Product-projective changes of affine connections in a locally product space

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Holomorphically projective changes of affine connections in an almost complex space have been studied by S. Ishihara, T. Ōtsuki, S. Tachibana, Y. Tashiro and others (cf. Chapter XII of [5]). Problems analogous to this arise in an almost product space, and S. Tachibana ([2]) and S. Yamaguchi ([3], [4]) have studied infinitesimal product-projective transformations of a locally decomposable Riemannian space.

We shall devote this paper to study product-projective changes of affine connections in a locally product space.

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§ 1. Locally product spaces and affine connections

Let M_n be an n-dimensional locally product space such that M_n is a locally product space $M_p \times M_q$ of p- and q-dimensional spaces M_p and M_q $(p+q=n)$. Then, M_n is covered by such a system of coordinate neighbourhoods $\{(U, x^h)\}$ that in any intersection of two coordinate neighbourhoods (U, x^h) and $(U', x^{h'})$ we have

$$x^{\alpha'} = x^{\alpha'}(x^{\alpha}) \quad \text{and} \quad x^{\kappa'} = x^{\kappa'}(x^{\kappa}) \tag{1.1}$$

with

$$|\partial_{\alpha} x^{\alpha'}| \neq 0 \quad \text{and} \quad |\partial_{\kappa} x^{\kappa'}| \neq 0 ,$$

where $\partial_h = \partial/\partial x^h$, the indices α, β, γ, δ, ε run over the range $1, 2, \cdots, p$, the indices κ, λ, μ, ν, ω run over the range $p+1, \cdots, p+q$ $(=n)$ and the Latin indices run over the range $1, 2, \cdots, n$. Such a coordinate system will be called a separating coordinate system of M_n.

If we define ϕ_i^h by

$$(\phi_i^h) = \begin{pmatrix} D_\beta^\alpha & 0 \\ 0 & -D_\lambda^\kappa \end{pmatrix} \tag{1.2}$$

in each separating coordinate neighbourhood, ϕ_i^h is a tensor field on M_n and satisfies

(1.3) $$\phi_j^h \phi_i^j = D_i^h ,$$

where D_i^h is a Kronecker's delta.

Conversely, consider an n-dimensional space M_n which admits a tensor field ϕ_i^h ($\neq D_i^h$) satisfying (1.2) for a certain system of coordinate neighbourhoods of M_n, then, this system is a separating coordinate system, therefore, M_n is a locally product space. Such a tensor field ϕ_i^h is called a locally product structure tensor field. From now on, we shall assume that the space is a locally product space.

Now suppose that there is given an affine connection Γ_{ij}^h on a space M_n with a locally product structure tensor field ϕ_i^h. In order that ϕ_i^h is covariantly constant with respect to Γ_{ij}^h, it is necessary and sufficient that locally subspaces M_p and M_q of $M_n = M_p \times M_q$ are parallel in M_n with respect to Γ_{ij}^h, that is, an arbitrary contravariant vector tangent to M_p (resp. M_q), displaced parallelly in any direction, is still tangent to M_p (resp. M_q) (cf. Chapter X of [5]). And it is equivalent to

(1.4) $$\Gamma_{\beta\varepsilon}^{\alpha} = \Gamma_{\kappa\beta}^{\alpha} = \Gamma_{\kappa\lambda}^{\alpha} = \Gamma_{\lambda\alpha}^{\kappa} = \Gamma_{\alpha\lambda}^{\kappa} = \Gamma_{\alpha\beta}^{\kappa} = 0 .$$

Such a symmetric affine connection will be called a ϕ-connection in what follows.

§ 2. ϕ-planar curves

We now consider the curve $x^h = x^h(t)$ in a space M_n with an affine connection Γ_{ij}^h satisfying the ordinary differential equations

(2.1) $$\frac{d^2x^h}{dt^2} + \Gamma_{ij}^h \frac{dx^i}{dt}\frac{dx^j}{dt} = f(t)\frac{dx^h}{dt} + g(t)\phi_i^h \frac{dx^i}{dt}$$

where $f(t)$ and $g(t)$ are certain functions of t, and we will call this curve a ϕ-planar curve. From the theory of ordinary differential equations, it follows that there exists uniquely a ϕ-planar curve through an arbitrary point Q of M_n such that the curve has an arbitrary tangent vector at Q as the vector tangent to the curve at Q.

