On the exact ranges of complex manifolds

By Shawich SATO

Introduction

In [3] Kobayashi gave the following interesting statement: let M be a complex manifold on which a complex lie group acts transitively, then M can not be hyperbolic. The method of the proof and the fact that the complex line C, which is a typical, non-taut, non-tight and non-hyperbolic space, plays an essential role in the proof suggested to us to consider the same circumstances from our point of view. Our aim is to show that the set of holomorphic mappings $H(C, M)$ can neither be normal nor equicontinuous. For this purpose we introduce the property (P) instead of "two-fold assigning" property. The property (P) is in a sense a localization of "two-fold assigning property". The property (P) was already introduced in [4] and made it possible to improve Wu's theorem, cf. Lemma 4.1, [4]. In this paper the property (P) and the notion of exact range, see Definition 2.3, which is laso a localization of the notion introduced in [4], will prove themselves effective for our purpose.

The purpose of § 3 is to make some remarks on Kobayashi's statement cited above.

§ 1. Preliminaries.

Through this paper complex manifolds are all assumed connected and second countable. For two complex manifolds M and N we denote by $H(M, N)$ the space of all holomorphic mappings of M to N. The space $H(M, N)$ can be topologized by so-called compact-open topology. Since by assumption M and N are second countable, $H(M, N)$ is also second countable, and the compactness of a subset of $H(M, N)$ is verified by its sequential compactness. By the same assumption the complex manifold N is metrizable and we can construct a distance function d_N on N which metrizes N, cf. Kelley [1]. So we can speak of the convergence of a sequence of $H(M, N)$ making use of the distance function d_N.

A sequence $\{f_i\}\subset H(M, N)$ converges *compact-uniformly* in M if and only if it converges uniformly on every compact subset of M. The compact-uniform limit of a sequence of $H(M, N)$ belongs to $H(M, N)$. A sequence $\{f_i\}\subset H(M, N)$ is said to be *compactly divergent* if and only if for any compact subset K in M and for any compact subset L in N there exists a number i_0 such that $f_i(K)\cap L=\phi$ for all $i\geqq i_0$.

By definition a subet F of $H(M, N)$ is called *normal* if and only if every sequence of F contains a subsequence which is either relatively compact in $H(M, N)$ or compactly divergent. Let d_N be a distance function on N. A subset F of $H(M, N)$ is called *equicontinuous* if and only if for any positive number ε and any point $x\in M$ there exists a neighborhood U of x such that $y\in U$ implies $d_N(f(x), f(y))<\varepsilon$ for all $f\in F$.

DEFINITION 1. 1. *A complex manifold N is called taut if and only if for every complex manifold M the space of holomorphic mappings $H(M, N)$ is normal. A complex manifold N with distance function d_N, which metrizes N, is called tight if and only if $H(M, N)$ is equicontinuous.*

§ 2. The exact range of complex manifold.

In [4] the author considered the two-fold assinging families of holomorphic mappings and obtained several interesting results, see Theorem 2. 2 and Theorem 2. 4.

A two-fold assigning families of holomorphic mapping is defined as follows. Let M and N be complex manifolds. A subset F of $H(M, N)$ is called *two-fold assigning* if and only if for any two different points p, $q\in M$ and for any two different points P, $Q\in N$ there exists an $f\in F$ such that $f(p)=P$ and $f(q)=Q$.

Combining Theorem 2. 2 and Theorem 2. 4 in [4] we can state the following

THEOREM 2. 1. *Let M and N be complex manifolds and let F be a subset of $H(M. N)$. If F is two-fold assigning, then F can neither be normal nor equicontinuous.*

The equicontinuity of a set of mappings is a local character. Also the normality is essentially a local character. Let M and N be complex manifolds and F a subset of $H(M, N)$. Let us say that F is normal at a point $x\in M$ if there exists a neighborhood U of x in which F is normal. Then, it is easily verified that F is normal if and only if F is normal at every point of M. With those remark we can replace the condition "*two-fold assigning*" in Therem 2. 1 by weaker condition which we shall call *property* (P),

(P): there exist a sequence $\{x_i\}$ of M converging to a point $x_0\in M$, a sequence $\{f_i\}$ of F and two different points P, $Q\in N$ such that $f_i(x_i)=P$ and $f_i(x_0)=Q$.

We have

PROPOSITION 2. 2. *Let M and N be complex manifolds. If a subset F of $H(M, N)$ possesses property* (P), *then it can neither be normal nor equicontinuous and hence N can neither be taut nor tight.*

PROOF. By assumption there exists a sequence $\{x_i\}$ of M converging to a point $x_0 \in M$, a sequence $\{f_i\}$ of F and two different points $P, Q \in N$ such that $f_i(x_i)=P$ and $f_i(x_0)=Q$.

