Tensor products of linear operators and the method of separation of variables

Dedicated to Professor Minoru Kurita on his 60th birthday

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Introduction

Many of familiar Banach spaces have natural tensor product representations $X\hat{\otimes}_\alpha Y$ with suitable Banach spaces X and Y and a suitable uniform resonable norm α. Consider linear operators with domains in such a Banach space $X\hat{\otimes}_\beta Y$ with a norm β and ranges in $X\hat{\otimes}_\alpha Y$ with another norm α, the set of which will be denoted by $\mathfrak{L}(X\hat{\otimes}_\beta Y, X\hat{\otimes}_\alpha Y)$ and if $\alpha=\beta$, by $\mathfrak{L}(X\hat{\otimes}_\alpha Y)$. Some of them are represented as "polynomials" of suitable, densely defined closed linear operators A in X and B in Y: if $P(\xi, \eta)=\sum_{jk}c_{jk}\xi^j\cdot\eta^k$,

$$P\{A\otimes I, I\otimes B\}\equiv\sum_{jk}c_{jk}A^j\otimes B^k. \tag{0.1}$$

In this case, they are often expected to inherit some properties from the operators A and B.

The aim of the present paper is to investigate, as one of such properties, their invertibility which will be derived from the properties of A and B. In this respect, our approach may give a meaning to the method of separation of variables (cf. [7]).

It has been shown by the author [16] that if $\alpha=\beta$, for a class of polynomials the spectral mapping theorem holds:

$$P\big(\sigma(A), \sigma(B)\big)=\sigma\big(P\{A\otimes I, I\otimes B\}\big)=\sigma\big(\tilde{P}\{A\otimes I, I\otimes B\}\big), \tag{0.2}$$

where $\tilde{P}\{A\otimes I, I\otimes B\}$ is a maximal extension of $P\{A\otimes I, I\otimes B\}$ in $X\hat{\otimes}_\alpha Y$. This has extended in particular the results of Ju. M. Berezanskiĭ [3] and L. and K. Maurin [23] for A and B selfadjoint, A. Brown and C. Pearcy [4], M. Schechter [28] for A and B bounded, and the author [14], [15] for A and B not necessarily bounded. Another investigation has been made by M. Reed and B. Simon [26].

The results have been applied to the operators $A\otimes I+I\otimes B$ in generalizing in [16] the result of V. P. Mihaĭlov [25] on the first boundary value problem for quasi-elliptic differential equations and to the spectral theory of many-body Schrödinger operators ([1], [29]).

In the present paper, we study, for one of A and B, say B, being a scalar type spectral operator in the sense of N. Dunford and W. G. Bade (see [6, Part III]), the invertibility of the operators (0. 1) and their maximal extensions (in fact, their closures), considered as operators in (a) $\mathfrak{L}(X\hat{\otimes}_{\pi}L)$, (b) $\mathfrak{L}(X\hat{\otimes}_{\epsilon}C)$, (c) $\mathfrak{L}(H_1\hat{\otimes}_{\alpha_0}H_2)$ and (d) $\mathfrak{L}(X\hat{\otimes}_{\epsilon}Y, X\hat{\otimes}_{\pi}Y)$, where L(resp. C) is an $\mathscr{L}_1$-space (resp. $\mathscr{L}_{\infty}$-space) of J. Lindenstrauss and A. Pełczyński [21], and H_i are Hilbert spaces. In particular, in the first three cases, we establish the spectral mapping theorem (0. 2) with $P(\sigma(A), \sigma(B))$ replaced by its closure, for a larger class of polynomials as that in [16]. In this context, an attempt was already made for some special cases in [17]. We present here a rather comprehensive study.

Applied to the operator $A\otimes I+I\otimes B$, our theory involves the partial differential equations of not only elliptic and parabolic but also hyperbolic type.

Here it should be noted that in the first three cases as above, if both A and B are scalar type spectral operators, then for any polynomials the spectral mapping theorem (0. 2) holds with $P(\sigma(A), \sigma(B))$ replaced by its closure.

In Section 1, there are summarized some definitions and useful results on linear operators, $\mathscr{L}_p$-spaces of J. Lindenstrauss and A. Pełczyński and reasonable norms on tensor products of Banach spaces. Section 2 is concerned with the Gowurin property of the bounded Boolean algebra of projections; this concept is employed to guarantee the existence of a certain integral with respect to the spectral measure. In Section 3, we introduce the polynomial operators to study their invertibility and in particular, to establish the spectral mapping theorem. The present class of admissible polynomials is strictly larger than that in [16], though the operators and the spaces are more restricted. The results are also restated for $A\otimes I+I\otimes B$. In Section 4, our results are applied to the initial value problem for an abstract wave equation such as was considered by L. Gårding and J. Leray for a strongly hyperbolic differential equation in a strip.

The author should like to express his hearty thanks to Professor Gårding for bringing his attention to their unpublished result.

1. Definitions and Preliminaries

In this section, we collect some definitions and basic facts on linear operators, $\mathscr{L}_p$-spaces and tensor products.

Linear operators. If Z and Z_1 are Banach spaces, $\mathfrak{L}(Z, Z_1)$ denotes the set of all linear operators T: $D[T]\subset Z\rightarrow Z_1$ with domain $D[T]$ in Z and range in Z_1, and $\mathfrak{L}(Z)=\mathfrak{L}(Z, Z)$. By $L(Z, Z_1)$ we denote the Banach space of all continuous linear operators of Z into Z_1, and $L(Z)=L(Z, Z)$. Every densely defined linear operator T: $D[T]\subset Z\rightarrow Z_1$ has a maximal extension $\tilde{T}$. The domain $D[\tilde{T}]$ of $\tilde{T}$ is the projection of the closure of the graph $G(T)\subset Z\times Z_1$ into Z ([20], [14]). The spectrum of a linear operator T: $D[T]\subset Z\rightarrow Z$ is denoted by $\sigma(T)$. The spectrum is unchanged under maximal extensions, in particular, under the closure operation [14]. The resolvent set of T is denoted by $\rho(T)$.

The $\mathscr{L}_p$-spaces. J. Lindenstrauss and A. Pełczyński [21] have introduced the $\mathscr{L}_p$-spaces.

For $1\leq p\leq\infty$ and a positive integer n, we denote by l_p^n the space of all $s=(s_1, s_2, \cdots, s_n)\in \boldsymbol{C}^n$ with norm $\|s\|=(\sum_i|s_i|^p)^{1/p}$ if $p<\infty$ and $\|s\|=\max|s_i|$ if $p=\infty$. For two isomorphic Banach spaces E and F, so that there exists an invertible continuous linear operator T of E onto F, their distance $d(E, F)$ is defined by inf $(\|T\|\,\|T^{-1}\|)$, where the infimum is taken over all invertible continuous linear operators T of E onto F.

A Banach space Z is called an *$\mathscr{L}_{p,\lambda}$-space* $(1\leq p\leq\infty,\ 1\leq\lambda<\infty)$ if for each finite-dimensional subspace $F\subset Z$ there exists a finite-dimensional subspace G with $F\subset G\subset Z$ such that $d(G, l_p^n)\leq\lambda$, where $n=\dim G$. A Banach space Z is called an *$\mathscr{L}_p$-space* $(1\leq p\leq\infty)$ if it is an $\mathscr{L}_{p,\lambda}$-space for some λ.

It is known ([21], [22]) that the space $L_p(\mu)$ of all (equivalence classes of) measurable functions on some measure space whose p-th power is integrable (resp. essentially bounded if $p=\infty$) is an $\mathscr{L}_p$-space, and the space $C(K)$ of all continuous functions on the compact Hausdorff space K is an $\mathscr{L}_\infty$-space. The L_p-spaces have many properties in common with the spaces $L_p(\mu)$, although unless $p=2$, there are $\mathscr{L}_p$-spaces which are not isomorphic to $L_p(\mu)$. If $p=2$, every $\mathscr{L}_2$-space is isomorphic to a Hilbert space. Every Banach space Z is an $\mathscr{L}_p$-space $(1\leq p\leq\infty)$ if and only if its dual space Z' is an $\mathscr{L}_{p'}$-space with $1/p+1/p'=1$. It is also known [18] that a separable $\mathscr{L}_p$-space $(1\leq p\leq\infty)$ has a Schauder basis.

In the present paper, we shall only concern with the cases $p=1$ and ∞. From the above mentioned, Banach spaces of type (L) and (C) in the sense of A. Grothendieck [12] are an $\mathscr{L}_1$-space and an $\mathscr{L}_\infty$-space, respectively, but this converse is not always true. However, throughout, we shall employ the symbols L and C to express an $\mathscr{L}_1$-space and an $\mathscr{L}_\infty$-space, respectively. By the way, H and H_i, $i=1, 2$, denote Hilbert spaces.

By a bounded Boolean algebra $\mathcal{E}$ of projections in a Banach space Z, we mean a Boolean algebra of commuting continuous projections E in Z

such that for some constant h one has $\sup \|E\| < h$ for all $E \in \mathcal{E}$ (see [6]).

The following result is due to J. Lindenstrauss and A. Pełczyński ([21], cf. [24]).

THEOREM 1.1. *Let Z be an $\mathcal{L}_1$-space (resp. $\mathcal{L}_\infty$-space), in particular, a Banach space of type (L) (resp. type (C)), and $\mathcal{E}$ a bounded Boolean algebra of projections in Z. Then there exists a constant k_0 such that for every finite collection $\{E_i\}_{i=1}^s \subset \mathcal{E}$ of disjoint projections one has*

$$\sum_{i=1}^s \|E_i z\| \le k_0 \|\sum_{i=1}^s E_i z\|, \quad z \in Z$$
$$(resp.\ \ \|\sum_{i=1}^s E_i z\| \le k_0 \max_{1 \le i \le s} \|E_i z\|, \quad z \in Z).$$

Tensor products. Let X and Y be complex Banach spaces, and X', Y' their topological dual spaces. Let $X \otimes Y$ be the algebraic tensor product of X and Y. On $X \otimes Y$ are defined suitable norms (see [11], [12], [27]).

A norm α on $X \otimes Y$ is said to be *reasonable* if it satisfies $\|x \otimes y\|_\alpha = \|x\| \|y\|$ for $(x, y) \in X \times Y$ and if the dual norm α' induced on $X' \otimes Y'$ by the topological dual of the space $X \otimes Y$ equipped with the norm α satisfies $\|x' \otimes y'\|_{\alpha'} = \|x'\| \|y'\|$ for $(x', y') \in X' \times Y'$. The dual norm α' of a reasonable norm α on $X \otimes Y$ is reasonable on $X' \otimes Y'$. The completion of $X \otimes Y$ (resp. $X' \otimes Y'$) with respect to the norm α (resp. α') is denoted by $X \hat{\otimes}_\alpha Y$ (resp. $X' \hat{\otimes}_{\alpha'} Y'$).

There exists on $X \otimes Y$ the greatest reasonable norm π and the smallest one ε. If $u \in X \otimes Y$, they are defined as follows:

$$\|u\|_\pi = \inf \sum_j \|x_j\| \|y_j\|,$$

where the infimum is taken over all representations of the form of finite sum $u = \sum_j x_j \otimes y_j$;

$$\|u\|_\varepsilon = \sup\left\{|\langle u, x' \otimes y' \rangle| \,;\ (x', y') \in X' \times Y',\ \|x'\| \le 1,\ \|y'\| \le 1\right\}.$$

A norm α is said to be *uniform* if one has $\|(A \otimes B)u\|_\alpha \le \|A\| \|B\| \|u\|_\alpha$ for all $u \in X \otimes Y$, whenever $(A, B) \in L(X) \times L(Y)$. The norms π and ε are uniform.

When both X and Y are Hilbert spaces, it is possible to equip $X \otimes Y$ with a pre-Hilbert space structure such that for $x_i \in X$ and $y_i \in Y$, $i = 1, 2$,

$$(x_1 \otimes y_1, x_2 \otimes y_2) = (x_1, x_2)(y_1, y_2).$$

If $u \in X \otimes Y$, we denote its prehilbertian norm by $\|u\|_{\alpha_0}$, so that the completion $X \hat{\otimes}_{\alpha_0} Y$ is a Hilbert space. The norm α_0 is reasonable and uniform.