We see directly from (2.1) that, for a ϕ-connection Γ_{ij}^h, a curve $x^h = x^h(t)$ is a ϕ-planar curve if and only if the 2-plane determined by two vector fields dx^h/dt and $\phi_i^h dx^i/dt$ is parallel along the curve itself. And for a ϕ-connection Γ_{ij}^h on $M_n = M_p \times M_q$, since $\Gamma_{\beta\gamma}^{\alpha}$ and $\Gamma_{\lambda\mu}^{\kappa}$ can be regarded as the connections on M_p and M_q, respectively, then, we see that a curve $x^h = x^h(t)$ is a ϕ-planar curve if and only if the curve $x^\alpha = x^\alpha(t)$ is a geodesic in M_p and the curve $x^\kappa = x^\kappa(t)$ is a geodesic in M_q.

§ 3. Product-projective changes of affine connections

We consider the conditions for two affine connections to have all ϕ-planar curves in common. Such connections will be called to be P-projectively related to each other.

THEOREM 1. *Two symmetric affine connections Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are P-projectively related to each other if and only if*

$$\Gamma^h_{ij} = \bar{\Gamma}^h_{ij} + 2U_{(i}D^h_{j)} + 2V_{(i}\phi^h_{j)} \tag{3.1}$$

holds for certain vector fields U_h and V_h.

In (3.1), parentheses mean taking a symmetric part with respect to indices in a parenthesis, for example,

$$T_{(hij)} = (T_{hij} + T_{ijh} + T_{jhi} + T_{ihj} + T_{jih} + T_{hji})/3\,!\,.$$

PROOF When Γ^h_{ij} is given by (3.1), it is obvious that Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are P-projectively related to each other. Conversely, we suppose that Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ have all ϕ-planar curves in common. Then, from the following equations

$$\frac{d^2x^h}{dt^2} + \Gamma^h_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = f(t)\frac{dx^h}{dt} + g(t)\phi^h_i\frac{dx^i}{dt}$$

and

$$\frac{d^2x^h}{dt^2} + \bar{\Gamma}^h_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = \bar{f}(t)\frac{dx^h}{dt} + \bar{g}(t)\phi^h_i\frac{dx^i}{dt}\,,$$

we have for any point and any vector field W^h

$$S^h_{ij}W^iW^j = f^*W^h + g^*\phi^h_iW^i \tag{3.2}$$

where f^* and g^* depend on both the point and W^h, and $S^h_{ij} = \Gamma^h_{ij} - \bar{\Gamma}^h_{ij}$. Multiplying W^k to (3.2) and taking the skew-symmetric part with respect to indices h and k, we have

$$S^{[h}_{ij}D^{k]}_l W^iW^jW^l = g^*\phi^{[h}_i D^{k]}_j W^iW^j \tag{3.3}$$

where brackets mean taking a skew-symmetric part with respect to indices in a bracket, for example,

$$T_{[hi]} = (T_{hi} - T_{ih})/2\,!\,.$$

If g^* vanishes identically, from the symmetric part of (3.3) with respect to indices i, j and l, we have

$$S^{[h}_{(ij}D^{k]}_{l)} = 0\,. \tag{3.4}$$

Contracting (3.4) by D^l_k, we can obtain

(3.5) $$S^h_{ij} = 2S^k_{k(i}D^h_{j)}/(n+1).$$

Therefore, it follows that Γ^h_{ij} and $\overline{\Gamma}^h_{ij}$ are projectively related to each other, that is, $U_h = S^i_{ih}/(n+1)$ and $V_h = 0$. If g^* does not vanish, eliminating g^* from (3.3), we have

(3.6) $$(S^{[h}_{ij}D^{k]}_l \phi^{[a}_b D^{c]}_d - \phi^{[h}_i D^{k]}_l S^{[a}_{bd} D^{c]}_j) W^i W^j W^l W^b W^d = 0.$$

Taking a symmetric part of (3.6) with respect to indices i, j, l, b and d, we get

(3.7) $$S^{[h}_{(ij}D^{k]}_l \phi^{[a}_b D^{c]}_{d)} - \phi^{[h}_{(i} D^{k]}_l S^{[a}_{bd} D^{c]}_{j)} = 0.$$

Contracting (3.7) by $D^l_k D^d_c \phi^b_a$, we have

(3.8) $$\begin{aligned} -(n^2+2n-\bar{n}^2-2)S^h_{ij} &= 2\left\{S^k_{kl}\phi^l_a + \bar{n}S^k_{ka} - (n+1)\phi^k_l S^l_{ka}\right\} D^a_{(i}\phi^h_{j)} \\ &+ 2\left\{\phi^k_l S^l_{kb}\phi^b_a + \bar{n}\phi^k_l S^l_{ka} - (n+1)S^k_{ka}\right\} D^a_{(i}D^h_{j)} \\ &+ 2\left\{2\phi^h_l S^l_{k(i}\phi^k_{j)} - S^h_{kl}\phi^k_i\phi^l_j\right\} - 2\bar{n}\left\{2S^h_{k(i}\phi^k_{j)} - S^k_{ij}\phi^h_k\right\}. \end{aligned}$$

