Notes on relatively harmonic immersions

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The notion of harmonic mappings was introduced and such mappings were studied by Eells and Sampson [1]. Recently, such mappings have been discussed by several authors (See [1], [2], [3], [4] and [5], for example) and many interesting results have been obtained. Yano and one of the present authors [5] have proved, concerning harmonic mappings, some theorems in which sufficient conditions for a harmonic mapping to be affine or homothetic are stated. To prove these theorems, they computed Laplacian $\Delta\|df\|^2$ of the square of the differential mapping df for a harmonic mapping f of a compact Riemannian space (M, g) into a Riemannian space $(N, \bar{g})$ and pinched in a certain sense the sum of eigenvalues of the tensor g^* induced in M from $\bar{g}$ by f. In the present paper, we define relatively harmonic immersions of a compact Riemannian space $(M, \bar{g})$ of dimension n into a Riemannian space $(N, \bar{g})$ of dimension $n+1$ (See § 1) and obtain some sufficient conditions for such an immersion to be relatively affine or homothetic by a similar way to that taken in [5]. The results will be stated in Theorems 4.1~4.5.

In § 1, notations and some concepts concerning immersions and relatively harmonic immersions will be defined and some propositions will be proved. In § 2 Laplacian $\Delta\|df\|^2$ will be computed and in § 3 some inequalities will be given for later use. The last § 4 is devoted to prove Theorems 4.1~4.5.

§1. Differentiable immersions of a Riemannian space into another

Let (M, g) and $(N, \bar{g})$ be two Riemannian spaces of dimension n and $n+1$ respectively, where $n \geqq 2$. Let there be given a differentiable immersion $f: M \to N$, that is, a differentiable mapping $f: M \to N$ whose rank is equal to n everywhere. Such an immersion will be sometimes denoted by $f:(M, g) \to (N, \bar{g})$. Manifolds, mappings and geometric objects we discuss are assumed to be differentiable and of class C^∞. Take a coordinate neighborhoods $\{U, x^h\}$ of M and $\{\bar{U}, y^\alpha\}$ of N in such a way that $f(U) \subset \bar{U}$, where local coordinates of M are denoted by $(x^h)=(x^1, \cdots x^n)$ and those of N by $(y^\alpha)=(y^{\bar{1}}, \cdots, y^{\overline{n+1}})$. The indices h, i, j, k, l, m, r, s run over the range $\{1, \cdots, n\}$ and the indices $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu$ over the range $\{\bar{1}, \cdots, \overline{n+1}\}$. The summation convention will be used with respect to these two systems of indices. Suppose that the mapping f is represented by equations

$$(1.1)\qquad y^{\alpha}=y^{\alpha}(x^1,\cdots,x^n)$$

with respect to $\{U, x^h\}$ and $\{\bar{U}, y^{\alpha}\}$. Differentiating (1.1), we now put in U

$$(1.2)\qquad A_i^{\alpha}=\partial_i y^{\alpha}(x^1,\cdots,x^n),$$

where $\partial_i=\partial/\partial x^i$. Then the differential mapping df of f is represented by the matrix (A_i^{α}) with respect to local coordinates (x^h) of M and those (y^{α}) of N.

When a function ρ, local or global, is given in N, we shall throughout this paper identify ρ with the function $\rho\circ f$ induced in M. We denote by g_{ji} components of the Riemannian metric g in M and by $\bar{g}_{\gamma\beta}$ those of the Riemannian metric $\bar{g}$ in N. We now put $(g^{ji})=(g_{ji})^{-1}$ and $(\bar{g}^{\gamma\beta})=(\bar{g}_{\gamma\beta})^{-1}$. Then

$$(1.3)\qquad g^*_{ji}=\bar{g}_{\gamma\beta}A_j^{\gamma}A_i^{\beta}$$

are components of the Riemannian metric $g^*=f^*\bar{g}$ induced in M from $\bar{g}$ by $f: M\to N$. The Christoffel's symbols $\left\{{h \atop ji}\right\}$, $\left\{{\alpha \atop \gamma\beta}\right\}$ and $\left\{{h \atop ji}\right\}^*$ are formed with g_{ji}, $\bar{g}_{\gamma\beta}$ and g^*_{ji}, respectively.

In this and the next sections, we denote by X, Y and Z arbitrary vector fields in M with local expressions $X=X^h\partial/\partial x^h$, $Y=Y^h\partial/\partial x^h$ and $Z=Z^h\partial/\partial x^h$, respectively. Then $(A_i^{\alpha}X^i)\partial/\partial y^{\alpha}$ is the local expression of the vector field $(df)X$ defined along $f(M)$. If we put in U

$$(1.4)\qquad A_{ji}^{\alpha}=\nabla_j A_i^{\alpha},$$

where we have defined $\nabla_j A_i^{\alpha}$ by

$$(1.5)\qquad \nabla_j A_i^{\alpha}=\partial_j A_i^{\alpha}+\left\{{\alpha \atop \gamma\beta}\right\}A_j^{\gamma}A_i^{\beta}-\left\{{h \atop ji}\right\}A_h^{\alpha},$$