A reasonable norm $\alpha \ge \varepsilon$ is said to be *faithful* on $X \otimes Y$ [9] if the natural continuous linear mapping j_ε^α of $X \hat{\otimes}_\alpha Y$ into $X \hat{\otimes}_\varepsilon Y$ is one-to-one.

It is obvious that the norm ε (resp. α_0) is faithful on $X\otimes Y$ for every pair of Banach (resp. Hilbert) spaces X and Y. Whether or not the norm π is faithful is the "problème de biunivocité" of A. Grothendieck [11]. The norm π is faithful on $X\otimes Y$ if either X or Y satisfies the approximation condition, in particular, has a Schauder basis. P. Enflo[1] has recently constructed a Banach space which does not satisfy the approximation condition. It follows by the equivalence of the approximation problem and the problème de biunivocité [11] that the norm π is not always faithful on $X\otimes Y$ for some pair of Banach spaces X and Y.

Further we shall use the concept of $\otimes$-norms due to A. Grothendieck (For the definition see [12]). If α is a $\otimes$-norm, it is defined on the tensor product $X\otimes Y$ for every pair of Banach spaces X and Y. If X_i and Y_i, $i=1, 2$, are Banach spaces, then for any $(A, B)\in L(X_1, X_2)\times L(Y_1, Y_2)$ one has $\|(A\otimes B)u\|_\alpha\leq\|A\|\|B\|\|u\|_\alpha$ if $u\in X_1\otimes Y_1$. The norms π and ε are $\otimes$-norms. Note the norm α_0 has also this property when X_i and Y_i, $i=1, 2$, are Hilbert spaces.

2. Gowurin Property

Throughout this section, X, X_1 and Y are complex Banach spaces with topological dual spaces X', X_1' and Y'. The identity operators in X and X_1 are denoted by I, I_1, respectively. α and β are uniform reasonable norms on $X\otimes Y$, $X_1\otimes Y$, respectively.

Let $\mathcal{E}$ be a bounded Boolean algebra of projections in Y. We are interested in $\mathcal{E}$ satisfying the following conditions:

(G): there exists a constant K such that for any finite collection $\{E_i\}_{i=1}^s\subset\mathcal{E}$ of disjoint projections and for any collection $\{A_i\}_{i=1}^s\subset L(X, X_1)$, one has

$$\|[\textstyle\sum_{i=1}^s A_i\otimes E_i]u\|_\beta\leq K\sup_i\|A_i\|\|u\|_\alpha,\quad u\in X\otimes Y; \tag{2.1}$$

(G'): there exists a constant K such that for any finite collection $\{E_i\}_{i=1}^s\subset\mathcal{E}$ of disjoint projections and for any collection $\{A_i\}_{i=1}^s\subset L(X, X_1)$, one has

$$\|[\textstyle\sum_{i=1}^s A_i'\otimes E_i]u'\|_{\alpha'}\leq K\sup_i\|A_i'\|\|u'\|_{\beta'},\quad u'\in X_1'\otimes Y'; \tag{2.2}$$

(g): there exists a constant K such that for any finite collection $\{E_i\}_{i=1}^s\subset\mathcal{E}$ of disjoint projections and for any collection $\{B_i\}_{i=1}^s\subset\mathfrak{L}(X, X_1)$ of densely defined linear operators with $D[B_i]\supset D$ and $D[B_i']\supset D'$, $1\leq i\leq s$, one has

[1] P. Enflo: A counterexample to the approximation problem in Banach spaces, Acta Math. 130, 309–317 (1973).

$$(2.3)\qquad \|[\sum_{i=1}^{s} B_i\otimes E_i](x\otimes y)\|_\beta \leq K \sup_i \|B_i x\|\,\|y\|,\quad x\otimes y\in D\otimes Y,$$

and

$$(2.4)\qquad \|[\sum_{i=1}^{s} B_i'\otimes E_i'](x'\otimes y')\|_{\alpha'} \leq K \sup_i \|B_i' x'\|\,\|y'\|,\quad x'\otimes y'\in D'\otimes Y'.$$

Here, for $1\leq i\leq s$, A_i' and B_i' are the adjoints of A_i and B_i, respectively, which are in $\mathfrak{L}(X_1', X')$, and D and D' are given fixed subspaces of X and X_1', respectively.

PROPOSITION 2.1. (G) *implies* (G'). *In the right member of* (2.1), u *may be replaced with a possibly other constant* K *by* $[I\otimes\sum_{j=1}^{s} E_j]u$; *in that of* (2.2), u' *by* $[I_1'\otimes\sum_{j=1}^{s} E_j']u'$; *in that of* (2.3), y *by* $[\sum_{j=1}^{s} E_j]y$ *and in that of* (2.4), y' *by* $[\sum_{j=1}^{s} E_j']y'$.

PROOF. The second half is evident. We show (G) implies (G'). The adjoints A_i' are in $L(X_1', X')$ with $\|A_i'\|=\|A_i\|$. If $u'\in X'\otimes Y'$, the left member of (2.2) equals

$$\sup\left\{|\langle u,\ [\textstyle\sum_{i=1}^{s} A_i'\otimes E_i']u'\rangle|;\ u\in X\otimes Y,\ \|u\|_\alpha\leq 1\right\}$$

$$=\sup\left\{|\langle[\textstyle\sum_{i=1}^{s} A_i\otimes E_i]u,\ u'\rangle|;\ u\in X\otimes Y,\ \|u\|_\alpha\leq 1\right\}$$

$$=\sup\left\{\|[\textstyle\sum_{i=1}^{s} A_i\otimes E_i]u\|_\beta;\ u\in X\otimes Y,\ \|u\|_\alpha\leq 1\right\}\cdot\|u'\|_{\beta'},$$

which is in virtue of (G) not greater than

$$K\sup_i\|A_i\|\,\|u'\|_{\beta'}=K\sup_i\|A_i'\|\,\|u'\|_{\beta'}.\qquad \text{Q.E.D.}$$

We say $\mathcal{E}$ has the *Gowurin property* in $L(X\hat{\otimes}_\alpha Y, X_1\hat{\otimes}_\beta Y)$ (cf. [10]) if $\mathcal{E}$ satisfies (G) (and consequently (G') by Proposition 2.1) and (g).

Examples of $L(X\hat{\otimes}_\alpha Y, X_1\hat{\otimes}_\beta Y)$ in which $\mathcal{E}$ has the Gowurin property are given by the following

THEOREM 2.2. *The bounded Boolean algebra* $\mathcal{E}$ *of projections in* Y *has the Gowurin property in*

(a) $L(X\hat{\otimes}_\pi L, X_1\hat{\otimes}_\pi L)$ (*for* $Y=L$),

(b) $L(X\hat{\otimes}_\varepsilon C, X_1\hat{\otimes}_\varepsilon C)$ (*for* $Y=C$),

(c) $L(H\hat{\otimes}_{\alpha_0} H_2, H_1\hat{\otimes}_{\alpha_0} H_2)$ (*for* $Y=H_2$), *and*

(d) $L(X\hat{\otimes}_\pi Y, X_1\hat{\otimes}_\varepsilon Y)$

for every pair of Banach spaces X, X_1 *and Hilbert spaces* H, H_1.

PROOF. In virtue of Propositson 2.1, we have only to show (G) and (g). Let $\{E_i\}_{i=1}^{s}$, $\{A_i\}_{i=1}^{s}$ and $\{B_i\}_{i=1}^{s}$ be those described in the conditions (G) and (g).

(a) For $u=\sum_k x_k\otimes y_k\in D\otimes L$, we obtain by Theorem 1.1

$$\begin{aligned}\|[\sum_{i=1}^{s} B_i \otimes E_i] u\|_\pi &= \sum_{ik} \|B_i x_k\| \, \|E_i y_k\| \\ &\leq \sum_k \sup_i \|B_i x_k\| \sum_{j=1}^{s} \|E_j y_k\| \\ &\leq h k_0 \sum_k \sup_i \|B_i x_k\| \, \|y_k\| \,.\end{aligned}$$

If $u = x \otimes y$, this implies (2. 3) for $\beta = \pi$. If $D = X$ and $B_i = A_i$, $1 \leq i \leq s$, this yields

$$\|[\sum_{i=1}^{s} A_i \otimes E_i] u\|_\pi \leq h k_0 \sup_i \|A_i\| \sum_k \|x_k\| \, \|y_k\| ,$$

whence follows (2. 1) for $\alpha = \beta = \pi$, by taking the infimum of the right member over all representations of u.

To derive (2. 4), let $x' \otimes y' \in D' \otimes L'$. Then since $\pi' = \varepsilon$, we obtain

$$\begin{aligned}&\|[\sum_{i=1}^{s} B_i' \otimes E_i'](x' \otimes y')\|_{\pi'} \\ &\quad = \|[\sum_{i=1}^{s} B_i' \otimes E_i'](x' \otimes y')\|_\varepsilon \\ &\quad = \sup \left\{ |\sum_i <x, B_i' x'> <y, E_i' y'>| \,;\ (x, y) \in X \times L,\ \|x\| \leq 1,\ \|y\| \leq 1 \right\} \\ &\quad = \sup \left\{ \|\sum_i E_i' <x, B_i' x'> y'\| \,;\ x \in X,\ \|x\| \leq 1 \right\} \\ &\quad = \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|[\sum_{i=1}^{s} E_i'][\sum_{j=1}^{s} E_j' <x, B_j' x'> y']\| \,.\end{aligned}$$

Since the dual space L' of the $\mathscr{L}_1$-space L is an $\mathscr{L}_\infty$-space, we obtain by Theorem 1. 1

$$\begin{aligned}\|[\sum_{i=1}^{s} B_i' \otimes E_i'](x' \otimes y')\|_{\pi'} &\leq k_0 \sup_{\substack{x \in X \\ \|x\| \leq 1}} \sup_i \|E_i' <x, B_i' x'> y'\| \\ &\leq h k_0 \sup_{\substack{x \in X \\ \|x\| \leq 1}} \sup_i \|<x, B_i' x'> y'\| \\ &= h k_0 \sup_i \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|<x, B_i' x'> y'\| \\ &= h k_0 \sup_i \|[B_i' \otimes I'](x' \otimes y')\|_\varepsilon \\ &= h k_0 \sup_i \|[B_i' \otimes I'](x' \otimes y')\|_{\pi'} ,\end{aligned}$$

which implies (2.4) for $\alpha = \pi$.

(b) If $u = \sum_k x_k \otimes y_k \in D \otimes C$, it will be possible to show in the same way as in (a) a stronger result

$$\|[\sum_{i=1}^{s} B_i \otimes E_i] u\|_\varepsilon \leq h k_0 \sup_i \|[B_i \otimes I] u\|_\varepsilon ,$$

which yields in particular (2. 1) and (2. 3) for $\alpha = \beta = \varepsilon$. We shall obtain (2. 4) similarly to the proof in (a), if we note $\varepsilon' \leq \pi$.