Therefore, we have

(3.9) $$\begin{cases} S^\alpha_{\beta\gamma} = 2\{S^h_{h(\gamma}D^\alpha_{\beta)} + \phi^h_i S^i_{h(\gamma}D^\alpha_{\beta)}\}/(n+\bar{n}+2), \\ S^\alpha_{\beta\kappa} = (S^h_{h\kappa} + \phi^h_i S^i_{h\kappa})D^\alpha_\beta/(n+\bar{n}), \\ S^\alpha_{\kappa\lambda} = S^\kappa_{\alpha\beta} = 0, \\ S^\kappa_{\lambda\alpha} = (S^h_{h\alpha} - \phi^h_i S^i_{h\alpha})D^\kappa_\lambda/(n-\bar{n}), \quad \text{and} \\ S^\kappa_{\lambda\mu} = 2\{S^h_{h(\mu}D^\kappa_{\lambda)} - \phi^h_i S^i_{h(\mu}D^\kappa_{\lambda)}\}/(n-\bar{n}+2). \end{cases}$$

On the other hand, putting

$$S^h_{ij} = U_i D^h_j + U^*_j D^h_i + V_i \phi^h_j + V^*_j \phi^h_i,$$

we have from (3.9)

$$\begin{cases} U_\alpha + V_\alpha = U^*_\alpha + V^*_\alpha = (S^h_{h\alpha} + \phi^h_i S^i_{h\alpha})/(n+\bar{n}+2), \\ U_\kappa + V_\kappa = U^*_\kappa + V^*_\kappa = (S^h_{h\kappa} + \phi^h_i S^i_{h\kappa})/(n+\bar{n}), \\ U_\alpha - V_\alpha = U^*_\alpha - V^*_\alpha = (S^h_{h\alpha} - \phi^h_i S^i_{h\alpha})/(n-\bar{n}), \quad \text{and} \\ U_\kappa - V_\kappa = U^*_\kappa - V^*_\kappa = (S^h_{h\kappa} - \phi^h_i S^i_{h\kappa})/(n-\bar{n}+2). \end{cases}$$

Hence, we can obtain

$$U_h = U^*_h = \frac{(2p+1)A_h - A_i\phi^i_h}{8p(p+1)} + \frac{(2q+1)B_h + B_i\phi^i_h}{8q(q+1)}$$

and

$$V_h = V^*_h = \frac{(2p+1)A_h - A_i\phi^i_h}{8p(p+1)} - \frac{(2q+1)B_h + B_i\phi^i_h}{8q(q+1)},$$

where $n=p+q$, $\bar{n}=p-q$, $A_h = S^i_{ih} + \phi^i_j S^j_{ih}$ and $B_h = S^i_{ih} - \phi^i_j S^j_{ih}$.

COROLLARY 1. *In order that two symmetric affine connections are P-projectively related to each other, it is necessary and sufficient that the following quantities Π^h_{ij} corresponding to these connections coincide*:

$$\begin{aligned}\Pi^h_{ij} = \Gamma^h_{ij} &- \frac{2(n+1)(n^2+2n-\bar{n}^2)}{(n^2-\bar{n}^2)((n+2)^2-\bar{n}^2)}\{\Gamma^k_{k(i}D^h_{j)} + \phi^k_l \Gamma^l_{k(i}\phi^h_{j)}\}\\ &+ \frac{2\bar{n}(n^2+2n+2-\bar{n}^2)}{(n^2-\bar{n}^2)((n+2)^2-\bar{n}^2)}\{\Gamma^k_{k(i}\phi^h_{j)} + \phi^k_l \Gamma^l_{k(i}D^h_{j)}\}\\ &- \frac{4\bar{n}(n+1)}{(n^2-\bar{n}^2)((n+2)^2-\bar{n}^2)}\{\Gamma^k_{kl}\phi^l_{(i}D^h_{j)} + \phi^k_l \Gamma^l_{km}\phi^m_{(i}\phi^h_{j)}\}\\ &+ \frac{2(n^2+2n+\bar{n}^2)}{(n^2-\bar{n}^2)((n+2)^2-\bar{n}^2)}\{\Gamma^k_{kl}\phi^l_{(i}\phi^h_{j)} + \phi^k_l \Gamma^l_{km}\phi^m_{(i}D^h_{j)}\}.\end{aligned}$$

THEOREM 2. *Let Γ^h_{ij} be a ϕ-connection and $\bar{\Gamma}^h_{ij}$ a symmetric affine connection to be P-projectively related to Γ^h_{ij}. In order for $\bar{\Gamma}^h_{ij}$ to be a ϕ-connection, it is necessary and sufficient that $U_h = \phi^i_h V_i$ in* (3. 1).

