A note on minimal submanifolds in Riemannian manifolds

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In this note we shall prove the following: Let $\bar{M}^{n+p}$ be a Riemannian manifold of constant curvature $\bar{c}$, and let M^n be a minimal submanifold in $\bar{M}$ of constant curvature c. Then either M is totally geodesic, i.e. $\bar{c}=c$, or $\bar{c}\geqq(2p-n+1)c/(p-n+1)$, in the latter case the equality arising only when $\bar{c}>0$. Our method is based on the Simons' type formula which has been given by Simons [4].

On the other hand, we shall study the Laplacian of the Ricci operator of a minimal submanifold of codimension 1 in a Riemannian manifold of constant curvature and give some inequality. And combing the theorems of Lawson [2], we shall prove some theorems for compact minimal hypersurfaces in a unit sphere.

1. Preliminaries

In this section we shall summarize the basic formulas for submanifolds in Riemannian manifolds.

Let $\bar{M}$ be a Riemannian manifold of dimension $n+p$, and let M be a submanifold of $\bar{M}$ of dimension n. Let $\langle\,,\rangle$ be the metric tensor field of $\bar{M}$ as well as the metric induced on M. We denote by $\bar{\nabla}$ the covariant differentiation in $\bar{M}$ and by ∇ the covariant differentiation in M determined by the induced metric on M. Then the Gauss-Weingarten formulas are given by

$$\bar{\nabla}_X Y=\nabla_X Y+B(X, Y), \qquad X, Y\in\mathfrak{X}(M),$$
$$\bar{\nabla}_X N=-A^N(X)+D_X N, \quad X\in\mathfrak{X}(M), \quad N\in\mathfrak{X}(M)^{\perp}$$

Where D is the linear connection in the normal bundle $T(M)^{\perp}$. We call A and B the second fundamental form of M and they satisfy $<B(X, Y), N>=<A^N(X), Y>$. The Riemannian curvature tensors of $\bar{M}$ and M will be denoted by $\bar{R}$ and R respectively. From the Gauss-Weingarten formulas, we have

$$\bar{R}_{X,Y}Z=R_{X,Y}Z-A^{B(Y,Z)}(X)+A^{B(X,Z)}(Y)+(\tilde{\nabla}_X B)(Y, Z)-(\tilde{\nabla}_Y B)(X, Z),$$

where $\tilde{\nabla}$ denotes the covariant differentiation for B. And we obtain the

Gauss-equation

$$(1.1)\qquad <\bar{R}_{X,Y}Z, W> = <R_{X,Y}Z, W> - <B(Y,Z), B(X,W)> + <B(X,Z), B(Y,W)> .$$

If $\bar{M}$ is of constant curvature, the Codazzi-equation is satisfied, that is $(\tilde{\nabla}_X B)(Y,Z)=(\tilde{\nabla}_Y B)(X,Z)$, and we have

$$(1.2)\qquad \bar{R}_{X,Y}Z = R_{X,Y}Z - A^{B(Y,Z)}(X) + A^{B(X,Z)}(Y) .$$

Let $e_1, \cdots, e_n$ be a frame for $T_m(M)$. Then the mean curvature K of M is defined by $K=\sum_{i=1}^{n} B(e_i, e_i)$. If $K=0$, a submanifold M is said to be minimal in $\bar{M}$. Let $v_1, \cdots, v_p$ be a frame for $T_m(M)^{\perp}$. Here we assume that $\bar{M}$ is of constant curvature $\bar{c}$, and M is minimal in $\bar{M}$. Then the Ricci tensor S of M is given by

$$(1.3)\qquad S(x,y) = (n-1)\bar{c}<x,y> - \sum_{i=1}^{p} <A^i A^i(x), y>$$

where $x, y \in T_m(M)$ and we denote A^i instead of A^{v_i} to simplify. From this the scalar curvature Sc of M is represented by

$$(1.4)\qquad Sc = n(n-1)\bar{c} - \|A\|^2$$

where $\|A\|$ is the length of the second fundamental form. If the second fundamental form is identically zero, M is said to be totally geodesic in M.

2. Minimal submanifolds of constant curvature

In this section we prove the following.