(c) For $u\in D\otimes H_2$ and $w\in H_1\otimes H_2$, we obtain

$$\begin{aligned}|([\textstyle\sum_{i=1}^{s}B_i\otimes E_i]u,\ w)| &= |\textstyle\sum_{i=1}^{s}([B_i\otimes E_i]u,\ [I_1\otimes E_i]w)|\\ &\leq \textstyle\sum_{i=1}^{s}\|[B_i\otimes E_i]u\|_{\alpha_0}\|[I_1\otimes E_i]w\|_{\alpha_0}\\ &\leq \left\{\textstyle\sum_{i=1}^{s}\|[B_i\otimes E_i]u\|_{\alpha_0}^2\right\}^{1/2}\left\{\textstyle\sum_{i=1}^{s}([I_1\otimes E_i]w,\ w)\right\}^{1/2}\\ &\leq h\left\{\textstyle\sum_{i=1}^{s}\|[B_i\otimes E_i]u\|_{\alpha_0}^2\right\}^{1/2}\|w\|_{\alpha_0}\,,\end{aligned}$$

whence

$$\|[\textstyle\sum_{i=1}^{s}B_i\otimes E_i]u\|_{\alpha_0}\leq h\left\{\textstyle\sum_{i=1}^{s}\|[B_i\otimes E_i]u\|_{\alpha_0}^2\right\}^{1/2}.$$

If $D=H$ and $B_i=A_i$, $1\leq i\leq s$, we obtain

$$\begin{aligned}\|[\textstyle\sum_{i=1}^{s}A_i\otimes E_i]u\|_{\alpha_0} &\leq h\left\{\textstyle\sum_{i=1}^{s}\|A_i\|^2\|[I\otimes E_i]u\|_{\alpha_0}^2\right\}^{1/2}\\ &\leq h\sup_i\|A_i\|\left\{\textstyle\sum_{j=1}^{s}([I\otimes E_j]u,\ u)\right\}^{1/2}\\ &\leq h^2\sup_i\|A_i\|\|u\|_{\alpha_0}\,,\end{aligned}$$

which proves (2. 1) for $\alpha=\beta=\alpha_0$.

If $u=x\otimes y$, then

$$\begin{aligned}\|[\textstyle\sum_{i=1}^{s}B_i\otimes E_i](x\otimes y)\|_{\alpha_0} &\leq h\left\{\textstyle\sum_{i=1}^{s}\|B_ix\otimes E_iy\|_{\alpha_0}^2\right\}^{1/2}\\ &= h\left\{\textstyle\sum_{i=1}^{s}\|B_ix\|^2\|E_iy\|^2\right\}^{1/2}\\ &\leq h\sup_i\|B_ix\|\left\{\textstyle\sum_{j=1}^{s}(E_jy,\ y)\right\}^{1/2}\\ &\leq h^2\sup\|B_ix\|\|y\|\,,\end{aligned}$$

which shows (2. 3) for $\beta=\alpha_0$ as well as (2. 4) for $\alpha=\alpha_0$.

(d) For $u=\sum_k x_k\otimes y_k\in D\otimes Y$, we obtain

$$\begin{aligned}&\|[\textstyle\sum_{i=1}^{s}B_i\otimes E_i]u\|_{\varepsilon}\\ &=\sup\left\{|\textstyle\sum_{ik}<B_ix_k,\ x'><E_iy_k,\ y'>|\,;\ (x',\ y')\in X_1'\times Y',\ \|x'\|\leq 1,\ \|y'\|\leq 1\right\}\\ &\leq\sup\left\{\textstyle\sum_{ik}\|B_ix_k\||<E_iy_k,\ y'>|\,;\ \ y'\in Y',\ \|y'\|\leq 1\right\}\\ &\leq\sup\left\{\textstyle\sum_k\sup_i\|B_ix_k\|\sum_{j=1}^{s}|<E_jy_k,\ y'>|\,;\ \ y'\in Y',\ \|y'\|\leq 1\right\}\\ &\leq 4h\textstyle\sum_k\sup_i\|B_ix_k\|\|y_k\|\,.\end{aligned}$$

This will yield (2. 1) and (2. 3) for $\alpha=\pi$ and $\beta=\varepsilon$ in the same way as in (a), and (2. 3) yields (2. 4) for $\alpha=\pi$. Q.E.D.

In what follows, let $B: D[B]\subset Y\rightarrow Y$ be a scalar type spectral operator in the sense of N. Dunford and W. G. Bade (see [6, Part III, Chap. 18]). There exists a countably additive resolution E of the identity defined on the field of Borel sets of the complex plane $\boldsymbol{C}$ such that for $k=1, 2, \cdots$,

$$B^k y=\int_{\sigma(B)}\eta^k E(d\eta)y\equiv\lim_{\nu\to\infty}\int_{\sigma(B)\cap\sigma_\nu}\eta^k E(d\eta)y, \quad y\in D[B^k].$$

Here and below, $\{\sigma_\nu\}_{\nu=1}^{\infty}$ is any increasing sequence of bounded Borel sets such that $E(\bigcup_{\nu=1}^{\infty}\sigma_\nu)=I$. We may assume the spectrum $\sigma(B)$ of B nonempty. The Boolean algebra of projections generated by the resolution E of the identity is bounded.

We shall now see that the Gowurin property in $L(X\hat{\otimes}_\alpha Y, X_1\hat{\otimes}_\beta Y)$ of the Boolean algebra of projections generated by E guarantees the existence of the improper Riemann integral

$$\int_{\sigma(B)}\left[F(\eta)\otimes E(d\eta)\right]v \tag{2. 5}$$

in $X_1\hat{\otimes}_\beta Y$ for $v\in X\otimes Y$, where $F(\eta): \sigma(B)\rightarrow L(X, X_1)$ is a operator-valued function continuous and uniformly bounded in $L(X, X_1)$.

First we show that for any bounded Borel set σ the Riemann integral

$$I(\sigma)v=\int_{\sigma(B)\cap\sigma}\left[F(\eta)\otimes E(d\eta)\right]v, \quad v\in X\otimes Y \tag{2. 6}$$

exists in the uniform operator topology of $L(X\hat{\otimes}_\alpha Y, X_1\hat{\otimes}_\beta Y)$ and further that there exists a constant K_1 such that for every bounded Borel set σ and $v\in X\otimes Y$

$$\|I(\sigma)v\|_\beta\leq K_1\|v\|_\alpha \tag{2. 7}$$

and

$$\|I(\sigma)v\|_\beta\leq K_1\left\|\left[I\otimes E(\sigma(B)\cap\sigma)\right]v\right\|_\alpha. \tag{2. 8}$$

Our proof follows the argument in [6, Part III, Chap. 16, 5.18, p. 2163]. Let $\Delta=\{e_1, \cdots, e_s\}$ and $\Delta'=\{e_1', \cdots, e_t'\}$ be two arbitrary partitions of $\sigma(B)\cap\sigma$ into disjoint Borel sets. Then observing (G) of the Gowurin property, if $\eta_i\in e_i$, $1\leq i\leq s$, and $\eta_j'\in E_j'$, $1\leq j\leq t$, we have for some constant K

$$\begin{aligned}&\left\|\left[\textstyle\sum_{i=1}^{s}F(\eta_i)\otimes E(e_i)-\sum_{j=1}^{t}F(\eta_j')\otimes E(e_j')\right]v\right\|_\beta\\&=\left\|\left[\textstyle\sum_{ij}(F(\eta_i)-F(\eta_j'))\otimes E(e_i\cap e_j')\right]v\right\|_\beta\\&\leq K\sup\|F(\eta_i)-F(\eta_j')\|\,\|v\|_\alpha,\end{aligned}$$

where the supremum is taken over those i and j for which $e_i \cap e'_j$ is not empty. Thus, if $|\Delta| = \max_{1\leq i\leq s} \operatorname{diam} e_i$ tends to zero,

$$\lim_{|\Delta|\to 0} \left[\sum_{i=1}^{s} F(\eta_i)\otimes E(e_i)\right]$$

exists in the uniform operator topology of $L(X\hat{\otimes}_\alpha Y, X_1\hat{\otimes}_\beta Y)$. This limit defines the Riemann integral (2. 6). It is clear that $I(\sigma)$ satisfies for $v\in X\otimes Y$

$$\begin{aligned}\|I(\sigma)v\|_\beta &\leq K \sup_{\eta\in\sigma(B)\cap\sigma} \|F(\eta)\| \|v\|_\alpha \\ &\leq K \sup_{\eta\in\sigma(B)} \|F(\eta)\| \|v\|_\alpha .\end{aligned}$$

This proves (2. 7) and by Proposition 2. 1 we have also (2. 8).

Now, let $\{\sigma_\nu\}_{\nu=1}^{\infty}$ be any increasing sequence of bounded Borel sets such that $E(\cup_{\nu=1}^{\infty}\sigma_\nu)=I$. Then if $x\otimes y\in X\otimes Y$, we obtain by (2. 8) for $\mu\geq\nu$

$$\begin{aligned}\|I(\sigma_\mu)(x\otimes y)-I(\sigma_\nu)(x\otimes y)\|_\beta &= \|I(\sigma_\mu\backslash\sigma_\nu)(x\otimes y)\|_\beta \\ &\leq K_1 \left\|\left[I\otimes E\big(\sigma(B)\cap(\sigma_\mu\backslash\sigma_\nu)\big)\right](x\otimes y)\right\|_\alpha \\ &\leq K_1\|x\| \left\|E\big(\sigma(B)\cap(\sigma_\mu\backslash\sigma_\nu)\big)\, y\right\|,\end{aligned}$$

which tends to zero as $\mu, \nu\to\infty$. It follows that the sequence $\{I(\sigma_\nu)(x\otimes y)\}_{\nu=1}^{\infty}$ and consequently $\{I(\sigma_\nu)v\}_{\nu=1}^{\infty}$ for $v\in X\otimes Y$ is convergent in $X_1\hat{\otimes}_\beta Y$ as $\nu\to\infty$.

Further it is easy to verify that the limit is independent of the choice of a sequence $\{\sigma_\nu\}_{\nu=1}^{\infty}$. This limit is by definition what is meant by the integral (2. 5) and defines on account of (2. 7) a continuous linear mapping of $X\otimes Y\subset X\hat{\otimes}_\alpha Y$ into $X_1\hat{\otimes}_\beta Y$.

Thus we have shown

PROPOSITION 2. 3. *Suppose the Boolean algebra of projections generated by E has the Gowurin property in $L(X\hat{\otimes}_\alpha Y, X_1\hat{\otimes}_\beta Y)$. If $F(\eta): \sigma(B)\to L(X, X_1)$ is an operator-valued function continuous and uniformly bounded in $L(X, X_1)$, then for $v\in X\otimes Y$ the improper Riemann integral* (2. 5) *exists in $X_1\hat{\otimes}_\beta Y$ and defines a continuous linear mapping of $X\otimes Y\subset X\hat{\otimes}_\alpha Y$ into $X_1\hat{\otimes}_\beta Y$.*

3. Polynomial Operators

We shall introduce polynomial operators defined between the tensor products of Banach spaces (cf. [15], [16]), and study their invertibiltiy, in particular, establish the spectral mapping theorem.

Throughout this section, X and Y are complex Banach spaces, and $A: D[A]\subset X\to X$ and $B: D[B]\subset Y\to Y$ are densely defined closed linear operators with nonempty resolvent sets. As in Section 2, assume further that B is a scalar type spectral operator with the countably additive resolution E of the identity. By the same symbol I, we shall denote the identity operators in both X and Y. α and β are uniform reasonable norms on $X\otimes Y$.

To each polynomial of degrees m in ξ and n in η

$$(3.1)\qquad P(\xi,\eta)=\sum_{jk}c_{jk}\xi^j\cdot\eta^k\,,$$

we assingn two kinds of *polynomial operators* in $\mathfrak{L}(X\hat{\otimes}_\beta Y, X\hat{\otimes}_\alpha Y)$:

$$(3.2)\qquad P\{A\otimes I, I\otimes B\}\equiv\sum_{jk}c_{jk}A^j\otimes B^k$$

with domain $D[A^m]\otimes D[B^n]$, and

$$(3.3)\qquad \sum_{jk}c_{jk}A^j\hat{\otimes}B^k$$

with domain $\bigcap_{j,k;c_{jk}\neq 0}D[A^j\hat{\otimes}B^k]$, where $A^j\hat{\otimes}B^k$ denotes a maximal extension of $A^j\otimes B^k$ as an operator in $\mathfrak{L}(X\hat{\otimes}_\beta Y, X\hat{\otimes}_\alpha Y)$. If $\beta=\alpha$, it will be written as $A^j\hat{\otimes}_\alpha B^k$, which is a linear operator in $\mathfrak{L}(X\hat{\otimes}_\alpha Y)$. Maximal extensions of (3.2) and (3.3) as operators in $\mathfrak{L}(X\hat{\otimes}_\beta Y, X\hat{\otimes}_\alpha Y)$ are denoted by $\tilde{P}\{A\otimes I, I\otimes B\}$, $(\sum_{jk}c_{jk}A^j\hat{\otimes}B^k)^\sim$, respectively. In this context, such a general case has not been considered in [15] and [16] but the case $\alpha=\beta$. However, it will be shown just in the same way as in [16, Theorem 1.1] that if α is faithful on $X\otimes Y$, then $P\{A\otimes I, I\otimes B\}$ (and hence $A^j\otimes B^k$) and $\sum_{jk}c_{jk}A^j\hat{\otimes}B^k$ are closable as operators in $\mathfrak{L}(X\hat{\otimes}_\beta Y, X\hat{\otimes}_\alpha Y)$, so that $\tilde{P}\{A\otimes I, I\otimes B\}$ (and hence $A^j\hat{\otimes}B^k$) and $(\sum_{jk}c_{jk}A^j\hat{\otimes}B^k)^\sim$ are nothing but their closures.