PROOF. When we denote by ∇ and $\bar{\nabla}$ the operators of covariant differentiation with respect to Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$, respectively, we have

$$\nabla_j \phi^h_i = \bar{\nabla}_j \phi^h_i + (D^h_j \phi^k_i - D^k_i \phi^h_j)(U_k - \phi^l_k V_l). \tag{3. 10}$$

If $U_h = \phi^i_h V_i$, it follows from (3. 10) that $\bar{\Gamma}^h_{ij}$ is a ϕ-connection. Conversely, if $\nabla_j \phi^h_i$ and $\bar{\nabla}_j \phi^h_i$ vanish, we have

$$(n\phi^j_h - \bar{n}D^j_h)(U_j - \phi^i_j V_i) = 0$$

by virtue of (3. 10). Since the matrix $(n\phi^j_h - \bar{n}D^j_h)$ is regular, we can obtain $U_h = \phi^i_h V_i$.

COROLLARY 2. *Let Γ^h_{ij} be a ϕ-connection and $\bar{\Gamma}^h_{ij}$ a symmetric affine connection to be P-projectively related to Γ^h_{ij}. In order for $\bar{\Gamma}^h_{ij}$ to be a ϕ-connection, it is necessary and sufficient that, in* (3. 1),

$$U_h = AS^i_{ih} + BS^i_{ij}\phi^j_h \quad \textit{and} \quad V_h = BS^i_{ih} + AS^i_{ij}\phi^j_h$$

where $A = (n+2)/((n+2)^2 - \bar{n}^2)$, $B = -\bar{n}/((n+2)^2 - \bar{n}^2)$ *and* $S^h_{ij} = \Gamma^h_{ij} - \bar{\Gamma}^h_{ij}$.

PROOF. If $\bar{\Gamma}^h_{ij}$ is a ϕ-connection, it follows from Theorem 2 that $V_h = \phi^i_h U_i$. Therefore, from (3. 1), we have

$$(3.11)\qquad \begin{cases} U_\alpha = V_\alpha = S^h_{h\alpha}/(n+\bar{n}+2) \text{ and} \\ U_\kappa = -V_\kappa = S^h_{h\kappa}/(n-\bar{n}+2)\,. \end{cases}$$

When we put

$$(3.12)\qquad U_h = AS^i_{ih} + BS^i_{ij}\phi^j_h\,,$$

from (3.11) and (3.12), we get

$$A = (n+2)/\left((n+2)^2-\bar{n}^2\right) \quad\text{and}\quad B = -\bar{n}/\left((n+2)^2-\bar{n}^2\right).$$

Conversely, for such U_h and V_h, it is obvious that $\bar{\Gamma}^h_{ij}$ is a ϕ-connection.

COROLLARY 3. *In order that two ϕ-connections are P-projectively related to each other, it is necessary and sufficient that the following quantities Π^h_{ij} corresponding to these connections coincide*:

$$\Pi^h_{ij} = \Gamma^h_{ij} - \frac{2(n+2)}{(n+2)^2-\bar{n}^2}\{\Gamma^k_{k(i}D^h_{j)} + \Gamma^k_{kl}\phi^l_{(i}\phi^h_{j)}\} + \frac{2\bar{n}}{(n+2)^2-\bar{n}^2}\{\Gamma^k_{k(i}\phi^h_{j)} + \Gamma^k_{kl}\phi^l_{(i}D^h_{j)}\}\,.$$

§ 4. The product-projective curvature tensor field

Let Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ be ϕ-connections to be P-projectively related to each other, $B^h{}_{ijk}$ and $\bar{B}^h{}_{ijk}$ the curvature tensor fields of Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$, respectively, and, B_{hi} and $\bar{B}_{hi}$ the Ricci tensor fields of Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$, respectively. Then, we get the relations

$$(4.1)\qquad B^h{}_{ijk} = \bar{B}^h{}_{ijk} + 2D^h_iP_{[jk]} + 2\phi^h_iP_{l[k}\phi^l_{j]} + 2P_{i[k}D^h_{j]} + 2\phi^l_iP_{l[k}\phi^h_{j]}$$

and

$$(4.2)\qquad B_{hi} = \bar{B}_{hi} + P_{ih} - (n+1)P_{hi} - \bar{n}\phi^j_hP_{ji} + 2\phi^j_h\phi^k_iP_{(jk)}$$

where $P_{hi} = \bar{\nabla}_iU_h - U_hU_i - V_hV_i$.

