A characterization of A_7 and M_{11}, II

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1. Introduction

In this paper we shall prove the following theorem.

THEOREM 1. *Let G be a doubly transitive group on the set $\Omega=\{1, 2, \cdots, n\}$. If the stabilizer $G_{1,2}$ of points* 1 *and* 2 *is isomorphic to a simple group $PSL(2, q)$, $q\equiv 3$ or 5 (mod 8), then one of the following holds:*

(1) *G has a regular normal subgroup,*
(2) *$n=7$ and G is the alternating group A_7 of degree seven,*
(3) *$n=12$ and G is the Mathieu group M_{11} of degree eleven.*

In [17] Yamaki proved Theorem in the case $q=5$. Therefore we may assume $q\geq 11$.

Let X be a subset of a permutation group. Let $F(X)$ denote the set of all fixed points of X and $\alpha(X)$ be the number of points in $F(X)$. $N_G(X)$ acts on $F(X)$. Let $\chi_1(X)$ and $\chi(X)$ be the kernel of this representation and its image, respectively. The other notation is standard.

2. Preliminaries

Let us assume G has no regular normal subgroup. Let $G_{1,2}$ be $PSL(2, q)$, $q\equiv 3$ or 5 (8). Let $K=\langle \tau, \tau' \rangle$ be a Sylow 2-subgroup of $G_{1,2}$. Let I be an involution of G with the cycle structure $(1, 2)\cdots$. Then I normalizes $G_{1,2}$ and hence we may assume I normalizes K and $[I, \tau]=1$. Let τ fix i points of Ω, say $1, 2, \cdots, i$. Every involution of G is conjugate to an involution in $IG_{1,2}$.

LEMMA 1. *It may be assumed that the action of I on $G_{1,2}$ is trivial or an outer automorphism.*

PROOF. Since $q-1\not\equiv 0$ (8), $[P\Gamma L(2, q) : PGL(2, q)]$ is odd. Let ϕ be the homomorphism of $\langle I, G_{1,2} \rangle$ into Aut $G_{1,2}$. If ker $\phi\neq 1$, we can replace I by an element ($\neq 1$) of ker ϕ.

LEMMA 2. *If I does not centralize $G_{1,2}$, then G has just one class of involutions.*

PROOF. Since $\langle I, G_{1,2} \rangle = PGL(2, q)$ has two classes of involutions, every involution in $IG_{1,2}$ is conjugate to I.

Let d be the number of elements in $G_{1,2}$ inverted by I. Set $\gamma=[G_{1,2}: C_G(\tau)\cap G_{1,2}]$. Let β be the number of involutions with the cycle structures $(1, 2)\cdots$ which are conjugate to τ. Let $g_1^*(2)$ and $g^*(2)$ be number of involutions which fix only the point 1 and which fix no point of Ω, respectively. Then $n=i(\beta i-\beta+\gamma)/\gamma, d=\beta+g_1^*(2)$ if n is odd and $d=\beta+g^*(2)/(n-1)$ if n is even (see [9]).

LEMMA 3. *Assume I centralizes $G_{1,2}$. Then every involution is conjugate to I or $I\tau$. If G has two classes of involutions, then $\alpha(I)=i$ and $\beta=1$ or $\alpha(I\tau)=i$ and $\beta=\gamma$. If G has just one class of involutions, then $\beta=\gamma+1$.*

PROOF. Trivial.

LEMMA 4. *Assume I does not centralize $G_{1,2}$. If $q\equiv 3$ (8), then $d=\beta=q(q+1)/2$ and $\gamma=q(q-1)/2$. If $q\equiv 5$ (8), then $d=\beta=q(q-1)/2$ and $\gamma=q(q+1)/2$.*

PROOF. *$\langle I, G_{1,2}\rangle$ is $PGL(2, q)$. Therefore all involutions in $IG_{1,2}$ are conjugate and $d=\beta$. The other part is trivial.*

LEMMA 5. *$\chi(\tau)$ has a regular normal subgroup.*

PROOF. It is trivial that $C_{G_{1,2}}(\tau)/\langle\tau\rangle$ is a dihedral group of order $2\times$(odd number). Assume $\chi(\tau)$ has no regular normal subgroup. Then $\chi(\tau)_{1,2}$ is of even order. If $|\chi(\tau)_{1,2}|=2$, then by [8] $\chi(\tau)$ is (1) A_5, $i=6$ or (2) $P\Gamma L(2, 8), i=28$. If $|\chi(\tau)_{1,2}|>2$, then by [9] and [10] $\chi(\tau)$ is (3) $S_5, i=5$ or (4) $PSL(2, 11), i=11$. If $\chi(\tau)$ has just one class of involutions, then G has also just one class of involutions.