In the present paper we assume further for simplicity that α and β are faithful, consequently they are faithful uniform reasonable norms on $X\otimes Y$, so that all polynomial operators are closable, considered as operators in $\mathfrak{L}(X\hat{\otimes}_\beta Y, X\hat{\otimes}_\alpha Y)$.

3.1. *Invertibility and closures of polynomial operators.*

Rewrite $P(\xi,\eta)$ of (3.1) in the following form

$$(3.1)'\qquad P(\xi,\eta)=c_m(\eta)\xi^m+c_{m-1}(\eta)\xi^{m-1}+\cdots+c_0(\eta)\,,$$

where $c_m(\eta)\not\equiv 0$. Then for η fixed

$$(3.4)\qquad P(A,\eta)=\sum_{j=0}^m c_j(\eta)A^j$$

is a densely defined closed linear operator in X with domain $D[A^{m(\eta)}]$, where $m(\eta)$ is the greatest integer with $c_{m(\eta)}(\eta)\neq 0$, $0\leq m(\eta)\leq m$.

To formulate our results, we shall consider $P(\xi,\eta)$ under one of the following conditions:

(P): $P(A, \eta)$, with $\eta \in \sigma(B)$, has an inverse $P(A, \eta)^{-1} \in L(X)$ which is uniformly bounded on $\sigma(B)$;

(P_1): there exists a nonempty open set whose complement CU is included in $\rho(A)$ and whose boundary ∂U, restricted to the closed disc $K(0; R)$ with center 0 and radius R for each $R>0$, consists of a finite number of rectifiable Jordan arcs and has a length of $O(R)$ such that $|P(\xi, \eta)|$ is bounded away from zero on $U \times \sigma(B)$ and the resolvent $R(\xi; A)$ is uniformly bounded in CU.

We remark that it will be shown in the same way as in [17, Theorem 2.2] that (P_1) implies (P).

For the following statements, we note that

$$\begin{aligned}
&\lim_{\nu\to\infty} P\left\{A\otimes E(\sigma(B)\cap\sigma_\nu),\ I\otimes BE(\sigma(B)\cap\sigma_\nu)\right\} u \\
&\quad \equiv \lim_{\nu\to\infty} \left[\sum_{jk} c_{jk} A^j \otimes \left(BE(\sigma(B)\cap\sigma_\nu)\right)^k\right] u \\
&\quad = \lim_{\nu\to\infty} \int_{\sigma(B)\cap\sigma_\nu} \left[P(A, \eta)\otimes E(d\eta)\right] u = \sum_{jk} c_{jk}\left[A^j \otimes \int_{\sigma(B)} \eta^k E(d\eta)\right] u \\
&\quad = P\{A\otimes I, I\otimes B\} u,
\end{aligned}$$

for $u = x\otimes y$ in $D[A^m]\otimes D[B^n]$ and hence for u in $D[A^m]\otimes[B^n]$, the limit being taken in the norm of $X\hat{\otimes}_\alpha Y$. Here and in the following, $\{\sigma_\nu\}_{\nu=1}^\infty$ is any increasing sequence of bounded Borel sets such that $E(\bigcup_{\nu=1}^\infty \sigma_\nu) = I$.

PROPOSITION 3.1. *Suppose that the Boolean algebra of projections generated by E has the Gowurin property in $L(X\hat{\otimes}_\alpha Y, X\hat{\otimes}_\beta Y)$. Let $P(\xi, \eta)$ be a polynomial of the form* (3.1) *satisfying* (P) *when $\alpha=\beta$ and* (P_1) *when $\alpha\neq\beta$.*

Then both the polynomial operators (3.2) *and* (3.3) *have the same closure as operators in $\mathfrak{L}(X\hat{\otimes}_\beta Y, X\hat{\otimes}_\alpha Y)$:*

$$\tilde{P}\{A\otimes I, I\otimes B\} = (\textstyle\sum_{jk} c_{jk} A^j \hat{\otimes} B^k)^\sim, \tag{3.5}$$

and the closed operator (3.5) *has an everywhere defined continuous inverse $\tilde{P}\{A\otimes I, I\otimes B\}^{-1} \in L(X\hat{\otimes}_\alpha Y, X\hat{\otimes}_\beta Y)$.*

PROOF. (1) First we show $\tilde{P}\{A\otimes I, I\otimes B\}$ has an inverse $\tilde{P}\{A\otimes I, I\otimes B\}^{-1}$ which lies in $L(X\hat{\otimes}_\alpha Y, X\hat{\otimes}_\beta Y)$.

By Proposition 2.3, the improper Riemann integral

$$\int_{\sigma(B)} \left[P(A, \eta)^{-1}\otimes E(d\eta)\right] v, \qquad v\in X\otimes Y,$$

exists in $X\hat{\otimes}_\beta Y$ and defines a continuous linear mapping of $X\otimes Y \subset X\hat{\otimes}_\alpha Y$ into $X\hat{\otimes}_\beta Y$. We denote its continuous extension to the entire space

$X\hat{\otimes}_\alpha Y$ by $\widetilde{P^{-1}}\{A\otimes I, I\otimes B\}$, which lies in $L(X\hat{\otimes}_\alpha Y, X\hat{\otimes}_\beta Y)$. We show $\widetilde{P^{-1}}\{A\otimes I, I\otimes B\}$ is the continuous inverse of $\tilde{P}\{A\otimes I, I\otimes B\}$.

Let $\{\sigma_\nu\}_{\nu=1}^\infty$ be any increasing sequence of bounded Borel sets such that $E(\cup_{\nu=1}^\infty \sigma_\nu)=I$. Then for u in $D[A^m]\otimes D[B^n]$ we have by definition

$$\widetilde{P^{-1}}\{A\otimes I, I\otimes B\}\,\tilde{P}\{A\otimes I, I\otimes B\}\,u$$
$$=\lim_{\nu\to\infty}\int_{\sigma(B)\cap\sigma_\nu}\left[P(A,\eta)^{-1}\otimes E(d\eta)\right]P\{A\otimes I, I\otimes B\}\,u\,,$$

where the limit is taken in the norm of $X\hat{\otimes}_\beta Y$.

For ν fixed, the integral

$$\int_{\sigma(B)\cap\sigma_\nu}\left[P(A,\eta)^{-1}\otimes E(d\eta)\right]$$

is the limit of Riemann sums

$$\sum_{i=1}^s\left[P(A,\eta_i)^{-1}\otimes E(e_i)\right]$$

in the uniform operator topology of $L(X\hat{\otimes}_\alpha Y, X\hat{\otimes}_\beta Y)$ as $|\varDelta|=\max\limits_{1\le i\le s}$ diam e_i tends to zero, where $\varDelta=\{e_1,\cdots,e_s\}$ is an arbitrary partition of $\sigma(B)\cap\sigma_\nu$ into disjoint Borel sets.

On the other hand, for $x\otimes y$ in $D[A^m]\otimes D[B^n]$ we obtain by (*g*) of the Gowurin property with Proposition 2. 1

$$\left\|\sum_{ij}\left[P(A,\eta_i)^{-1}P(A,\eta_j')\otimes E(e_i\cap e_j')\right](x\otimes y)-\left[I\otimes E(\sigma(B)\cap\sigma_\nu)\right](x\otimes y)\right\|_\beta$$
$$=\left\|\sum_{ij}\left(P(A,\eta_i)^{-1}P(A,\eta_j')-I\right)x\otimes E(e_i\cap e_j')y\right\|_\beta$$
$$\le K\sup\left\|\left(P(A,\eta_i)^{-1}P(A,\eta_j')-I\right)x\right\|\left\|\sum_{hk}E(e_h\cap e_k')y\right\|$$
$$=K\sup\left\|\left(P(A,\eta_i)^{-1}P(A,\eta_j')-I\right)x\right\|\left\|E\left(\sigma(B)\cap\sigma_\nu\right)y\right\|,$$

where the supremum is taken over those i and j for which $e_i\cap e_j'$ is not empty. If both $|\varDelta|$ and $|\varDelta'|$ tend to zero,

$$\sup\left\|\left(P(A,\eta_i)^{-1}P(A,\eta_j')-I\right)x\right\|$$

converges to zero. It follows that if $u=x\otimes y$ is in $D[A^m]\otimes D[B^n]$,

$$\int_{\sigma(B)\cap\sigma_\nu}\left[P(A,\eta)^{-1}\otimes E(d\eta)\right]P\{A\otimes I, I\otimes B\}\,u=\left[I\otimes E(\sigma(B)\cap\sigma_\nu)\right]u\,.$$

On tending ν to infinity, we obtain

$$\widetilde{P^{-1}}\{A\otimes I, I\otimes B\}\,\tilde{P}\{A\otimes I, I\otimes B\}\,u=u \tag{3.6}$$

for $u=x\otimes y$ in $D[A^m]\otimes D[B^n]$ and further by continuity of $\widetilde{P^{-1}}\{A\otimes I, I\otimes B\}$ for all u in the domain of $\tilde{P}\{A\otimes I, I\otimes B\}$.

The proof of

$$\tilde{P}\{A\otimes I, I\otimes B\}\widetilde{P^{-1}}\{A\otimes I, I\otimes B\}v=v \tag{3.7}$$

for $v\in X\hat{\otimes}_\alpha Y$ will be divided into two cases $\alpha=\beta$ and $\alpha\neq\beta$.

For case $\alpha=\beta$, first note that $P(A, \eta)^{-1}$ maps $D[A^{2m}]$ onto $D[A^{2m-m(\eta)}]$ $\subset D[A^m]$ and $E\ D[B^n]\subset D[B^n]$.

We shall be able to see similarly to the proof of (3.6) that if $v=x\otimes y$ is in $D[A^{2m}]\otimes D[B^n]$, the Riemann integral

$$\int_{\sigma(B)\cap\sigma_\nu}\left[P(A, \eta)^{-1}\otimes E(d\eta)\right]v \tag{3.8}$$

is approximated by a sequence $\{u_\mu\}_{\mu=1}^\infty$ in $D[A^m]\otimes D[B^n]$ of Riemann sums such that $P\{A\otimes I, I\otimes B\}u_\mu$ is convergent in $X\hat{\otimes}_\alpha Y$ to $[I\otimes E(\sigma(B)\cap\sigma_\nu)]v$ as $\mu\to\infty$. Then by closedness of $\tilde{P}\{A\otimes I, I\otimes B\}$, the integral (3.8) belongs to the domain of $\tilde{P}\{A\otimes I, I\otimes B\}$ and

$$\tilde{P}\{A\otimes I, I\otimes B\}\int_{\sigma(B)\cap\sigma_\nu}\left[P(A, \eta)^{-1}\otimes E(d\eta)\right]v=\left[I\otimes E(\sigma(B)\cap\sigma_\nu)\right]v\,.$$

Then tending ν to infinity, we obtain (3.7) by closedness for all $v=x\otimes y$ in $D[A^{2m}]\otimes D[B^n]$ and hence all v in $D[A^{2m}]\otimes D[B^n]$. Further, since $D[A^{2m}]\otimes D[B^n]$ is dense in $X\hat{\otimes}_\alpha Y$, we see by continuity of $\widetilde{P^{-1}}\{A\otimes I, I\otimes B\}$ and by closedness of $\tilde{P}\{A\otimes I, I\otimes B\}$ that (3.7) is valid also for all v in $X\hat{\otimes}_\alpha Y$. Thus we have shown that when $\alpha=\beta$, $\widetilde{P^{-1}}\{A\otimes I, I\otimes B\}$ is the everywhere defined continuous inverse of $\tilde{P}\{A\otimes I, I\otimes B\}$.