LEMMA 1. *For ϕ-connections Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ to be P-projectively related to each other, P_{hi} in* (4.1) *and* (4.2) *satisfies*

$$(4.3)\qquad \left((n+\bar{n})^2-4\right)\left((n-\bar{n})^2-4\right)P_{hi} = n(4-n^2+\bar{n}^2)T_{hi} + 2(4-n^2-\bar{n}^2)\{T_{[ih]} + \phi^j_h\phi^k_iT_{(jk)}\} + \bar{n}(4+n^2-\bar{n}^2)\phi^j_hT_{ji} + 4n\bar{n}\{\phi^j_hT_{[ij]} + \phi^j_iT_{(hj)}\}\,,$$

where $T_{hi} = B_{hi} - \bar{B}_{hi}$.

PROOF. From (4.2), we have

$$(4.4)\qquad \begin{cases} P_{\alpha\beta}=((n+\bar{n})T_{\alpha\beta}+2T_{\beta\alpha})/(4-(n+\bar{n})^2), \\ P_{\alpha\kappa}=-T_{\alpha\kappa}/(n+\bar{n}+2), \\ P_{\kappa\alpha}=-T_{\kappa\alpha}/(n-\bar{n}+2), \quad \text{and} \\ P_{\kappa\lambda}=((n-\bar{n})T_{\kappa\lambda}+2T_{\lambda\kappa})/(4-(n-\bar{n})^2). \end{cases}$$

On the other hand, when we put

$$P_{hi}=AT_{hi}+BT_{ih}+\phi_h^j(CT_{ji}+DT_{ij}) \\ +\phi_i^j(ET_{hj}+FT_{jh})+\phi_h^j\phi_i^k(GT_{jk}+HT_{kj}),$$

calculating $P_{\alpha\beta}$, $P_{\alpha\kappa}$, $P_{\kappa\alpha}$ and $P_{\kappa\lambda}$, and comparing to (4. 4), we get

$$A=(n+\bar{n}-1)/2\left(4-(n+\bar{n})^2\right)+(n-\bar{n}-1)/2\left(4-(n-\bar{n})^2\right),$$

$$B=G=H=1/2\left(4-(n+\bar{n})^2\right)+1/2\left(4-(n-\bar{n})^2\right),$$

$$C=(n+\bar{n}-1)/2\left(4-(n+\bar{n})^2\right)-(n-\bar{n}-1)/2\left(4-(n-\bar{n})^2\right), \quad \text{and}$$

$$D=E=F=1/2\left(4-(n+\bar{n})^2\right)-1/2\left(4-(n-\bar{n})^2\right).$$

Therefore, we can obtain (4. 3).

THEOREM 3. *Let Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$ be ϕ-connections to be P-projectively related to each other. Then, the tensor fields defined by*

$$(4.5)\qquad P^h{}_{ijk}=B^h{}_{ijk}+2Q_{i[j}D_{k]}^h+2\phi_i^l Q_{l[j}\phi_{k]}^h+2D_i^h Q_{[kj]}+2\phi_i^h Q_{l[j}\phi_{k]}^l$$

are equal to each other for such connections, where Q_{hi} is defined by

$$(4.6)\qquad \left((n+\bar{n})^2-4\right)\left((n-\bar{n})^2-4\right)Q_{hi} \\ =n(4-n^2+\bar{n}^2)B_{hi}+2(4-n^2-\bar{n}^2)\{B_{[ih]}+\phi_h^j\phi_i^k B_{(jk)}\} \\ +\bar{n}(4+n^2-\bar{n}^2)\phi_h^j B_{ji}+4n\bar{n}\{\phi_h^j B_{[ij]}+\phi_i^j B_{(hj)}\}.$$

We call such a tensor field $P^h{}_{ijk}$ the product-projective curvature tensor field, briefly, $P-P$ curvature tensor field. The proof of Theorem 3 follows from (4. 1) and Lemma 1.

If a tensor field T_{hi} satisfies $\phi_h^j\phi_i^k T_{jk}=T_{hi}$, we say that T_{hi} is pure.