THEOREM 2.1. *Let $\bar{M}$ be a Riemannian manifold of dimension $n+p$ and constant curvature $\bar{c}$, and let M be a minimal submanifold of $\bar{M}$ of dimension n and constant curvature c. If $p>n-1$, then either M is totally geodesic, i.e. $\bar{c}=c$, or $\bar{c} \geqq (2p-n+1)c/(p-n+1)$, in the latter case the equality arising only when $\bar{c}>0$. If $p=n-1$, then M is flat.*

PROOF. Since M is minimal, the second fundamental form A of M satisfies (cf. [5], p 93)

$$\nabla^2 A = -A \circ \tilde{A} - \underset{\sim}{A} \circ A + n\bar{c}A$$

where the operators $\tilde{A}$ and $\underset{\sim}{A}$ are defined by setting

$$\tilde{A} = {}^t A \circ A \quad \text{and} \quad \underset{\sim}{A} = \sum_{i=1}^{p} adA^i adA^i .$$

If $\bar{M}$ and M are both of constant curvature, then the length of the second fundamental form is constant and we obtain

(2. 1) $$<\nabla A, \nabla A> = <A\circ\tilde{A}, A> + <\underset{\sim}{A},\circ A, A> - n\bar{c}\|A\|^2 .$$

Let $x, y\in T_m(M)$ and $w\in T_m(M)^{\perp}$. From (1. 2), we have

$$<A^{\tilde{A}(w)}(x), y> = \sum_{j=1}^{p} <A^j A^w A^j(x), y> + (\bar{c}-c)<A^w(x), y> ,$$

which implies

(2. 2) $$<A\circ\tilde{A}, A> = \sum_{t=1}^{n}\sum_{i,j=1}^{p} <A^j A^i A^j(e_t), A^i(e_t)> + (\bar{c}-c)\|A\|^2 .$$

On the other hand, we can see

(2. 3) $$<\underset{\sim}{A}\circ A, A> = \sum_{i,j=1}^{p} \|[A^i, A^j]\|^2$$
$$= 2\sum_{t=1}^{n}\sum_{i,j=1}^{p} (<A^j A^j A^i(e_t), A^i(e_t)> - <A^j A^i A^j(e_t), A^i(e_t)>) .$$

By (1. 3), the first term of the right hand side of (2. 3) becomes $2(n-1)(\bar{c}-c)\|A\|^2$ and consequently (2. 1), (2. 2) and (2. 3) imply

(2. 4) $$<\nabla A, \nabla A> = (\bar{c}-2c)n\|A\|^2 - <A\cdot\tilde{A}, A> .$$

Since $\tilde{A}$ is symmetric, positive semi-definite operator, we can choose a frame $v_1\cdots v_p$ in $T_m(M)^{\perp}$ such that

$$\tilde{A}(v_i) = \lambda_i^2 v_i \quad \text{and} \quad \|A\|^2 = \sum_{i=1}^{p}\lambda_i^2 .$$

Then we have the following

$$<A\circ\tilde{A}, A> = \sum_{i=1}^{p}\lambda_i^4 \geqq \frac{1}{p}\left(\sum_{i=1}^{p}\lambda_i^2\right)^2 = \frac{1}{p}\|A\|^4 .$$

Noticing that $<\underset{\sim}{A}\circ A, A>\geqq 0$, we have $<A\circ\tilde{A}, A>\leqq \dfrac{1}{n-1}\|A\|^4$ by (2. 2) and (2. 3). Therefore if $p<n-1$, $<A\circ\tilde{A}, A>=0$, which shows that M is totally geodesic in $\bar{M}$. If $p=n-1$, then $<\underset{\sim}{A}\circ A, A>=0$ and M has trivial normal connection and moreover M is flat (see Cartan, Oeuvres Completes, partie III, vol. 1, p. 417 and John Moore's Berkeley Thesis).

Let $p>n-1$. Then the equation (2. 4) implies the following

(2. 5) $$<\nabla A, \nabla A> \leqq \frac{n}{p}\Big((p-n+1)\bar{c} - (2p-n+1)c\Big)\|A\|^2 .$$

Suppese $\bar{c}\leqq(2p-n+1)c/(p-n+1)$. Then the right hand side of this inequality is zero. Therefore M is totally geodesic, *i.e.* $\bar{c}=c$, or $\bar{c}=(2p-n+1)c/(p-n+1)$. Since $\bar{c}\geqq c$ always, the latter case arising only when $\bar{c}>0$. Except for these posibilities, we obtain $\bar{c}>(2p-n+1)c/(p-n+1)$. This completes our assertion.