To treat case $\alpha\neq\beta$, we need

LEMMA 3.2. *Suppose $P(\xi, \eta)$ satisfies (P_1). If $\xi_0\notin U$ so that $\xi_0\in\rho(A)$, then for u in $D[A^{m+2}]\otimes Y$ the improper Riemann integral*

$$(2\pi i)^{-1}\int_{\partial U}(\xi-\xi_0)^{-(m+2)}\left[R(\xi\,;A)\otimes P(\xi, B)^{-1}\right]\left[(A-\xi_0 I)^{m+2}\otimes I\right]v\,d\xi \tag{3.9}$$

exists in $X\hat{\otimes}_\beta Y$ and coincides with (3.6).

PROOF. The condition (P_1) implies that $P(\xi, B)$, for $\xi\in U$, is a closed operator in Y with inverse $P(\xi, B)^{-1}\in L(Y)$ which is uniformly bounded in U and hence on its closure $\bar{U}$. $R(\xi\,;A)$ is uniformly bounded in CU and hence on ∂U. Then it is clear that the integral (3.9) exists in $X\hat{\otimes}_\beta Y$, in fact, even in $X\hat{\otimes}_\pi Y$.

In a similar way to the proof of Proposition 2.3 with the aid of (G)

of the Gowurin property, we shall see that for v in $D[A^{m+2}]\otimes Y$ the improper double Riemann integral

$$(3.10)\qquad (2\pi i)^{-1}\int_{\partial U}\int_{\sigma(B)}(\xi-\xi_0)^{-(m+2)}P(\xi,\eta)^{-1} \cdot\left[R(\xi;A)d\xi\otimes E(d\eta)\right]\left[(A-\xi_0 I)^{m+2}\otimes I\right]v$$

exists in $X\hat{\otimes}_\beta Y$. Further, it is easy to verify by spectrality of B that (3.10) coincides with (3.9) and by the Cauchy integral theorem that (3.10) coincides with (3.6), if $v\in D[A^{m+2}]\otimes Y$. Q.E.D.

End of Proof of Proposition 3.1. Now we show (3.7) in case $\alpha\neq\beta$. Let v be in $D[A^{2m+2}]\otimes D[B^n]$. Then with the aid of Lemma 3.2 we obtain by closedness of $\tilde{P}\{A\otimes I, I\otimes B\}$

$$\begin{aligned}&\tilde{P}\{A\otimes I, I\otimes B\}\widetilde{P^{-1}}\{A\otimes I, I\otimes B\}v\\ &=(2\pi i)^{-1}\int_{\partial U}(\xi-\xi_0)^{-(m+2)}\left[R(\xi;A)\otimes P(\xi,B)^{-1}\right]\\ &\qquad\cdot P\{A\otimes I, I\otimes B\}\left[(A-\xi_0 I)^{m+2}\otimes I\right]v\,d\xi,\end{aligned}$$

where the integral is convergent in the norm of $X\hat{\otimes}_\alpha Y$.

Hence

$$\begin{aligned}&\tilde{P}\{A\otimes I, I\otimes B\}\widetilde{P^{-1}}\{A\otimes I, I\otimes B\}v\\ &=(2\pi i)^{-1}\int_{\partial U}(\xi-\xi_0)^{-(m+2)}\left[R(\xi;A)(A-\xi_0 I)^{m+2}\otimes I\right]v\,d\xi\\ &\quad+(2\pi i)^{-1}\int_{\partial U}(\xi-\xi_0)^{-(m+2)}\left[R(\xi;A)\otimes P(\xi,B)^{-1}\right]\\ &\qquad\cdot\left[P\{A\otimes I, I\otimes B\}-I\otimes P(\xi,B)\right]\left[(A-\xi_0 I)^{m+2}\otimes I\right]v\,d\xi\\ &=v,\end{aligned}$$

since the integrand of the second integral above is holomorphic in U so that this integral turns out to vanish with the aid of the Cauchy integral theorem.

Since $D[A^{2m+2}]\otimes D[B^n]$ is dense in $X\hat{\otimes}_\alpha Y$, we see by continuity and by closedness that (3.7) is valid for all v in $X\hat{\otimes}_\alpha Y$. This proves that $\widetilde{P^{-1}}\{A\otimes I, I\otimes B\}$ is the everywhere defined continuous inverse of $\tilde{P}\{A\otimes I, I\otimes B\}$ when $\alpha\neq\beta$.

(2) We show next that both the closures of (3.2) and (3.3) coincide. Since the right member of (3.5) is obviously a closed extension of the left member, we must show the converse.

As shown above, the closed operator $\tilde{P}\{A\otimes I, I\otimes B\}$, as an operator in

$\mathfrak{L}(X\hat{\otimes}_\beta Y,\ X\hat{\otimes}_\alpha Y)$, has an everywhere defined continuous inverse. Therefore it suffices to show that $(\sum_{jk} c_{jk}A^j\hat{\otimes}B^k)^\sim$ is one-to-one, or that its adjoint has a dense range in $(X\hat{\otimes}_\beta Y)'$ with respect to the weak topology defined by the dual pair $<(X\hat{\otimes}_\beta Y)', X\hat{\otimes}_\beta Y>$.

The adjoint B' of B is also a scalar type spectral operator in Y' with the resolution E' of the identity. Since the dual norms α' and β' are faithful on $X'\otimes Y'$, a similar argument to the above concludes with the aid of Proposition 2.1 that $\tilde{P}\{A'\otimes I', I'\otimes B'\}$, as a closed operator in $\mathfrak{L}(X'\hat{\otimes}_{\alpha'} Y', X'\hat{\otimes}_{\beta'} Y')$, has an everywhere defined continuous inverse, in particular, it has the range $X'\hat{\otimes}_{\beta'} Y'$. Since β is faithful so that $X'\otimes Y'$ is dense in $(X\hat{\otimes}_\beta Y)'$ with respect to the weak topology defined by the dual pair $<(X\hat{\otimes}_\beta Y)', X\hat{\otimes}_\beta Y>$, the adjoint of $(\sum_{jk} c_{jk}A^j\hat{\otimes}B^k)^\sim$, which is an extension of $\tilde{P}\{A'\otimes I', I'\otimes B'\}$ both as operators in $\mathfrak{L}((X\hat{\otimes}_\alpha Y)', (X\hat{\otimes}_\beta Y)')$, has a range dense in $(X\hat{\otimes}_\beta Y)'$ in the weak topology defined by the dual pair $<(X\hat{\otimes}_\beta Y)', X\hat{\otimes}_\beta Y>$.

Q.E.D.

To formulate Theorem 3.3 below, both α and β are assumed to be $\otimes$-norms, or also allowed to be the prehilbertian norm α_0 when both the spaces on whose tensor product they are defined are Hilbert spaces.

Let X_1 be a Banach space which is a subspace of X with a continuous injection: $X_1 \hookrightarrow X$. Assume that either X or Y and either X_1 or Y satisfy the approximation condition, so that α and β are faithful on $X\otimes Y$, $X_1\otimes Y$, respectively, and there is a continuous injection: $X_1\hat{\otimes}_\beta Y \hookrightarrow X\hat{\otimes}_\beta Y$. Then in virtue of Proposition 3.1, we see that (3.2) and (3.3) have the same closure (3.5) as operators in $\mathfrak{L}(X\hat{\otimes}_\beta Y, X\hat{\otimes}_\alpha Y)$, if $P(\xi, \eta)$ satisfies (P) or (P_1).

We consider the problem of finding a u in $X\hat{\otimes}_\beta Y$ which satisfies

$$\tilde{P}\{A\otimes I, I\otimes B\}u = (\textstyle\sum_{jk} c_{jk}A^j\hat{\otimes}B^k)^\sim u = f, \qquad f\in X\hat{\otimes}_\alpha Y.$$

To solve this problem, let $A_1 = A|X_1$ be the closed operator in X_1 obtained from A by restricting the domain and range of A to X_1, i.e. A_1 has the domain $D[A_1] = \{x\in D[A];\ x\in X_1,\ Ax\in X_1\}$ and is defined by $A_1x = Ax$ for $x\in D[A_1]$. So, $I_1 = I|X_1$.

Then we can state and prove

THEOREM 3.3. *Assume the Boolean algebra of projections generated by the resolution E of the identity of B has the Gowurin property in $L(X\hat{\otimes}_\alpha Y, X\hat{\otimes}_\beta Y)$ and $L(X\hat{\otimes}_\alpha Y, X_1\hat{\otimes}_\beta Y)$. Let $P(\xi, \eta)$ be a polynomial of the form* (3.1) *satisfying not only (P) when $\alpha=\beta$ and (P_1) when $\alpha\neq\beta$, but also the $\eta^k A^j P(A, \eta)^{-1}$ are in $L(X, X_1)$ and uniformly bounded on $\sigma(B)$ in $L(X, X_1)$ for all $(j, k)\in J$, where J is a given double-index subset of $\{(j, k);\ 0\leq j\leq m, 0\leq k\leq n\}$ such that $(j, k)\in J$ implies $(p, q)\in J$ for $0\leq p\leq j$, $0\leq q\leq k$.*

Then the closed operator (3.5), *considered as an operator in* $\mathfrak{L}(X\hat{\otimes}_\beta Y, X\hat{\otimes}_\alpha Y)$, *has an inverse which is a one-to-one continuous linear mapping of* $X\hat{\otimes}_\alpha Y$ *onto* $D[\tilde{P}\{A\otimes I, I\otimes B\}]\subset\bigcap_{(j,k)\in J}D[A_1{}^j\hat{\otimes}_\beta B^k]\subset X_1\hat{\otimes}_\beta Y$, *where* $D[\tilde{P}\{A\otimes I, I\otimes B\}]$ *is equipped with the norm* $\sum_{(j,k)\in J}\|A_1{}^j\hat{\otimes}_\beta B^k u\|_\beta$.

REMARK 3.4. It is also true under the assumption of Theorem 3.3 that (3.5) has an inverse which is a one-to-one continuous linear mapping of $X\hat{\otimes}_\alpha Y$ onto $D[\tilde{P}\{A\otimes I, I\otimes B\}]\subset\bigcap_{(j,k)\in J}D[A^j\hat{\otimes}_\beta B^k]\subset X\hat{\otimes}_\beta Y$, where $D[\tilde{P}\{A\hat{\otimes}I, I\hat{\otimes}B\}]$ is in its turn equipped with the norm $\sum_{(j,k)\in J}\|A^j\hat{\otimes}_\beta B^k u\|_\beta$. However, note $D[A_1{}^j\hat{\otimes}_\beta B^k]\subset D[A^j\hat{\otimes}_\beta B^k]$ for $(j,k)\in J$, as seen from the following proof.

PROOF of THEOREM 3.3. In virtue of Proposition 3.1, the closed operator (3.5) has an inverse $\tilde{P}\{A\otimes I, I\otimes B\}^{-1}\in L(X\hat{\otimes}_\alpha Y, X\hat{\otimes}_\beta Y)$.

To prove Theorem 3.3, it suffices to show that if $(j,k)\in J$, then

$$D\left[\tilde{P}\{A\otimes I, I\otimes B\}\right]\subset D[A_1^j\hat{\otimes}_\beta B^k]$$

and

$$\|A_1^j\hat{\otimes}_\beta B^k u\|_\beta\leq K\|\tilde{P}\{A\otimes I, I\otimes B\}u\|_\alpha \tag{3.11}$$

for $u\in D[\tilde{P}\{A\otimes I, I\otimes B\}]$, with a constant K independent of u.