LEMMA 2. *Let Γ_{ij}^h and $\bar{\Gamma}_{ij}^h$ be ϕ-connections to be P-projectively related to each other. The following conditions are equivalent to each other*:

i) *$B_{hi}-\bar{B}_{hi}$ is symmetric and pure.*

ii) *U_h and V_h in* (3. 1) *are gradient vector fields.*

PROOF. From (4. 3) and the definition of P_{hi} in (4. 1), we have

$$P_{[hi]} = \{\bar{n} T_{j[h}\phi^j_{i]} + (n+2) T_{[hi]}\} \big/ \left((n+2)^2 - \bar{n}^2\right),$$

$$P_{j[i}\phi^j_{h]} = \{(n+2) T_{j[h}\phi^j_{i]} + \bar{n} T_{[hi]}\} \big/ \left((n+2)^2 - \bar{n}^2\right),$$

$$P_{[hi]} = \bar{\nabla}_{[i} U_{h]}, \quad \text{and} \quad P_{j[i}\phi^j_{h]} = \bar{\nabla}_{[i} V_{h]},$$

where $T_{hi} = B_{hi} - \bar{B}_{hi}$. By means of these equations, we have

$$T_{[hi]} = T_{j[h}\phi^j_{i]} = 0 .$$

Therefore, the proof is completed.

COROLLARY 4. *Let Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ be ϕ-connections to be P-projectively related to each other. If U_h and V_h in* (3. 1) *are gradient vector fields, i.e., $B_{hi} - \bar{B}_{hi}$ is symmetric and pure, the P–P curvature tensor field has the following form:*

$$P^h{}_{ijk} = B^h{}_{ijk} + \frac{2(n-2)}{(n-2)^2 - \bar{n}^2} \{B_{i[k}D^h_{j]} + \phi^l_i B_{l[k}\phi^h_{j]}\} - \frac{2\bar{n}}{(n-2)^2 - \bar{n}^2} \{B_{i[k}\phi^h_{j]} + \phi^l_i B_{l[k}D^h_{j]}\} . \tag{4.7}$$

The proof of this corollary is followed from Theorem 3 and Lemma 2. And (4. 7) coincides with the product-projective curvature tensor field of [2].

§ 5. A *P*-projectively flat connection.

The ϕ-connection which is P-projectively related to a flat connection will be called the P-projectively flat connection. Now let Γ^h_{ij} be a P-projectively flat connection. Then, since $\bar{B}^h{}_{ijk}$ and $\bar{B}_{hi}$ of a flat connection $\bar{\Gamma}^h_{ij}$ vanish, it follows from Theorem 3 that $\bar{P}^h{}_{ijk}$ vanishes and

$$\begin{aligned} 0 &= \nabla_h P^h{}_{ijk} \\ &= \nabla_k B_{ij} - \nabla_j B_{ik} + \nabla_k Q_{ij} - \nabla_j Q_{ik} + 2\nabla_i Q_{[kj]} \\ &\quad + 2\phi^l_i \{\phi^h_k \nabla_{(h} Q_{l)j} - \phi^h_j \nabla_{(h} Q_{l)k}\} . \end{aligned} \tag{5.1}$$

Therefore, we have

$$\nabla_\gamma B_{\alpha\beta} - \nabla_\beta B_{\alpha\gamma} + 2(\nabla_\gamma Q_{\alpha\beta} - \nabla_\beta Q_{\alpha\gamma}) + 4\nabla_\alpha Q_{[\gamma\beta]} = 0 , \tag{5.2}$$

$$\nabla_\kappa B_{\alpha\beta} - \nabla_\beta B_{\alpha\kappa} - 4\nabla_{(\alpha} Q_{\beta)\kappa} = 0 , \tag{5.3}$$

$$\nabla_\lambda B_{\alpha\kappa} - \nabla_\kappa B_{\alpha\lambda} = \nabla_\beta B_{\kappa\alpha} - \nabla_\alpha B_{\kappa\beta} = 0 , \tag{5.4}$$

$$\nabla_\lambda B_{\kappa\alpha} - \nabla_\alpha B_{\kappa\lambda} + 4\nabla_{(\lambda} Q_{\kappa)\alpha} = 0 , \quad \text{and} \tag{5.5}$$

$$\nabla_\mu B_{\kappa\lambda} - \nabla_\lambda B_{\kappa\mu} + 2(\nabla_\mu Q_{\kappa\lambda} - \nabla_\lambda Q_{\kappa\mu}) + 4\nabla_\kappa Q_{[\mu\lambda]} = 0 . \tag{5.6}$$

On the other hand, from (4. 6), we have

(5. 7)
$$\begin{cases} B_{\alpha\beta} = 2Q_{\beta\alpha} - (n+\bar{n})Q_{\alpha\beta}, \\ B_{\alpha\kappa} = -(n+\bar{n}+2)Q_{\alpha\kappa}, \\ B_{\kappa\alpha} = -(n-\bar{n}+2)Q_{\kappa\alpha}, \quad \text{and} \\ B_{\kappa\lambda} = 2Q_{\lambda\kappa} - (n-\bar{n})Q_{\kappa\lambda}. \end{cases}$$