First let us assume that F is normal. Then we can choose a subsequence of $\{f_i\}$ which is either compact-uniformly convergent or compactly divergent in M. We shall show that the sequence $\{f_i\}$ does not contain any compactly divergent subsequence. Let K be a compact neighborhood of x_0. Then there exists an i_0 such that $x_i \in K$ for all $i \geqq i_0$. Put $L=\{P, Q\}$. Then obviously L is compact in L and $f_i(K) \cap L \supset \{f_i(x_i\} \cup (f_i(x_0)\} = \{P, Q\} \neq \phi$, for all $i \geqq i_0$. Thus, the sequence $\{f_i\}$ can not contain any compactly divergent subsequence. $\{f_i\}$ therefore should contain a compact-uniformly convergent subsequence, say $\{f_j\}$. Since we topologized $H(M, N)$ by compact-open topology, the limit f_0 of the subsequence $\{f_j\}$ belongs to $H(M, N)$. Let us choose a subsequence $\{x_j\}$ of $\{x_i\}$ corresponding to the choice of $\{f_j\}$.

Since $f_j \to f_0$ and $x_j \to x_0$ as $j \to \infty$, we have $f_j(x_j) \to f_0(x_0)$ as $j \to \infty$. On the other hand, by the choice of original $\{f_i\}$, $f_j(x_j)=P$ and $f_j(x_0)=Q$. Thus $P=Q$, which contradicts the assumption that P and Q are different. Hence, F can not be normal.

Secondly, let us assume that F is equicontinuous. Then by definition for any positive number ε there exists a neighbornood V of x_0 such that $x \in V$ implies $d(f(x), f(x_0)) < \varepsilon$ for all $f \in F$, where d is the distance function metrizing N. Since $x_i \to x_0$ as $i \to \infty$, we can find an i_0 such that $x_i \in V$ for all $i \geqq i_0$. Specializing f to the element of $\{f_i\}$ we have $d(f_i(x_i), f_i(x_0)) < \varepsilon$ for all $i \geqq i_0$. Let us choose ε so that $\varepsilon < d(P, Q)$. Then, by the choice of the sequences $\{f_i\}$ and $\{x_i\}$ that $f_i(x_i)=P$ and $f_i(x_0)=Q$, we have $d(f_i(x_i), f_i(x_0)) = d(P, Q) < \varepsilon < d(P, Q)$. Thus F can not be equicontinuous.

In [4] we introduced the notion of exact range of complex manifold which is defined as follows. A complex manifold N is called an *exact range* of complex manifold M if and only if for any two different points P, Q of N there exists an $f \in H(M, N)$ such that $f(M)$ contains both P and Q. This definition of exact range is too strong for the qurpose of the present paper. So we introduce here another definition.

DEFINITION 2. 3. *A complex manifold N is called the exact range of* *complex manifold M if and only if for any point P of N there exists a neighborhood U such that for every $Q \in U$ there exists an $f \in H(M, N)$ satisfying $f(M) \ni P, Q$.*

Property (P) introduced above and Definition 2. 3 are the localizations of two-fold assigning property and the notion of exact range in [4], respectively.

Under Definition 2. 3 we can prove Theorem 4. 5 in [4] anew. Namely we have

THEOREM 2. 4. *Let M be a complex manifold and the group Aut (M) of automorphisms of M act two-fold transitively on M. If a complex manifold N is the exact range of M, then N can neither be taut nor tight, and there is non-constant holomorphic mapping from N to neither a taut manifold nor a tight manifold.*

PROOF. First we shall show that the space $H(M, N)$ possesses the property (P). Since N is the exact range of M, there exist an $f \in H(M, N)$ and two different points $P, Q \in N$ such that $f(M) \in P, Q$. We can choose two different points p, q of M so that $f(p)=P$ and $f(q)=Q$.

Now, choose a sequence $\{x_n\}$ of M converging to a point $x_0 \in M$. Since Aut (M) acts two-fold transitively on M we can find a sequence $\{\sigma_n\}$ of Aut (M) such that $\sigma_n(x_0)=p$ and $\sigma_n(x_n)=q$.

Thus, the sequence of holomorphic mappings $\{f \circ \sigma_n\}$ possesses the property (P) and hence $H(M, N)$ also does. Then by Proposition 2. 2 N can neither be taut nor tight.

Now, for the proof of the second half let us assume that there exist a taut or tight manifold L and a non-constant holomorphic mapping g of N to L. Then, since g is non-constant, it is possible to choose a point P of N such that for any neighborhood U of P there is a point Q satisfying $g(P) \neq g(Q)$. Since by assumption N is the exact range of M the set $g \circ H(M, N) = \{g \circ f : f \in H(M, N)\}$ possesses the property (P). According to Proposition 2. 2 again L can neither be taut nor tight. This is a contradiction. Thus, g should be constant.