By assumption, $\eta^k A^j P(A,\eta)^{-1}\in L(X, X_1)$ if $(j,k)\in J$ and $\eta\in\sigma(B)$. In particular, $A^p P(A,\eta)^{-1}\in L(X, X_1)$ for $0\leq p\leq j$. Hence

$$D[A^m]\subset D[A^{m(\eta)}]=D\left[P(A,\eta)\right]=P(A,\eta)^{-1}X\subset X_1.$$

Here we see $j\leq m(\eta)\leq m$ for $\eta\in\sigma(B)$. Thus we have $A^p P(A,\eta)^{-1}=A_1^p P(A,\eta)^{-1}$ for $0\leq p\leq j$. As $\eta^k A^j P(A,\eta)^{-1}$ is uniformly bounded in $L(X, X_1)$ on $\sigma(B)$, so is $\eta^k A_1^j P(A,\eta)^{-1}$. It is seen by the Gowurin property and by Proposition 2.3 that the integral

$$\int_{\sigma(B)}\left[\eta^k A_1^j P(A,\eta)^{-1}\otimes E(d\eta)\right]v, \qquad v\in X\otimes Y,$$

exists as an element of $X_1\hat{\otimes}_\beta Y$ and defines a continuous linear mapping of $X\otimes Y\subset X\hat{\otimes}_\alpha Y$ into $X_1\hat{\otimes}_\beta Y$. We denote its continuous extension to $X\hat{\otimes}_\alpha Y$ by $\mathscr{A}$. There exists a constant K such that $\|\mathscr{A}v\|_\beta\leq K\|v\|_\alpha$ for $v\in X\otimes Y$.

By the definition of the integral, it is shown similarly to the proof of Proposition 3.1 that for $u\in D[A^m]\otimes D[B^n]$

$$\mathscr{A}\left(\tilde{P}\{A\otimes I, I\otimes B\}u\right)=\mathscr{A}\left(P\{A\otimes I, I\otimes B\}u\right)=(A_1^j\otimes B^k)u,$$

which belongs to $X_1\hat{\otimes}_\beta Y$. Hence for $u\in D[A^m]\otimes D[B^n]$

$$\|(A_1^j \otimes B^k)u\|_\beta \leq \|\mathscr{A}\| \|\tilde{P}\{A\otimes I, I\otimes B\}u\|_\alpha$$
$$\leq K\|\tilde{P}\{A\otimes I, I\otimes B\}u\|_\alpha .$$

Since $D[A^m]\otimes D[B^n]$ is dense in the domain of $\tilde{P}\{A\otimes I, I\otimes B\}$ in its graph norm, we obtain by closedness of $A_1^j\hat{\otimes}_\beta B^k$ the inequality (3.11) Q.E.D.

REMARK 3.5. Although the hypothesis of Theorem 3.3 may not be sufficient to guarantee

$$A_1^j\hat{\otimes}_\beta B^k=(A_1^j\hat{\otimes}_\beta I)(I_1\hat{\otimes}_\beta B^k)=(I_1\hat{\otimes}_\beta B^k)(A_1^j\hat{\otimes}_\beta I)$$

as linear operators in $\mathfrak{L}(X_1\hat{\otimes}_\beta Y)$ (cf. [16, Theorem 1.8]), it is seen from the proof above that $A_1^j\hat{\otimes}_\beta B^k$ in the statement of Theorem 3.3 may be replaced by $(A_1^j\hat{\otimes}_\beta I)(I_1\hat{\otimes}_\beta B^k)$ or $(I_1\hat{\otimes}_\beta B^k)(A_1^j\hat{\otimes}_\beta I)$, which is in turn identical with $(A_1\hat{\otimes}_\beta I)^j(I_1\hat{\otimes}_\beta B)^k$ or $(I_1\hat{\otimes}_\beta B)^k(A_1\hat{\otimes}_\beta I)^j$, respectively.

From Theorem 3.3 combined with Theorem 2.2, we have

COROLLARY 3.6. *The assertion of Theorem* 3.3 *is valid for those* X, X_1, Y *and* α, β *which come from one of the cases* (a), (b), (c) *and* (d) *in Theorem* 2.2, *provided either* X *or* Y *and* X_1 *or* Y *satisfy the approximation condition.*

3.2. *The class* $\mathscr{P}'(A, B)$ *of polynomials.*

In order to formulate our main results, we introduce two classes of polynomials (see [17]) for which the assumptions in Section 3.1 are valid.

Given two subsets G_1 and $G_2\neq\emptyset$ of the complex plane $\boldsymbol{C}$ and a polynomial $P(\xi, \eta)$, we can define $P(G_1, G_2)$ and its closure $\overline{P(G_1, G_2)}$ in an obvious way, if G_1 is not empty when $P(\xi, \eta)$ is dependent on ξ, and otherwise we understand $P(G_1, G_2)=\overline{P(G_1, G_2)}=\emptyset$. We set $W_\varepsilon=\{\xi;\ \mathrm{dist}\,(\xi, \overline{P(\sigma(A), \sigma(B))})<\varepsilon\}$ if $\overline{P(\sigma(A), \sigma(B))}\neq\boldsymbol{C}$ and denote by $K(0; R)$ the closed disk $\{\xi;\ |\xi|\leq R\}$ with radius R.

Let $\mathscr{P}'(A, B)$ be the class of polynomials $P(\xi, \eta)$ of degrees m in ξ and n in η satisfying the following condition: for any $\varepsilon>0$ (resp. for any $R>0$ when $\sigma(A)$ is empty) there exists a nonempty open set U whose complement $\complement U$ is included in $\rho(A)$ such that

(i)′ $P(U, \sigma(B))\subset W_\varepsilon$ (resp. $P(U, \sigma(B))\subset\complement K(0; R)$, when $\sigma(A)$ is empty), and

(ii)′ the resolvent $R(\xi; A)$ is uniformly bounded in $\complement U$.

Further, we say $P(\xi, \eta)$ belongs to the class $\mathscr{P}_1'(A, B)$ if in the definition of $\mathscr{P}'(A, B)$, the open set U is chosen such that the boundary ∂U, restricted to $K(0; R)$ for each $R>0$, consists of a finite number of rectifiable Jordan arcs and has a length of $O(R)$.

For $P \in \mathscr{P}'(A, B)$ or $P \in \mathscr{P}_1'(A, B)$, the set $P(\sigma(A), \sigma(B))$ may not be closed, so that these classes are strictly larger than the class $\mathscr{P}(A, B)$ in [16].

Then we have from Proposition 3. 1

THEOREM 3. 7. *Let α and β be faithful uniform reasonable norms on $X \otimes Y$. Let A be a densely defined closed linear operator in X with non-empty resolvent set and B a scalar type spectral operator in Y with non-empty resolvent set having the countably additive resolution E of the identity, the Boolean algebra of projections generated by which has the Gowurin property in $L(X \hat{\otimes}_\alpha Y, X \hat{\otimes}_\beta Y)$.*

Then when $\alpha = \beta$, for $P \in \mathscr{P}'(A, B)$ both (3. 2) *and* (3. 3), *as operators in $\mathfrak{L}(X \hat{\otimes}_\alpha Y)$, have the same closure* (3. 5) *provided that $\overline{P(\sigma(A), \sigma(B))} \neq \boldsymbol{C}$, and the spectral mapping theorem holds for $P \in \mathscr{P}'(A, B)$:*

$$(3.12) \qquad \overline{P\big(\sigma(A), \sigma(B)\big)} = \sigma\big(P\{A \otimes I, I \otimes B\}\big) = \sigma\big(\tilde{P}\{A \otimes I, I \otimes B\}\big)$$
$$= \sigma\big(\textstyle\sum_{jk} c_{jk} A^j \hat{\otimes}_\alpha B^k\big) = \sigma\Big(\big(\textstyle\sum_{jk} c_{jk} A^j \hat{\otimes}_\alpha B^k\big)^{\sim}\Big)\,.$$

By this is meant that (3. 12) *holds valid if $\sigma(A)$ is not empty, and the spectra of the operators* (3. 2) *and* (3. 3) *and their closures are empty if and only if $\sigma(A)$ is empty.*

When $\alpha \neq \beta$, for $P \in \mathscr{P}_1'(A, B)$ both (3. 2) *and* (3. 3), *as operators in $\mathfrak{L}(X \hat{\otimes}_\beta Y, X \hat{\otimes}_\alpha Y)$, have the same closure provided that $0 \notin \overline{P(\sigma(A), \sigma(B))}$, and the closed operator* (3. 5) *has an everywhere defined continuous inverse.*

PROOF. It is possible to show (see [17, Theorem 2. 2]) that if $\lambda \notin \overline{P(\sigma(A), \sigma(B))}$, then $(P(A, \eta) - \lambda I)^{-1}$ is in $L(X)$ and uniformly bounded on $\sigma(B)$. Therefore, Theorem 3. 7 follows immediately from Proposition 3. 1, if we only recall ([15], cf. [16]) that the inclusion

$$\overline{P\big(\sigma(A), \sigma(B)\big)} \subset \sigma\big(P\{A \otimes I, I \otimes B\}\big) = \sigma\big(\tilde{P}\{A \otimes I, I \otimes B\}\big)$$

is valid if $\sigma(A)$ and $\sigma(B)$ are nonempty. Note also that the spectrum of a linear operator is unchanged under maximal extensions, in particular, under the closure operation if it is closable [14]. Q.E.D.

For $P \in \mathscr{P}'(A, B)$, consider the polynomial operators (3. 2) and (3. 3) as operators in

(a) $\mathfrak{L}(X \hat{\otimes}_\pi L)$ $\quad (Y = L,\ \alpha = \beta = \pi)$,
(b) $\mathfrak{L}(X \hat{\otimes}_\varepsilon C)$ $\quad (Y = C,\ \alpha = \beta = \varepsilon)$,
(c) $\mathfrak{L}(H_1 \hat{\otimes}_{\alpha_0} H_2)$ $\quad (X = H_1,\ Y = H_2,\ \alpha = \beta = \alpha_0)$ and
(d) $\mathfrak{L}(X \hat{\otimes}_\varepsilon Y, X \hat{\otimes}_\pi Y)$ $\quad (\alpha = \pi,\ \beta = \varepsilon)$,

corresponding to the four cases in Theorem 2. 2. Here we assume that in cases (a) and (d), π is faithful on $X\otimes L$, $X\otimes Y$, respectively.

Then the following corollary is a direct consequence of Theorem 3.7 combined with Theorem 2.2.

COROLLARY 3.8. *In cases* (a), (b) *and* (c), *for* $P\in\mathscr{P}'(A, B)$ *both* (3.2) *and* (3.3) *have the same closure, provided that* $\overline{P(\sigma(A), \sigma(B))}\neq \boldsymbol{C}$, *and the spectral mapping theorem* (3.12) *holds. In case* (d), *if* $P\in\mathscr{P}_1'(A, B)$ *and* $0\notin\overline{P(\sigma(A), \sigma(B))}$, *both* (3.2) *and* (3.3), *as operators in* $\mathfrak{L}(X\hat{\otimes}_\varepsilon Y, X\hat{\otimes}_\pi Y)$, *have the same closure* (3.5) *which has in turn an everywhere defined continuous inverse.*

3.3. *The case* $P(\xi, \eta)=\xi+\eta$.

We consider the polynomial operators $A\otimes I+I\otimes B$ and $A\hat{\otimes}I+I\hat{\otimes}B$, corresponding to the polynomial $P(\xi, \eta)=\xi+\eta$, in the same situation as in Corollary 3.8, i.e. as operators in (a) $\mathfrak{L}(X\hat{\otimes}_\pi L)$, (b) $\mathfrak{L}(X\hat{\otimes}_\varepsilon C)$, (c) $\mathfrak{L}(H_1\hat{\otimes}_{\alpha_0}H_2)$ and (d) $\mathfrak{L}(X\hat{\otimes}_\varepsilon Y, X\hat{\otimes}_\pi Y)$, where we assume that in case (a), π is faithful on $X\otimes L$, and that in case (d), π is also faithful on $X\otimes Y$.

Then from Proposition 3.1 and Corollary 3.8 follows immediately the following corollary, which extends the results of Ju. M. Berezanskiĭ [3] and L. and K. Maurin [22] (cf. [16], [17]).