Since Γ_{ij}^h is a ϕ-connection, from (1. 4) and (5. 7), we get

(5. 8) $$\nabla_h B_{\alpha\beta} = 2\nabla_h Q_{\beta\alpha} - (n+\bar{n})\nabla_h Q_{\alpha\beta},$$

(5. 9) $$\nabla_h B_{\alpha\kappa} = -(n+\bar{n}+2)\nabla_h Q_{\alpha\kappa},$$

(5. 10) $$\nabla_h B_{\kappa\alpha} = -(n-\bar{n}+2)\nabla_h Q_{\kappa\alpha}, \quad \text{and}$$

(5. 11) $$\nabla_h B_{\kappa\lambda} = 2\nabla_h Q_{\lambda\kappa} - (n-\bar{n})\nabla_h Q_{\kappa\lambda}.$$

Substituting (5. 8) in (5. 2), we have

(5. 12)
$$\begin{aligned} &\nabla_\alpha Q_{\gamma\beta} - \nabla_\beta Q_{\gamma\alpha} + \nabla_\gamma Q_{\beta\alpha} - \nabla_\alpha Q_{\beta\gamma} + \nabla_\beta Q_{\alpha\gamma} - \nabla_\gamma Q_{\alpha\beta} \\ &= (n+\bar{n}-4)(\nabla_\gamma Q_{\alpha\beta} - \nabla_\beta Q_{\alpha\gamma})/2. \end{aligned}$$

Permuting the ordered indices $\{\alpha, \beta, \gamma\}$ by $\{\beta, \gamma, \alpha\}$ and $\{\gamma, \alpha, \beta\}$ of (5. 12), and adding them to (5. 12), we can obtain

(5. 13) $$\nabla_\gamma Q_{\alpha\beta} - \nabla_\beta Q_{\alpha\gamma} = 0$$

for p and q greater than 2. Next, substituting (5. 8) and (5. 9) in (5. 3), we have

(5. 14) $$2(\nabla_\kappa Q_{\beta\alpha} - \nabla_\alpha Q_{\beta\kappa}) = (n+\bar{n})(\nabla_\kappa Q_{\alpha\beta} - \nabla_\beta Q_{\alpha\kappa}).$$

Exchanging indices α and β in (5. 14), and adding it to (5. 14), we can obtain

(5. 15) $$\nabla_\kappa Q_{\alpha\beta} - \nabla_\beta Q_{\alpha\kappa} = 0.$$

From (5. 4), (5. 9) and (5. 10), we have

(5. 16) $$\nabla_\lambda Q_{\alpha\kappa} - \nabla_\kappa Q_{\alpha\lambda} = \nabla_\beta Q_{\kappa\alpha} - \nabla_\alpha Q_{\kappa\beta} = 0.$$

By means of calculations similar to (5. 13) and (5. 15), we have

(5. 17) $$\nabla_\alpha Q_{\lambda\kappa} - \nabla_\kappa Q_{\lambda\alpha} = \nabla_\mu Q_{\kappa\lambda} - \nabla_\lambda Q_{\kappa\mu} = 0$$

for p and q greater than 2. Therefore, from (5. 13), (5. 15), (5. 16) and (5.17), we get

THEOREM 4. *If Γ_{ij}^h is a P-projectively flat connection, for p and q greater than* 2,

(5. 18) $$\nabla_j Q_{hi} - \nabla_i Q_{hj} = 0.$$

COROLLARY 5. *Let M_n be a locally product space of M_p and M_q admitting a symmetric affine connection, and p and q greater than 2. If M_n is P-projectively flat, then*

$$\nabla_j B_{hi} - \nabla_i B_{hj} = 0 .$$

It is obvious that Theorem 4 and this corollary are the generalization of Theorem 9 of [2].