COROLLARY 2. 2. *Under the same assumption as in Theorem* 2. 1, *if*

$p=n$, then M is totally geodesic, or $\bar{c}\geqq(n+1)c$, in the latter case the equality arising only when $\bar{c}>0$.

REMARK: Len M^n be a compact minimal submanifold in a unit sphere S^{n+p} of constant curvature c satisfyisg $c=(p-n+1)/(2p-n+1)$. If $n=2$, then by the main theorem of Chern, do Carmo and Kobayashi [1], M is the Veronese surface and $c=1/3$.

3. Minimal hypersurfaces

First we prepare some lemmas for latter use.

Let $\bar{M}^{n+1}$ be a Riemannian manifold of constant curvature $\bar{c}$, and let M^n be a minimal hypersurface of $\bar{M}$. We denote by Q the Ricci operator of M, which satisfies $S(x, y)=\langle Qx, y\rangle$. Generally we have the following

LEMMA 3.1 (Nomizu [3]). *If the Ricci operator Q satisfies the Codazzi-equation*

$$(3.1) \qquad (\nabla_X Q)Y=(\nabla_Y Q)X, \quad X, Y\in\mathfrak{X}(M),$$

then the scalar curvature Sc is constant.

REMARK: The Ricci operator Q satisfies the Codazzi-equation if and only if $(\nabla_X S)(Y, Z)=(\nabla_Y S)(X, Z)$ for any $X, Y, Z\in\mathfrak{X}(M)$.

LEMMA 3.2. *Let $x, y\in T_m(M)$, and let $e_1, \cdots, e_n$ be a frame for $T_m(M)$. If the Ricci operator Q satisfies the Codazzi-equation, then we have*

$$(3.2) \qquad \nabla^2(S)(x, y)=\sum_{i=1}^{n} R_{e_i,x}(S)(e_i, y).$$

PROOF. Let $E_1, \cdots, E_n$ be local, orthonormal vector fields which extend $e_1, \cdots, e_n$, and which are covariant constant with respect to ∇ at $m\in M$. Let X, Y be local extensions of x, y which are also covariant constant with respect to ∇. Using (3.1) and Lemma 3.1, we have

$$\begin{aligned}\nabla^2(S)(x, y)&=\sum_{i=1}^{n}\nabla_{E_i}\nabla_{E_i}(S)(x, y)=\sum_{i=1}^{n}\nabla_{E_i}\nabla_X(S)(e_i, y)\\&=\sum_{i=1}^{n}\Big(R_{e_i,x}(S)(e_i, y)+\nabla_X\big(\nabla_Y(S)(E_i, E_i)\big)\Big)\\&=\sum_{i=1}^{n}R_{e_i,x}(S)(e_i, y).\end{aligned}$$

Let v be a unit normal. Hereafter we denote A^v by A to simplify. First we have the following

$$\sum_{i=1}^{n}R_{e_i,x}(S)(e_i, y)=-\sum_{i=1}^{n}\Big(S(R_{e_i,x}e_i, y)+S(e_i, R_{e_i,x}y)\Big),$$

and (1. 2) implies

$$(3.3)\qquad \nabla^2(S)(x,y)=-\sum_{i=1}^{n}\left\{\begin{array}{l} S(\bar{R}_{e_i,x}e_i,y)+S(A^{B(x,e_i)}(e_i),y) \\ +S(\bar{R}_{e_i,x}y,e_i)+S(A^{B(x,y)}(e_i),e_i) \\ -S(A^{B(e_i,y)}(x),e_i) \end{array}\right\}.$$

On the other hand, $QA=(n-1)\bar{c}A-A^3=AQ$ by (1.3), and hence

$$-\sum_{i=1}^{n}\Big(S\big(A^{B(x,e_i)}(e_i),y\big)-S\big(A^{B(e_i,y)}(x),e_i\big)\Big)$$
$$=-<QA^2(x),y>+<AQA(x),y>=0.$$

From (1. 3), we obtain

$$-\sum_{i=1}^{n}S\big(A^{B(x,y)}(e_i),e_i\big)=TrA^3<A(x),y>,$$

and we have also

$$-\sum_{i=1}^{n}\Big(S(\bar{R}_{e_i,x}e_i,y)+S(\bar{R}_{e_i,x}y,e_i)\Big)=\bar{c}n<Qx,y>-\bar{c}Sc<x,y>.$$