Case (1) $\chi(\tau)=A_5$. All involutions are conjugate. Assume I does not centralizes $G_{1,2}$. Then $n=30(q\pm 1)/(q\mp 1)+6$ by Lemma 4. Thus $(q\mp 1)/2$ is a factor of 15 and hence $q=11$ and $n=42$ or $q=29$ and $n=34$. Let P be a Sylow q-subgroup of $G_{1,2}$. Then $[G_1: N_{G_1}(P)]$ is a factor of $(n-1)(q+1)$ and it is divisible by $2(n-1)$, which contradicts the Sylow's theorem. Next assume I centralizes $G_{1,2}$. Since all involutions are conjugate, $i(i-1)/\gamma=60/q(q\pm 1)$ is an integer, which is contradiction.

Case (2) $\chi(\tau)=P\Gamma L(2, 8)$. All involutions of G are conjugate. If I does not centralize $G_{1,2}$ then by Lemma 4 $n=28\cdot 27(q\pm 1)/(q\mp 1)+28$. Thus $(q\mp 1)/2$ is a factor of $27\cdot 7$ and hence $q=19$, 43, 379, 13 or 53. By [4] and [7, II. 8. 27] $G_1=0(G_1)G_{1,2}$ since $n-1$ is not divisible by q. By a theorem of Brauer-Wielandt [15] $|0(G_1)||C_{0(G_1)}(K)|=|C_{0(G_1)}(\tau)|^3$. Thus $n-1$ is a factor of $(i-1)^3=27^3$. This is a contradiction.

Next assume I centralizes $G_{1,2}$. If $q\equiv 3$ (8), then $2i(i-1)/q(q-1)$ must be integral since $\beta=\gamma+1$. Thus $q(q-1)/2$ is a factor of $7\cdot 27$. This is a contradiction. If $q\equiv 5$ (8), then $28\cdot 27/q(q+1)$ must be integral. This is a contradiction.

Case (3) $\chi(\tau)=S_5$. If I does not centralize $G_{1,2}$, then $n=5\cdot 4(q\pm 1)/(q\mp 1)+5$. Thus $q=11$ and $n=29$. Let P be a Sylow 11-subgroup of G. $\alpha(P)=18$ or 7. By a theorem of Witt $|N_G(P)|=18\cdot 17\cdot 11\cdot 5$ or $7\cdot 6\cdot 11\cdot 5$. Since $|G|$ is not divisible by 17, $\alpha(P)=7$ and $|N_G(P)|=7\cdot 6\cdot 11\cdot 5$. Let Q be a Sylow 7-subgroup of $N_G(P)$. Then $[Q,P]=1$. Thus $\alpha(Q)>2$. This is a contradiction. Next assume I centralizes $G_{1,2}$. If $\gamma=1$ or $\gamma+1$, then $5\cdot 4/q(q\mp 1)$ must be integral. This is a contradiction.

Case (4) $\chi(\tau)=PSL(2,11)$. All involutions of G are conjugate. If I does not centralize $G_{1,2}$, then $n=11\cdot 10(q\pm 1)/(q\mp 1)+11$. Thus $q=11$ and $n=143$ or $q=109$ and $n=119$. If I centralizes $G_{1,2}$, then $11\cdot 10/q(q\pm 1)$ must be integral since $\beta=\gamma+1$. Thus $q=11$ and $n=123$. Let P be a Sylow $(n-1)/2$-subgroup of G_1. $C_{G_1}(P)=P$. By the theorem of Sylow P is normal in G_1. Thus K normalizes P and there exists an involution which centralizes P. This contradicts $\alpha(\tau)=11$.

This completes the proof of Lemma 5.

LEMMA 6. *If every involution is conjugate to τ, then I does not centralize $G_{1,2}$.*

PROOF. Assume I centralizes $G_{1,2}$. If G has an element of order 4, then so does $\langle I, G_{1,2}\rangle$, which is a contradiction. Thus a Sylow 2-subgroup of G is elementary abelian. By [14] G has a normal subgroup G' of odd index isomorphic to $PSL(2,2^m)$, the Janko group of order 175, 560 or a group of Ree type since G has one class of involutions. By [7, II. 8. 27] $G'\neq PSL(2,2^m)$. If $C_{G'}(\tau)=\chi_1(\tau)$, then $\chi(\tau)$ is cyclic of odd order, which is a contradiction. Thus $C_{G'}(\tau)/\chi_1(\tau)$ has a regular normal subgroup since $\chi(\tau)$ contains a regular normal subgroup by Lemma 5. Since $C_{G'}(\tau)/\langle\tau\rangle$ is simple, it is regular and it must be a regular normal subgroup of $\chi(\tau)$, which is a contradiction.