COROLLARY 3.9. *In cases* (a), (b) *and* (c), *suppose that if* $\sigma(A)$ *is not empty one has* $\|R(\xi; A)\|\leq C_\delta$ *outside* $U_\delta=\{\xi;\ dist\,(\xi, \sigma(A))<\delta\}$ *for any* $\delta>0$, *and that if* $\sigma(A)$ *is empty, for any* $R>0$ *there exists a nonempty open set* U *for which* $U+\sigma(B)\subset \complement K(0; R)$ *and* $\|R(\xi; A)\|\leq C_R$ *in* $\complement U$, *with constants* C_δ *and* C_R *depending only on* δ, R, *respectively. Then* $A\otimes I+I\otimes B$ *and* $A\hat{\otimes}_\alpha I+I\hat{\otimes}_\alpha B$ *have the same closure and it holds*

$$\begin{aligned}\overline{\sigma(A)+\sigma(B)} &= \overline{\sigma(A\hat{\otimes}_\alpha I)+\sigma(I\hat{\otimes}_\alpha B)}\\ &=\sigma(A\otimes I+I\otimes B)=\sigma\big((A\otimes I+I\otimes B)^{\sim}\big)\\ &=\sigma(A\hat{\otimes}_\alpha I+I\hat{\otimes}_\alpha B)=\sigma\big(((A\hat{\otimes}_\alpha I+(I\hat{\otimes}_\alpha B))^{\sim}\big),\end{aligned}$$

where in case (a): $\alpha=\pi$, (b): $\alpha=\varepsilon$ *and* (c): $\alpha=\alpha_0$.

In case (d), *if there exists a nonempty open set* U *with* $\complement U\subset\rho(A)$ *whose boundary* ∂U, *restricted to* $K(0; R)$ *for each* $R>0$, *consists of a finite number of rectifiable Jordan arcs and has a length of* $O(R)$ *such that the complement of* $U+\sigma(B)$ *includes a neighbourhood of* 0 *and* $R(\xi; A)$ *is uniformly bounded in* $\complement U$, *then* $A\otimes I+I\otimes B$ *and* $A\hat{\otimes}I+I\hat{\otimes}B$ *have the same closure which has in turn an everywhere defined continuous inverse.*

The following corollary extends Theorem 3.2 in [17] (cf. Theorem 4.6

(1) in [16]), and is shown just in the same way. The sector $\{\xi;\ |\arg\xi|\leq\theta\}$ is denoted by $S(\theta)$.

COROLLARY 3.10. *Consider cases* (a), (b) *and* (c). *Suppose, for* $0\leq\theta_A$, $\theta_B<\pi$ *with* $0\leq\theta_A+\theta_B<\pi$, *that the resolvent set* $\rho(A)$ *includes the complement of the sector* $S(\theta_A)$ *and* $\|\xi(R;A)\|\leq M_\theta$ *outside* $S(\theta)$ *for each* θ *with* $\theta_A<\theta<\pi$, *where* M_θ *is a constant depending only on* θ, *and the resolvent set* $\rho(B)$ *includes the complement of the sector* $S(\theta_B)$.

Then the closure of the operator $A\otimes I+I\otimes B$ *coincides with* $A\hat{\otimes}I+I\hat{\otimes}B$. *It holds*

$$\begin{aligned}\sigma(A)+\sigma(B)&=\sigma(A\hat{\otimes}_\alpha I)+\sigma(I\hat{\otimes}_\alpha B)\\&=\sigma(A\otimes I+I\otimes B)=\sigma\big((A\otimes I+I\otimes B)^{\sim}\big)\\&=\sigma(A\hat{\otimes}_\alpha I+I\hat{\otimes}_\alpha B),\end{aligned}$$

where in case (a): $\alpha=\pi$, (b): $\alpha=\varepsilon$ *and* (c): $\alpha=\alpha_0$.

4. Applications

In [16], we have already indicated an application to the first boundary value problem of a class of quasi-elliptic differential equations such as considered by V. P. Mihaĭlov [25].

In the present paper, we are concerned with the initial value problem of the abstract wave equation

$$P[u]=[d^2/dt^2+\Lambda]u(t)=f(t) \tag{4.1}$$

in a bounded interval $(0, T)$, $T>0$, with the initial condition

$$u(0)=u_t(0)=0, \tag{4.2}$$

where Λ is a nonnegative selfadjoint operator in a Hilbert space H.

4.1. *Banach spaces of vector-valued distributions.*

We introduce some Banach spaces of vector-valued distributions, some of which to be represented by tensor products of Banach spaces. The equalities indicate isometrical isomorphisms between the spaces involved with their usual norms.

Let $I=[0, T]$ be a bounded closed interval and $B(I)$ the Borel field of I. Let Y be a complex Banach space, whose norm is denoted by $\|\cdot\|$.

(a) $L_p(I, Y)$, $1\leq p<\infty$, *and* $C(I, Y)$.

By $L_p(I, Y)$, $1\leq p<\infty$, we denote the Banach space of all (equivalence classes of) vector-valued, p-th power integrable functions $f(t)\colon I\to Y$, equipped with norm $\|f\|_{L_p}=(\int_0^T\|f(t)\|^p dt)^{1/p}$.

By $C(I, Y)$, we denote the Banach space of all vector-valued continuous functions $f(t)$: $I\to Y$, equipped with norm $\|f\|_C=\sup_{t\in I}\|f(t)\|$.

Set $L_p(I, \boldsymbol{C})=L_p(I)$ and $C(I, \boldsymbol{C})=C(I)$.

Then these spaces are represented as follows [11]:

$$L_1(I, Y)=L_1(I)\hat{\otimes}_\pi Y; \tag{4.3}$$

$$C(I, Y)=C(I)\hat{\otimes}_\varepsilon Y; \tag{4.4}$$

$$L_2(I, H)=L_2(I)\hat{\otimes}_{\alpha_0} H, \tag{4.5}$$

where H is a Hilbert space.

(b) *$M(I, Y)$ and $M^1(I, Y)$.*

Given a mapping m: $B(I)\to Y$, we define for $y'\in Y'$ a set function $\mu_{y'}$ by $\mu_{y'}(e)=<\mu(e), y'>$ for $e\in B(I)$. We say that m is a *regular vector-valued measure* on I (cf. [6, Part I, Chap 4, 10]) if for each $y'\in Y'$, $\mu_{y'}$ is a regular scalar-valued measure on I. A vector-valued measure m on I said to be *of bounded semi-variation* if

$$\|m\|_M=\sup\|\textstyle\sum_i\alpha_i m(e_i)\|$$

is finite, and *of bounded variation* if

$$\|m\|_{M^1}=\sup\textstyle\sum_i\|m(e_i)\|$$

is finite, where the supremum is taken over all finite partitions $\{e_i\}$ of I into disjoint Borel sets and all collections $\{\alpha_i\}$ of complex numbers with $|\alpha_i|\leq 1$ (cf. [5]).

The space $M(I, Y)$ of all regular vector-valued measures m: $B(I)\to Y$ of bounded semi-variation is a Banach space under the norm $\|m\|_M$. The space $M^1(I, Y)$ of all $m\in M(I, Y)$ of bounded variation is a Banach space under the norm $\|m\|_{M^1}$.

If $Y=\boldsymbol{C}$, then both $M(I, \boldsymbol{C})$ and $M^1(I, \boldsymbol{C})$ are nothing but the space $M(I)$ of all regular scalar-valued measures on I of bounded variation, which is the dual space of $C(I)$: $M(I)=C(I)'$.

J. Gil de Lamadrid [8] has shown that $M(I)\hat{\otimes}_\varepsilon Y$ is a closed subspace of $M(I, Y)$, and that if Y is reflexive, then

$$M(I, Y)=L\big(C(I), Y\big)=\big(C(I)\hat{\otimes}_\pi Y'\big)' \tag{4.6}$$

and

$$M^1(I, Y)=C(I, Y')'=\big(C(I)\hat{\otimes}_\varepsilon Y'\big)'=M(I)\hat{\otimes}_\pi Y. \tag{4.7}$$

(c) *$BV(I, Y)$ and $BV^1(I, Y)$.*

A vector-valued function $f(t)$: $I \to Y$ is said to be *of bounded semi-variation* if

$$\|f\|_{BV} = \sup \left\| \sum_i \alpha_i \Big(f(t_i) - f(t_{i-1}) \Big) \right\|$$

is finite, and *of bounded variation* if

$$\|f\|_{BV^1} = \sup \sum_i \|f(t_i) - f(t_{i-1})\|$$

is finite, where the supremum is taken over all finite partitions Δ: $0 = t_0 \leq t_1 \leq \cdots \leq t_r = T$ of I and all collections $\{\alpha_i\}$ of complex numbers with $|\alpha_i| \leq 1$. Our concepts of bounded semi-variation and bounded variation of $f(t)$ are coincident with those of bounded variation and strong bounded variation, respectively, in [13, Chap. III, 3. 2, Def. 3. 2. 4].

A vector-valued function $f(t)$: $I \to Y$ of bounded semi-variation (resp. bounded variation) is *normalized* if $f(t)$ is continuous on the right at each interior point of I in the weak topology (resp. in the norm) and $w\text{-}\lim_{t \to +0} f(t) = 0$ (resp. $s\text{-}\lim_{t \to +0} f(t) = 0$).

The space $BV(I, Y)$ (resp. $BV^1(I, Y)$) of all normalized vector-valued functions $f(t)$: $I \to Y$ of bounded semi-variation (resp. bounded variation) is a Banach space under the norm $\|f\|_{BV}$ (resp. $\|f\|_{BV^1}$).

If $Y = \boldsymbol{C}$, then both $BV(I, \boldsymbol{C})$ and $BV^1(I, \boldsymbol{C})$ is the space $BV(I)$ of all normalized functions of bounded variation on I.

Observe that $BV(I) \otimes Y$ can be considered to be an algebraic subspace of $BV(I, Y)$ and $BV^1(I, Y)$ by the identification

$$\sum_k x_k(t) \otimes y_k = \sum_k x_k(t) y_k \,. \tag{4. 8}$$

We claim that $BV(I) \hat{\otimes}_\varepsilon Y$ is a closed subspace of $BV(I, Y)$. To see this, it suffices to show that the norm of $BV(I, Y)$ induces the norm ε on $BV(I) \otimes Y$. Let $u \in BV(I) \otimes Y$ be represented as in (4. 8). Then

$$\begin{aligned} \|u\|_{BV} &= \|\sum_k x_k(t) y_k\|_{BV} \\ &= \sup_\Delta \left\| \sum_i \alpha_i \sum_k \Big(x_k(t_i) - x_k(t_{i-1}) \Big) y_k \right\| \\ &= \sup_\Delta \sup_{\substack{y' \in Y' \\ \|y'\| \leq 1}} \left| < \sum_i \alpha_i \sum_k \Big(x_k(t_i) - x_k(t_{i-1}) \Big) y_k,\, y' > \right| \end{aligned}$$

where the supremum in the above two expressions is taken over all finite partitions Δ: $0 = t_0 \leq t_1 \leq \cdots \leq t_r = T$ of I and all collections $\{\alpha_i\}$ of complex numbers with $|\alpha_i| \leq 1$. Hence

$$\|u\|_{BV} = \sup_{\Delta} \sup_{\substack{y' \in Y' \\ \|y'\| \leq 1}} \sum_i \left| \sum_k \left(x_k(t_i) - x_k(t_{i-1}) \right) < y_k, y' > \right|$$

$$= \sup_{y' \in Y', \|y'\| \leq 1} \| \sum_k x_k < y_k, y' > \|_{BV(I)}$$

$$= \|u\|_\varepsilon .$$

This proves our claim.

As concerns $BV^1(I, Y)$, if Y is reflexive, the following representation holds:

$$BV^1(I, Y) = BV(I) \hat{\otimes}_\pi Y . \tag{4.9}$$

For proof, note that, by the representation theorems of Riesz type (see e.g. [2]), $M(I, Y) = C(I, Y')'$ is isometrically isomorphic to $BV^1(I, Y)$. In particular, there exists an isometrical isomorphism T: $M(I) \rightarrow BV(I)$, so that $M(I) \hat{\otimes}_\pi Y$ is isomorphic to $BV(I) \hat{\otimes}_\pi Y$ (use [11, § 1, n°2, Proposition 4]). Thus to prove (4.9), it suffices to show that the norm of $BV^1(I, Y)$ induces the norm π on $BV(I) \otimes Y$. It is easy to verify $\|u\|_{BV^1} \leq \|u\|_\pi$ for $u \in BV(I) \otimes Y$.

