiating (3.4) covariantly with respect to y^c, substituting its result and (3.4) into (1.5), we have

$$\bar{R}_{ijkl} B_a{}^i N_P{}^j B_b{}^k B_c{}^l = g_{ac} \nabla_b \rho_P - g_{ab} \nabla_c \rho_P + \sum_Q e_Q \rho_Q (L_{PQc} g_{ab} - L_{PQb} g_{ac}). \tag{3.9}$$

Furthermore, differentiating (3.8) covariantly with respect to y^f and using (1.6), (3.8), (3.9) and (2.1),

$$\begin{aligned} \nabla_f R_{abcd} = {} & K_m B_f{}^m \left[R_{abcd} - H_i H^i (g_{bc} g_{ad} - g_{ac} g_{bd}) \right] \\ & + \frac{1}{m(m-1)} (B_f{}^m \nabla_m \bar{R} - B_f{}^m K_m \bar{R}) (g_{bc} g_{ad} - g_{ac} g_{bd}) \end{aligned} \tag{3.10}$$

$$+\nabla_f(H_iH^i)(g_{bc}g_{ad}-g_{ac}g_{bd})$$
$$+\frac{1}{2}\Big[\nabla_a(H_iH^i)(g_{bc}g_{fd}-g_{bd}g_{fc})+\nabla_b(H_iH^i)(g_{da}g_{fc}-g_{ca}g_{fd})$$
$$+\nabla_c(H_iH^i)(g_{ad}g_{fb}-g_{bd}g_{fa})+\nabla_d(H_iH^i)(g_{bc}g_{fa}-g_{ac}g_{fd})\Big].$$

We assume that the mean curvature is a constant ($\neq 0$), then we have

$$\nabla_f R_{abcd}=K_m B_f^m\big[R_{abcd}-H_iH^i(g_{bc}g_{ad}-g_{ac}g_{bd})\big]$$
$$+\frac{1}{m(m-1)}(B_f^m\nabla_m\bar{R}-B_f^mK_m\bar{R})(g_{bc}g_{ad}-g_{ac}g_{bd}). \tag{3.11}$$

Contracting (3.11) with g^{bc}, we get

$$\nabla_f R_{ad}=K_mB_f^m\big[R_{ad}-(n-1)H_iH^ig_{ad}\big]$$
$$+\frac{1}{m(m-1)}(B_f^m\nabla_m\bar{R}-B_f^mK_m\bar{R})g_{ad}. \tag{3.12}$$

Transvecting (3.12) with g^{ad}, we have

$$\nabla_f R=K_mB_f^m\big[R-n(n-1)H_iH^i\big]$$
$$+\frac{n(n-1)}{m(m-1)}(B_f^m\nabla_m\bar{R}-B_f^mK_m\bar{R}). \tag{3.13}$$

From the above equations, we can consider the following two cases:

$$\text{(A)}\quad K_mB_f^m=K_f\neq 0, \qquad \text{(B)}\quad K_mB_f^m=0.$$

The case of (B) means that the recurrence vector K_m is orthogonal to the V_n immersed in the V_m.

§ 4. The subspace with non-orthogonal recurrence vector to the V_n.

In this section, let us consider that the recurrence vector is not orthogonal to the V_n. First we shall prove the following theorem.

THEOREM 4.1. *Let V_n be a totally umbilical surface immersed in a projective recurrent space and let the recurrence vector be not orthogonal to the V_n. If the mean curvature is a constant* ($\neq 0, n\geqq 3$), *then the V_n is a projective recurrent space.*

PROOF. Substituting (A) into (3.11) and (3.12), we have

$$\nabla_f R_{abcd}=K_f\big[R_{abcd}-H_iH^i(g_{bc}g_{ad}-g_{ac}g_{bd})\big]$$
$$+\frac{1}{m(m-1)}(\nabla_f\bar{R}-K_f\bar{R})(g_{bc}g_{ad}-g_{ac}g_{bd}), \tag{4.1}$$

$$(4.2)\qquad \nabla_f R_{ad}=K_f\left[R_{ad}-(n-1)H_iH^i g_{ad}\right]+\frac{n-1}{m(m-1)}(\nabla_f\bar{R}-K_f\bar{R})\,g_{ad}\,,$$

from which we have

$$K_fH_iH^i g_{ad}=\frac{1}{n-1}(K_fR_{ad}-\nabla_f R_{ad})+\frac{1}{m(m-1)}(\nabla_f\bar{R}-K_f\bar{R})\,g_{ad}\,.$$

Substituting this equation into (4.1), we find

$$\nabla_f R_{abcd}-\frac{1}{n-1}(\nabla_f R_{ad}g_{bc}-\nabla_f R_{ac}g_{bd})$$
$$=K_f\left[R_{abcd}-\frac{1}{n-1}(R_{ad}g_{bc}-R_{ac}g_{bd})\right],$$

that is, $\nabla_f P_{abcd}=K_f P_{abcd}$. This completes the proof.