THEOREM 5. *A ϕ-connection Γ^h_{ij} is a P-projectively flat connection if and only if the P–P curvature tensor field of Γ^h_{ij} vanishes, where p and q are greater than 2.*

PROOF. If Γ^h_{ij} is a P-projectively flat connection, it follows from Theorem 3 that $P^h{}_{ijk}$ vanishes. Conversely, suppose that $P^h{}_{ijk}$ vanishes. Now, let us consider the integrability of

$$\nabla_i U_h = U_h U_i + \phi^j_h U_j \phi^k_i U_k - Q_{hi} \tag{5.19}$$

where Q_{hi} is given by (4.6). From (4.5), we have

$$-U_h B^h{}_{ijk} = 2Q_{i[j}U_{k]} + 2\phi^l_i Q_{l[j}\phi^h_{k]}U_h + 2U_i Q_{[kj]} + 2\phi^h_i U_h Q_{l[j}\phi^l_{k]} . \tag{5.20}$$

On the other hand, covariantly derivating (5.19), we have

$$\begin{aligned}\nabla_{[k}\nabla_{j]}U_i = {} & Q_{i[j}U_{k]} + \phi^l_i Q_{l[j}\phi^h_{k]}U_h + U_i Q_{[kj]} \\ & + \phi^h_i U_h Q_{l[j}\phi^l_{k]} - \nabla_k Q_{ij} + \nabla_j Q_{ik} .\end{aligned} \tag{5.21}$$

Therefore, it follows from (5.20) and (5.21) that the condition of integrability of (5.19) is

$$\nabla_j Q_{hi} - \nabla_i Q_{hj} = 0 . \tag{5.22}$$

Since we have already obtained (5.22) from (5.13), (5.15), (5.16) and (5.17), it follows that there exists a vector field U_h satisfying (5.19). Now, putting

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + 2D^h_{(i}U_{j)} + 2\phi^h_{(i}\phi^k_{j)}U_k ,$$

from (4.2) and (5.19), we have

$$\bar{B}_{hi} = B_{hi} - Q_{ih} + (n+1)Q_{hi} + \bar{n}\phi^j_h Q_{ji} - 2\phi^j_h\phi^k_i Q_{(jk)} . \tag{5.23}$$

Using (5.7), we have that $\bar{B}_{hi}$ vanishes. Therefore, from Theorem 3, it follows that $\bar{B}^h{}_{ijk}$ vanishes, that is, $\bar{\Gamma}^h_{ij}$ is a flat connection.

It is well known that a necessary and sufficient condition for a symmetric affine connection Γ^h_{ij} on a general space M_n to be projectively flat is that the curvature tensor field is given by

$$B^h{}_{ijk} = \left\{D^h_k(nB_{ij} + B_{ji}) - D^h_j(nB_{ik} + B_{ki})\right\}/(n^2-1) - D^h_i(B_{jk} - B_{kj})/(n+1)$$

for n greater than 2 (cf. Chapter III of [1]).

THEOREM 6. *Let M_n be a locally product space of M_p and M_q admitting a symmetric affine connection, and p and q greater than 2. M_n is P-projectively flat if and only if M_p and M_q are projectively flat.*

PROOF. When M_n is P-projectively flat, M_p and M_q are projectively flat, because $P^{\alpha}{}_{\beta\gamma\delta}$ and $P^{\kappa}{}_{\lambda\mu\nu}$ are regarded as the projective curvature tensor fields of M_p and M_q, respectively. Conversely, when we denote projectively flat connections of M_p and M_q by ${}_1\Gamma^{\alpha}_{\beta\gamma}$ and ${}_2\Gamma^{\kappa}_{\lambda\mu}$, respectively, it follows that there exist vector fields A_α and B_κ satisfying

$$ {}_1\Gamma^{\alpha}_{\beta\gamma} = 2D^{\alpha}_{(\beta}A_{\gamma)} \quad \text{and} \quad {}_2\Gamma^{\kappa}_{\lambda\mu} = 2D^{\kappa}_{(\lambda}B_{\mu)} . $$

Now, putting

$$ U_\alpha = A_\alpha/2 , \quad U_\kappa = B_\kappa/2 , \quad V_\alpha = A_\alpha/2 , \quad V_\kappa = -B_\kappa/2 $$
$$ \Gamma^{\alpha}_{\beta\gamma} = {}_1\Gamma^{\alpha}_{\beta\gamma}, \quad \Gamma^{\kappa}_{\lambda\mu} = {}_2\Gamma^{\kappa}_{\lambda\mu}, \quad \text{and} \quad \Gamma^{h}_{\alpha\kappa} = \Gamma^{h}_{\kappa\alpha} = \Gamma^{\kappa}_{\alpha\beta} = \Gamma^{\alpha}_{\kappa\lambda} = 0 , $$

then,

$$ \Gamma^{h}_{ij} = 2D^{h}_{(i}U_{j)} + 2\phi^{h}_{(i}V_{j)} \quad \text{and} \quad V_h = \phi^{i}_{h}U_i . $$

Therefore, it follows that Γ^{h}_{ij} is a P-projectively flat connection on M_n.

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