The following proposition is almost trivial, but implies very interesting consequence.

PROPOSITION 2.5. *An exact range of the complex number space C^n is the exact range of the complex number plane C.*

COROLLARY 2. 6 (Wu [5]). *Any holomorphic mapping of C^n to a tight or taut complex manifold is constant.*

PROOF. It is obvious that *Aut* (C) acts two-fold transitively on C. On the other hand C^n is the exact range of C^n itself. Then by Proposition 2.5 Corollary 2.6 holds. More directly, it suffices to notice that C^n is the exact range of C.

In general any complex manifold is its exact range. Therefore, the fact that $Aut\,(C)$ is two-fold transitive implies classical Liouville's theorem on bounded holomorphic function.

§ 3. The exact ranges of the complex number plane C

In [3] Kobayashi gave the following proposition, see Example 3, p. 49:

PROPOSITION 3.1. *If M is a complex manifold on which a complex Lie group acts transitively, then M can not hyperbolic.*

The purpose of this section is to observe this proposition from our setting and make a remark which seems fundamental.

PROPOSITION 3.2. *Let M be a complex manifold on which a complex Lie group L acts trasitively. Then M is the exact range of the complex line C.*

PROOF. The idea of proof is wholly due to Kobayashi. Since L acts transitively on M, to any point $p \in M$ we can choose a neighborhood U_p such that for any point $q \in U_p$ there exists a complex 1-parameter subgroup L_p of L such that its orbit contains p and q.

This means that there exists a holomorphic mapping f_q of complex line C to M such that $f_q(C) \in p, q$. Thus M is an exact range of C. Then Theorem 2.4 applies.

The following statement is a direct consequence of Theorem 2.4.

COROLLARY 3.3. *Let M be a complex manifold on which a complex Lie group L acts transitively. Then M can neither be taut nor tight. Any holomorphic mapping of M to either taut or tight manifold is constant.*

By the results in Kiernan, [2] the first half of Corrollary 3.3 is equivalent to Proposition 3.1.

Now, we want to make a remark on Proposition 3.2. Any complex Lie group L acts transitively on itself by left ranslation. Then we can apply Proposition 3.2 to every complex Lie group regarded as a complex manifold. We obtain trivially.

THEOREM 3.4. *Any complex Lie group can neither be taut nor tight. Any holomorphic mapping from a complex Lie group to either a taut or tight manifold is constant.*

In Theorem 3.4. the structure of "complex Lie group" is important.

We shall give in the following an example of tight complex manifold on which a structure of real Lie group can be defined and furthermore its left translation is holomorphic.

Let D be the unit disk on the complex number plane C. We define the group operation in D as follows: for two element α, $z \in D$ the product $\alpha \cdot z$ is given by

$$(G) \qquad \alpha \cdot z = \frac{(1-\alpha)z+(1-\bar{\alpha})\alpha}{(1-\alpha)\bar{\alpha}z+(1-\bar{\alpha})}.$$

From (G) we see that the left translation $l_\alpha(z)=\alpha \cdot z$ is a meromorphic function. Since $|l_\alpha(z)|<1$, $l_\alpha(z)$ is bounded and therefore is holomorphic. On the other hand D is a typical tight manifold.

Now, we want to improve Proposition 3. 1 by the aid of Theorem 3. 4. For that we give precise definition of the action of a complex Lie group on a complex monifold. By definition a complex Lie group L acts on a complex manifold M if (1) there exists a group homomorphism ρ from L to Aut (M), and (2) the mapping Φ: $L \times M \rightarrow M$, defined by $\Phi(l, x)=\rho[l](x)$ the operation of the automorphism $\rho[l] \in$ Aut (M) on $x \in M$, is holomorphic.

PROPOSITION 3. 5. *Let a connected complex Lie group L act on a complex manifold M. If M is taut or tight, then the action of L on M is trivial.*

PROOF. Let $\Phi(l, x)=\rho[l](x)$ be the action of L on M. Put $f_\alpha(l)= \rho[l](\alpha)$ for any $\alpha \in M$. Then f_α is a holomorphic mapping of L to M. Since M is taut or tight and L is connected, Theorem 3. 4 implies that f_α is constant, say $f_\alpha(l)=\beta$, $\beta \in M$. If l is the identity of L, then $\rho[l]$ is also the identity of Aut (M). Hence $\beta=\alpha$. This means that ρ is trivial. The proof is completed.

Department of Mathematics,
Kumamoto University

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