On the other hand, we can see easily $\nabla^2(S)(x,y)=<\nabla^2(Q)x,y>$ for any $x, y\in T_m(M)$. Consequently (3. 3) implies

$$\nabla^2 Q=\bar{c}(nQ-ScI)+(TrA^3)A.$$

Here we assume that M is cempact and $\bar{c}>0$. Then we have

$$(3.4)\qquad 0\leqq\int_M<\nabla Q,\nabla Q>=-\int_M<\nabla^2 Q,Q>=\int_M\{\bar{c}(Sc^2-n\|Q\|^2)+(TrA^3)^2\}.$$

Using (1. 3) and (1. 4), this becomes

$$(3.5)\qquad \int_M<\nabla Q,\nabla Q>=\int_M\{\bar{c}\big((TrA^2)^2-nTrA^4\big)+(TrA^3)^2\}.$$

Therefore we have the following

THEOREM 3. 1. *Let $\bar{M}^{n+1}$ be a Riemannian manifold of constant curvature $\bar{c}>0$, and let M^n be a compact minimal hypersuface of $\bar{M}$. If the Ricci operator Q of M satisfies the Codazzi-equation, and if the second fundamental form A of M satisfies $\bar{c}(TrA^2)^2+(TrA^3)^2\leqq\bar{c}nTrA^4$, then the Ricci operator Q of M is covariariant constant.*

From this and Theorem 2 of Lawson [2], we obtain the following

COROLLARY 3. 2. *Let M^n be a compact minimal hypersurface in a unit sphere S^{n+1}. If the Ricci operator Q of M satisfies the Codazzi-equation, and if $(TrA^2)^2+(TrA^3)^2\leqq nTrA^4$, then, up to rotations of S^{n+1}, M^n is one of the minimal products of spheres*

$$S^{k}\left(\sqrt{\frac{k}{n}}\right)\times S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right): \qquad k=0, \cdots, \left[\frac{n}{2}\right].$$

THEOREM 3.3. *Let $\bar{M}$ and M be as in Theorem 3.1. If the Ricci operator Q of M satisfies the Codazzi-equation, and if $TrA^3=0$, then M is an Einstein manifold.*

PROOF. By (3.4), we have

$$0\leqq\int_M <\nabla Q, \nabla Q> = \bar{c}\int_M (Sc^2-n\|Q\|^2)\,.$$

But we have always $Sc^2\leqq n\|Q\|^2$, hance we get $Sc^2=n\|Q\|^2$, which shows that M is Einstein.

COROLLARY 3.4. *Let M^n be a compact minimal hypersurface in a unit sphere S^{n+1}. If Q satisfies the Codazzi-equation, and if $TrA^3=0$, then M is totaly geodesic, or $n=2k$, and it is*

$$S^{k}\left(\frac{1}{\sqrt{2}}\right)\times S^{k}\left(\frac{1}{\sqrt{2}}\right).$$

If the sealar curvature Sc of M is constant, and if the Weyl conformal tensor field satisfies the 2nd Bianchi's identity, then the Ricci operator satisfies the Codazzi-equation ([3], p. 344). From this we have the following

COROLLARY 3.5. *Let M^n $(n\geqq 3)$ be a compact minimal hypersurface with constant scalar curvature in a Riemannian manifold $\bar{M}^{n+1}$ of constant curvature $\bar{c}$. If M is conformally flat and $TrA^3=0$, then M is totally geodesic.*

PROOF. If $TrA^3=0$, then M is Einstein and hence M is of constant curvature. Hence by the condition of codimension, M is totally geodesic.

PROPOSITION 3.6. *Let $\bar{M}^{n+1}$ be a Riemannian manifold of constant curvature $\bar{c}<0$, and let M^n be a minimal hypersurface in $\bar{M}$ with parallel Ricci tensor. Then M is Einstein.*

PROOF. If the Ricci tensor of M is parallel, then we get

$$0 = <\nabla Q, \nabla Q> = \bar{c}(Sc^2-n\|Q\|^2)+(TrA^3)^2\,,$$

therefore we obtain

$$0\geqq \bar{c}(n\|Q\|^2-Sc^2)=(TrA^3)^2\geqq 0\,,$$

which shows that $n\|Q\|^2=Sc^2$ and M is Einstein.

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