LEMMA 7. *If I does not centralize $G_{1,2}$, then there is no K-orbit of length 2, i.e., $F(K)=F(\tau)$.*

PROOF. By Lemma 4 every involution is conjugate to τ. Let $\{a,b\}$ be a K-orbit contained in $F(\tau)$. Let $\Omega^{(2)}$ be the set of unordered pairs of points in Ω. Then G is transitive on $\Omega^{(2)}$ and $G_{\{1,2\}}=\langle I, G_{1,2}\rangle$. If $\alpha(\langle I,\tau\rangle)\leq 1$, then K satisfies the condition of Witt and $N_G(K)$ is transitive on the set of fixed points of K on $\Omega^{(2)}$. Therefore there must exist an element g of $N_G(K)$ with $\{1,2\}^g=\{a,b\}$, which is a contradiction. If $\alpha(\langle I,\tau\rangle)=\alpha(K)$, then every four-subgroup is conjugate to K. Since $\langle I,K\rangle$ is dihedral, $\chi(\tau)$ has two classes of involutions and $K\chi_1(\tau)/\chi_1(\tau)$ is not a central involution by [11]. Thus K is not normal in any Sylow 2-subgroup of $C_G(\tau)$ and hence a Sylow 2-subgroup of G contains no normal four-group. By [3, Th. 5. 4. 10] it is dihedral or semi-dihedral. If i is even, $i=4$ and $\chi(\tau)=S_4$ since $\chi(\tau)$ contains a regular normal subgroup by Lemma 5. Since $n=i(\beta(i-1)+\gamma)/\gamma$ with $\beta=q(q\pm 1)/2$ are $\gamma=q(q\mp 1)/2$, $4\cdot 3(q\pm 1)/(q\mp 1)$ must be integral and hence $q=7$ or 5, which is a contradiction. If i is odd, $\chi(\tau)=0(\chi(\tau))\,C_{\chi(\tau)}(IK\chi_1(\tau))$ by [1] and [11]. By [4] $\chi(\tau)/0(\chi(\tau))$ is 2-group and $C_G(\tau)$ is $2'$-closed. By [2] and [12] this is a contradiction.

3. The case n is odd

Since $\chi(\tau)$ contains a regular normal subgroup by Lemma 5, $\alpha(C_{G_{1,2}}(\tau))$ is odd.

LEMMA 8. *If* $g_1^*(2)\neq 0$, *then* $\alpha(G_{1,2})$ *is odd.*

PROOF. Let a be the point in $F(<I, C_{G_{1,2}}(\tau)>)$. If a is contained in $F(G_{1,2})$, then the lemma is trivial. Let Δ be the $G_{1,2}$-orbit containing a. Since I centralizes $G_{1,2}$ by Lemma 4, Δ is contained in $F(I)$. Since $C_{G_{1,2}}(\tau)$ is maximal in $G_{1,2}$, $G_{1,2,a}=C_{G_{1,2}}(\tau)$. Let x be an element of $N_{G_{1,2}}(K)$ of order 3. Then $a^x(\neq a)$ is contained in $F(K)$. Thus $|F(K)\cap\Delta|>2$ and $\alpha(<K, I>)>2$. This is a contradiction.

By Lemma 6, 8 and [11] we may assume $g_1^*(2)=0$, *i.e.*, every involution is conjugate to τ and I does not centralize $G_{1,2}$. Since by Lemma 7 $F(K)=F(\tau)$, $(C_G(\tau)\cap N_G(K))\,\chi_1(\tau)/\chi_1(\tau)$ is 2-transitive on $F(\tau)$. Since $\chi(\tau)$ contains a regular normal subgroup, a Sylow 2-subgroup of $\chi(\tau)$ is cyclic or (generalized) quaternion. Thus a Sylow 2-subgroup of G is dihedral of order 8. This is a contradiction by [4] and [12].

4. The case n is even

By [5] we may assume $i>2$.

LEMMA 9. *I centralizes* $G_{1,2}$ *and* $\alpha(K)<\alpha(\tau)$.

PROOF. If I does not centralizes $G_{1,2}$, $F(K)=F(\tau)$ by Lemma 7. If $F(K)=F(\tau)$, then G contains a regular normal subgroup by [13].

By this lemma $\alpha(\tau)>\alpha(K)$. By Lemma 6 $g^*(2)\neq 0$. Let N be a normal subgroup of $C_G(\tau)$ containing $\chi_1(\tau)$ such that $N/\chi_1(\tau)$ is a regular normal subgroup of $\chi(\tau)$. Let S be a Sylow 2-subgroup of N. Since $i>2$ and $\chi_1(\tau)$ is cyclic, $N=S\times 0(\chi_1(\tau))$ and S is elementary abelian of order $2i$.