The following lemma is well known [4]:

LEMMA 3. (M. Matsumoto [4]) *In a projective recurrent space a recurrence vector K_l is gradient.*

From this lemma, after easy calculation, we have

LEMMA 4. *The vector K_f defined by (A) is gradient.*

THEOREM 4.2. *Let V_n be a totally umbilical surface immersed in a projective recurrent space and let the recurrence vector be not orthogonal to the V_n. If the mean curvature is constant ($\neq 0, n\geqq 3$), then V_n is an Einstein space, or a recurrent space.*

PROOF. Substituting (A) into (3.13), we have

$$(4.3)\qquad \nabla_f R=K_f\left[R-n(n-1)H_iH^i\right]+\frac{n(n-1)}{m(m-1)}(\nabla_f\bar{R}-K_f\bar{R})\,.$$

From (4.3), we get

$$(4.4)\qquad \nabla_f\bar{R}-K_f\bar{R}=\frac{m(m-1)}{n(n-1)}(\nabla_f R-K_fR)+m(m-1)K_fH_iH^i\,.$$

Substituting (4.4) into (4.1) and (4.2), we have

$$(4.5)\qquad \nabla_f R_{abcd}=K_fR_{abcd}+\frac{1}{n(n-1)}(\nabla_f R-K_fR)(g_{bc}g_{ad}-g_{ac}g_{bd}),$$

$$(4.6)\qquad \nabla_f R_{ad}=K_fR_{ad}+\frac{1}{n}(\nabla_f R-K_fR)\,g_{ad}\,.$$

Differentiating (4.6) covariantly with respect to y^e, we have

$$\nabla_e\nabla_f R_{ad}=\nabla_e K_f R_{ad}$$
$$+K_fK_eR_{ad}-\frac{1}{n}K_fK_eRg_{ad}-\frac{1}{n}\nabla_eK_fRg_{ad}+\frac{1}{n}\nabla_e\nabla_f Rg_{ad}\,.$$

Exchanging the indices e and f and using the lemma 4, and subtracting the equation obtained from the last result, we get $\nabla_f \nabla_e R_{ad} - \nabla_e \nabla_f R_{ad} = 0$. Applying Ricci's identity to the left hand side of the last equation, we have $R_{bd} R_{fea}{}^b + R_{ab} R_{fed}{}^b = 0$. Differentiating this equation covariantly with respect to y^c and substituting (4. 5) and (4. 6) into its equation, we have

$$(4.7) \qquad (\nabla_c R - K_c R)(R_{fd} g_{ea} - R_{ed} g_{fa} + R_{af} g_{ed} - R_{ae} g_{fd}) = 0 .$$

Transvecting (4. 7) with g_{fd}, we have $(\nabla_c R - K_c R)(R g_{ae} - n R_{ae}) = 0$. It follows that $\nabla_c R - K_c R = 0$, or $R g_{ae} = n R_{ae}$. If the former equation holds, then V_n is a recurrent space according to (4. 5). If the latter equation holds, then V_n is an Einstein space. This completes the proof.

COROLLARY 1. *Let V_n be a totally geodesic surface immersed in a projective recurrent space and let the recurrence vector be not orthogonal to the V_n. Then V_n is a recurrent space, or an Einstein space.*

COROLLARY 2. *Let V_n be a totally geodesic surface immersed in a projective recurrent space and let the recurrence vector be not orthogonal to the V_n, and V_n be not an Einstein space. Then V_m is a recurrent space.*

§ 5. The subspace with orthogonal recurrence vector to the V_n

In this section, let us consider thst the recurrence vector is orthogonal to the V_n.

THEOREM 5. 1. *Let V_n be a totally umbilical surface immersed in a projective recurrent space and let the recurrence vector be orthogonal to the V_n. If the mean curvature is a constant ($\neq 0, n \geqq 3$), then V_n is symmetric in the sense of Cartan.*

PROOF. From (4. 1) and (4. 3), we have

$$(5.1) \qquad \nabla_f R_{abcd} = \frac{1}{m(m-1)} \nabla_f \bar{R} (g_{bc} g_{ad} - g_{ac} g_{bd}) ,$$

$$(5.2) \qquad \nabla_f R = \frac{n(n-1)}{m(m-1)} \nabla_f \bar{R} , \qquad \nabla_f \bar{R} = \frac{m(m-1)}{n(n-1)} \nabla_f R .$$

Substituting (5. 2) into (5. 1), we have

$$(5.3) \qquad \nabla_f R_{abcd} - \frac{1}{n(n-1)} \nabla_f R (g_{bc} g_{ad} - g_{ac} g_{bd}) = 0 .$$

The contraction with respect to g^{ad} in (5. 3) gives $\nabla_f R_{bc} - \frac{1}{n} \nabla_f R g_{bc} = 0$. Transvecting this equation with g_{ac}, we get $\nabla_b R = 0$, that is, R=constant.

Therefore, from (5. 3) we find $\nabla_f R_{abcd}=0$. This completes the proof.

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