To prove the reverse inequality, let $u \in BV(I) \otimes Y$ be represented as in (4.8), where we may assume both the x_k and the y_k are linearly independent. First we show that u can be approximated by elements of $BV(I) \otimes Y$ of the form

$$\text{v} = \sum_{j=1}^s \xi_j(t) \otimes \eta_j = \sum_{j=1}^s \xi_j(t) \eta_j \tag{4.10}$$

satisfying $\|v\|_{BV^1} = \|v\|_\pi$ and

$$\xi_j(t) = \begin{cases} 0, & \text{for } 0 \leq t \leq t_{j-1}, \\ \xi_j(t_j), & \text{for } t_j \leq t \leq T \qquad (1 \leq j \leq s), \end{cases}$$

for some finite partition Δ: $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_s = T$ of I.

Set $\mu_k = T^{-1} x_k \in M(I)$ for each k, so that $m = \sum_k \mu_k \otimes y_k \in M(I) \otimes Y$.

Let $\delta > 0$. In virtue of a lemma due to P. Saphar[2]), there exists $\nu \in M(I) = C(I)'$, a partition Δ: $0 = t_0 \leq t_1 \leq \cdots \leq t_s = T$ of I and $\{\eta_j\}_{j=1}^s \subset Y$ such that

$$\|m - m_\delta\|_\pi < \delta , \quad m_\delta = \sum_j \chi_j \nu \otimes \eta_j ,$$

where $\chi_j(t)$ is a characteristic function of the interval $[t_{j-1}, t_j)$ in I.

Set $T(\chi_j \nu) = \xi_j$ and $u_\delta = (T \otimes I) m_\delta = \sum_{j=1}^s \xi_j(t) \otimes \eta_j$. Note the $\xi_j(t)$ satisfy

2) See Lemma 3.2 in P. Saphar: Produits tensoriels d' espaces de Banach et classes d' applications linéaires, Studia Math. 38, 71-100 (1970).

the required condition. Then we have

$$\|u-u_\delta\|_\pi \leq \|T\| \|m-m_\delta\|_\pi < \|T\|\delta = \delta .$$

It is easy to show $\|u_\delta\|_{BV^1} \geq \|u_\delta\|_\pi$, whence $\|u_\delta\|_{BV^1} = \|u_\delta\|_\pi$.

Thus there exists a sequence $\{u_n\}_{n=1}^\infty$ in $BV(I)\otimes Y$ of the form (4. 10) which converges to u in the norm π. As it is a Cauchy sequence in the norm π, so is it in the norm of $BV^1(I, Y)$. It follows that $\|u\|_{BV^1} = \|u\|_\pi$. This proves (4. 9).

4. 2. *The initial value problem* (4. 1) *and* (4. 2).

We consider the initial value problem (4. 1) and (4. 2).

Let s be real if $0 \notin \sigma(\Lambda)$, otherwise $s \geq 0$. The domain $H^s = D[\Lambda^{s/2}] \subset H$ of $\Lambda^{s/2}$ is a Hilbert space under the norm $\|u\|_s = \|u\|_H + \|\Lambda^{s/2}u\|_H$, where $\|\cdot\|_H$ denotes the norm of H.

We shall employ the following notations:

$$L_{p,2}^{0,s}(I;\Lambda) = L_p(I, H^s) ;$$
$$C^{0,s}_{2}(I;\Lambda) = C(I, H^s) ;$$
$$M^{0,s}_{2}(I;\Lambda) = M^1(I, H^s) ;$$
$$B^{0,s}_{2}(I,\Lambda) = BV(I, H^s) .$$

Further for $X = L_p$, C, M and B, let

$$X^{1,s}_{2}(I;\Lambda) = \left\{f;\, f\in X^{0,s+1}_{2}(I;\Lambda),\ df/dt \in X^{0,s}_{2}(I;\Lambda)\right\}.$$

Then, applying Theorem 3. 3 or Corollary 3. 6, we obtain Theorem 4. 1 below. L. Gårding and J. Leray proved (unpublished) the statement (1) and a stronger one than (2), with $B^{1,s}_{2}(I;\Lambda)$ replaced by

$$\left\{f;\, f\in BV^1(I, H^{s+1}),\ df/dt \in BV^1(I, H^s)\right\},$$

for the initial value problem of a strongly hyperbolic differential equation in a strip. The statement (3) is well-known.

By P we denote also the closure of the operator associated with the problem (4. 1) and (4. 2) in the respective spaces.

THEOREM 4. 1. *P^{-1} is a continuous linear mapping*

(1) *of $L_{1,2}^{0,s}(I;\Lambda)$ into $C^{1,s}_{2}(I;\Lambda)$,*

(2) *of $M^{0,s}_{2}(I;\Lambda)$ into $B^{1,s}_{2}(I;\Lambda)$, and*

(3) *of $L_{2,2}^{0,s}(I;\Lambda)$ into $L_{2,2}^{1,s}(I;\Lambda)$.*

REMARK 4. 2. Note it is easy to verify by Corollary 3. 9 (cf. [17]) that P^{-1} is a continuous linear mapping

(1) of $L^{0;s}_{1;2}(I; \Lambda)$ into $L(I)\hat{\otimes}_\varepsilon H^s$,

(2) of $M^{0;s}_{2}(I; \Lambda)$ into $M(I)\hat{\otimes}_\varepsilon H^s$, and

(3) of $L^{0;s}_{2;2}(I; \Lambda)$ into itself.

PROOF of THEOREM 4.1. Let X, X_1 be, respectively, $L_1(I)$, $C(I)$; $M(I)$, $BV(I)$ and $L_2(I)$, $L_2(I)$. We see $X_1 \subsetneq X$ in these three cases. Let $A = d/dt$ in X with domain

$$D[A] = \left\{\varphi \in X;\ d\varphi/dt \in X,\ \varphi(0) = 0\right\},$$

and let $B = \Lambda^{1/2}$ in H^s with domain

$$D[B] = D[\Lambda^{1/2}] = \{h \in H^s;\ \Lambda^{1/2}h \in H^s\}.$$

Then A is a densely defined closed linear operator in X with empty spectrum $\sigma(A)$, and B is a selfadjoint in H^s with the spectrum $\sigma(B)$ being a closed subset of the nonnegative real line.

Observe the following tensor product representations of the spaces as in Section 4.1:

$$L^{0;s}_{1;2}(I; \Lambda) = L_1(I)\hat{\otimes}_\pi H^s;$$
$$M^{0;s}_{2}(I; \Lambda) = M(I)\hat{\otimes}_\pi H^s;$$
$$L^{0;s}_{2;2}(I; \Lambda) = L_2(I)\hat{\otimes}_{\alpha_0} H^s.$$

In our setting, P is considered as the closure of $A^2\hat{\otimes}I + I\hat{\otimes}B^2$ in the respective spaces.

Let $\delta > 0$. Then it is easy to verify that $|\xi^2 + \eta^2| \geq \delta^2$ for $(\xi, \eta) \in U \times \sigma(B)$, where $U = \{\xi;\ \mathrm{Re}\,\xi > \delta\}$, and $R(\xi; A)$ is uniformly bounded in $\complement U = \{\xi;\ \mathrm{Re}\,\xi > \delta\}$.

If $\eta \geq 0$, we have for $\varphi \in X$

$$(A^2 + \eta^2 I)^{-1}\varphi = \int_0^t \eta^{-1} \sin \eta(t-s)\varphi(s)ds$$

and

$$A(A^2 + \eta^2 I)^{-1}\varphi = \int_0^t \cos \eta(t-s)\varphi(s)ds,$$

$$\eta(A^2 + \eta^2 I)^{-1}\varphi = \int_0^t \sin \eta(t-s)\varphi(s)ds.$$

In all three cases, it is shown that $(A^2 + \eta^2 I)^{-1}$ belongs to $L(X)$, and that all $(A^2 + \eta^2 I)^{-1}$, $A(A^2 + \eta^2 I)^{-1}$ and $\eta(A^2 + \eta^2 I)^{-1}$ belong to $L(X, X_1)$ for $\eta \geq 0$ and are uniformly bounded in $L(X, X_1)$ for $\eta \geq 0$.

Let A_1 be the operator in X_1 obtained from A by restricting the domain and range of A to X_1 and let $I_1=I|X_1$.

(1) In case $X=L_1(I)$ and $X_1=C(I)$, we see in virtue of Theorem 3.3 or Corollary 3.6 for $\alpha=\pi$ and $\beta=\varepsilon$ that P^{-1} is a one-to-one continuous linear mapping of $L_{1;2}^{0;s}(I;\Lambda)$ onto $D[P]\subset D[A_1\hat{\otimes}_\varepsilon I]\cap D[I_1\hat{\otimes}_\varepsilon B]\subset C(I)\hat{\otimes}_\varepsilon H^s =C^{0;s}_{\ 2}(I;\Lambda)$, where $D[P]$ is equipped with the norm

$$\|u\|_C+\|A_1\hat{\otimes}_\varepsilon Iu\|_C+\|I_1\hat{\otimes}_\varepsilon Bu\|_C=\|u\|_C+\|du/dt\|_C+\|\Lambda^{1/2}u\|_C.$$

Since $D[A_1\hat{\otimes}_\varepsilon I]\cap D[I_1\hat{\otimes}_\varepsilon B]\subset C^{1;s}_{\ 2}(I;\Lambda)$, we have the assertion for (1).

(2) In case $X=M(I)$ and $X_1=BV(I)$, it is seen similarly for $\alpha=\pi$ and $\beta=\varepsilon$ that P^{-1} is a one-to-one continuous linear mapping of $M^{0;s}_{\ 2}(I;\Lambda)$ onto $D[P]\subset D[A_1\hat{\otimes}_\varepsilon I]\cap D[I_1\hat{\otimes}_\varepsilon B]\subset BV(I)\hat{\otimes}_\varepsilon H^s\subsetneq B^{0;s}_{\ 2}(I;\Lambda)$, where $D[P]$ is equipped with norm

$$\|u\|_{BV}+\|A_1\hat{\otimes}_\varepsilon Iu\|_{BV}+\|I_1\hat{\otimes}_\varepsilon Bu\|_{BV}=\|u\|_{BV}+\|du/dt\|_{BV}+\|\Lambda^{1/2}u\|_{BV}.$$

Since $D[A_1\hat{\otimes}_\varepsilon I]\cap D[I_1\hat{\otimes}_\varepsilon B]\subset B^{1;s}_{\ 2}(I;\Lambda)$, the assertion for (2) has been shown.

(3) In case $X=X_1=L_2(I)$, applying Theorem 3.3 or Corollary 3.6 for $\alpha=\beta=\alpha_0$, we see that P^{-1} is a one-to-one continuous linear mapping of $L_{2;2}^{0;s}(I;\Lambda)$ onto $D[P]\subset D[A\hat{\otimes}_{\alpha_0}I]\cap D[I\hat{\otimes}_{\alpha_0}B]\subset L_2(I)\hat{\otimes}_{\alpha_0}H^s=L_{2;2}^{0;s}(I;\Lambda)$, where $D[P]$ is equipped with the norm

$$\|u\|_{L_2}+\|A\hat{\otimes}_{\alpha_0}Iu\|_{L_2}+\|I\hat{\otimes}_{\alpha_0}Bu\|_{L_2}=\|u\|_{L_2}+\|du/dt\|_{L_2}+\|\Lambda^{1/2}u\|_{L_2}.$$

The assertion for (3) follows, since $D[A\hat{\otimes}_{\alpha_0}I]\cap D[I\hat{\otimes}_{\alpha_0}B]$ is included in $L_{2;2}^{1;s}(I;\Lambda)$. Q.E.D.

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