Since $C_G(\tau)$ is solvable, by [6] $\chi(\tau)_{1,2}$ is cyclic. Since $C_{G_{1,2}}(\tau)$ is dihedral, $|\chi(\tau)_{1,2}|=2$. Moreover $i=\alpha(K)^2$.

LEMMA 10. *S is a unique abelian 2-subgroup of $C_G(\tau)$ of order $2i$.*

PROOF. Let T be a maximal abelian 2-subgroup of $C_G(\tau)$ containing τ'. Since $i=\alpha(K)^2$, $|T|=4\sqrt{i}$. If $|T|\geq|S|$, then $i=4$ and $n=4\cdot 3\cdot\beta/\gamma+4$. Since $\gamma=q(q\pm1)/2$ and $\beta=\gamma$ or 1, $\gamma=\beta$ and $n=16$. Thus $q=11$, 13 or 27. If $q=11$ or 13, let P be a Sylow q-subgroup of $G_{1,2}$. If $q=27$, let P be a Sylow 13-subgroup of $G_{1,2}$. There exists just one non-trivial P-orbit in Ω. Since $[I, P]=1$, $\alpha(I)\geq 5$. This contradicts $\alpha(I)\leq 4$.

LEMMA 11. *Every involution of S which is conjugate to τ is already conjugate to τ in $N_G(S)$.*

PROOF. Let $\eta=\tau^g$ be an involution of S. Since S is abelian, $S^{g^{-1}}$ is contained in $C_G(\tau)$. By Lemma 10 g is contained in $N_G(S)$.

COROLLARY 11. $|N_G(S)|=i^2(i-1)|C_{G_{1,2}}(\tau)|$.

PROOF. Trivial.

COROLLARY 12. $\beta=\gamma$, $g^*(2)=n-1$ *and* $n=i^2$.

PROOF. By Lemma 3 $\beta=\gamma$ or $\beta=1$. By Corollary 11 n in divisible by i^2. If $\beta=1$, $n=i(i-1+\gamma)/\gamma$, which is a contradiction.

By [4] and [7, II. 8. 27] let G_1' be a normal subgroup of G_1 such that $G_1'/0(G_1)$ is $PSL(2, q^s)$ and $G_1/0(G_1)$ is a subgroup of $P\Gamma L(2, q^s)$.

LEMMA 13. *Every involution of G_1 acts trivially on $0(G_1)$.*

PROOF. Assume $0(G_1)\neq 1$. By a theorem of Brauer-Wielandt [15] $|0(G_1)||C_{0(G_1)}(K)|^2=|C_{0(G_1)}(\tau)|^3$. Since $0(G_1)\cap G_{1,2}=1$, $|0(G_1)|$ is a factor of $i-1$. Assume $0(G_1)$ is not contained in $C_{G_1}(\tau)$. If $q\equiv 5$ (8), $[0(G_1):C_{0(G_1)}(\tau)]$ is a factor of $q(q+1)$ since $[G_1:C_G(\tau)]$ is a factor of $(i+1)q(q+1)$. Let p be a prime factor of $[0(G_1):C_{0(G_1)}(\tau)]$. On the other hand $[G_1/0(G_1):C_{G_1}(\tau)$ $0(G_1)/0(G_1)]$ is divisible by $q(q+1)/2$. Thus $[G_1:C_{G_1}(\tau)]=(i+1)q(q+1)/2$ must be divisible by $pq(q+1)/2$, which is a contradiction. Similarly we have a contradiction when $q\equiv 3$ (8). Thus $0(G_1)$ is contained in $C_{G_1}(\tau)$. $0(G_1)$ is contained in $C_{G_1}(\tau)$.

If $s=1$, then $G_1'=0(G_1)\times G_{1,2}$ by this lemma, which is a contraction. Thus $s\geq 3$.

LEMMA 14. *If $q\equiv 5$ (8), $i+1=q^s(q^s+1)/q(q+1)$ and $i-1=|0(G_1)|(q^s-1)|G_1|/(q-1)|G_1'|$. If $q\equiv 3$ (8), $i+1=q^s(q^s-1)/q(q-1)$ and $i-1=|0(G_1)|(q^s+1)|G_1|/(q+1)|G_1'|$. $\alpha(K)-1=\sqrt{i}-1=|0(G_1)||G_1/G_1'|$.*

PROOF. This sollows from Lemma 13.

If $q\equiv 5$ (8), $\sqrt{i}+1=(i-1)/(\sqrt{i}-1)=(q^s-1)/(q-1)$ and hence $\sqrt{i}\equiv 0(q)$. Thus $i+1\equiv 1$ (q). If $q\equiv 3$ (8), again $\sqrt{i}\equiv 0$ (q) and $i+1\equiv 1$ $(q.)$ This contradicts $s\geq 3$.

This completes a proof of Theorem 1.

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