then $(A_{ji}^{\alpha}X^jY^i)\partial/\partial y^{\alpha}$ is the local expression of a vector field B defined along $f(M)$. Denoting by $C^{\alpha}\partial/\partial y^{\alpha}$ a local vector field along U which is unit and normal to $f(M)$, we can put

$$(1.6)\qquad A_{ji}^{\alpha}=D_{ji}^h A_h^{\alpha}+H_{ji}C^{\alpha},$$

where D_{ji}^h are components of a tensor field D of type (1, 2) in M and H_{ji} components of the second fundamental tensor H of the isometric immersion $f:(M, g^*)\to(N, \bar{g})$. Thus we can easily verify

$$(1.7)\qquad D_{ji}^h=\left\{{h \atop ji}\right\}^*-\left\{{h \atop ji}\right\}.$$

If we put

$$\nabla_j C^\alpha = \partial_j C^\alpha + \left\{ {\alpha \atop \gamma\beta} \right\} A_j^\gamma C^\beta,$$

then, using $A_i^\beta C^\alpha \bar{g}_{\beta\alpha}=0$, we obtain

(1.8) $$\nabla_j C^\alpha = -k_j^h A_h^\alpha,$$

where

(1.9) $$k_j^h g^*_{hi} = H_{ji}.$$

Summing up (1.6) and (1.8), we now have

(1.10) $$\nabla_j A_i^\alpha = D_{ji}^h A_h^\alpha + H_{ji} C^\alpha, \quad \nabla_j C^\alpha = -k_j^h A_h^\alpha.$$

Consider a curve $\gamma: I\to M$ in M, I being an interval, and denote by $\bar{\gamma}=f\circ\gamma: I\to N$ the image of γ by f. Then we have easily

$$\ddot{\bar{\gamma}} = (df)\ddot{\gamma} + A(\dot{\gamma}, \dot{\gamma}),$$

where $A(\dot{\gamma}, \dot{\gamma})=(A_{ji}^\alpha \dot{\gamma}^j \dot{\gamma}^i)\partial/\partial y^\alpha$, $\dot{\gamma}^h$ being components of $\dot{\gamma}$. Thus we see that $f:(M, g)\to(N, \bar{g})$ is *affine*, i.e., for any geodesic γ in (M, g) (for any curve γ satisfying $\ddot{\gamma}=0$) its image $\bar{\gamma}$ is also a geodesic in $(N, \bar{g})$, if and only if $A_{ji}^\alpha=0$. We say that $f:(M, g)\to(N, \bar{g})$ is *relatively affine* when any geodesic γ in (M, g) is also a geodesic in $(M, g)^*$. When $g^*=\rho^2 g$ with function $\rho^2>0$, $f:(M, g)\to(N, \bar{g})$ is called a *relatively conformal* immersion. When $g^*=\rho^2 g$ with constant $\rho^2>0$, $f:(M, g)\to(N, \bar{g})$ is said to be *relatively homothetic*. We now have by using (1.7)

PROPOSITION 1.1 *$f:(M, g)\to(N, \bar{g})$ is relatively affine if and only if $D=0$, i.e., $D_{ji}^h=0$.*

PROPOSITION 1.2 *$f:(M, g)\to(N, \bar{g})$ is relatively homothetic if and only if it is relatively affine and at the same time relatively conformal.*

On putting

(1.11) $$A^\alpha = g^{ji} A_{ji}^\alpha,$$

we have

(1.12) $$A^\alpha = E^h A_h^\alpha + hC^\alpha,$$

where

(1.13) $$E^h = g^{ji} D_{ji}^h, \quad h = H_{ji} g^{ji}.$$

Then we can easily see that A^α are components of a vector field T defined along $f(M)$, E^h are components of a vector field E in M and h is a local function defined in each coordinate neighborhood and globally defined up to sign. The vector fields T, E and the function h are called the *tension field*, the *relative tension field* and the *relative mean curvature* of $f:(M, g)\to(N, \bar{g})$, respectively. By the way, the local function

$$\bar{h}=H_{ji}g^{*ji}, \tag{1.14}$$

where $(g^{*ji})=(g^{*}_{ji})^{-1}$, is the *mean curvature* of the isometric immersion $f:(M, g^{*})\to(N, \bar{g})$. We now put for later use

$$\nabla_j A^{\alpha}=\partial_j A^{\alpha}+\left\{{\alpha \atop \gamma\beta}\right\}A^{\gamma}_j A^{\beta}. \tag{1.15}$$

Let $I=(-a, a)$ be an interval. Consider a mapping $F: M\times I\to N$ such that $F(p, 0)=f(p)$ for any $p\in M$. Such a mapping F is called a *variation* of f. If we suppose that F has the local expression

$$y^{\alpha}=y^{\alpha}(x^{h}, t),\ (t\in I),$$

then $v^{\alpha}=(\partial y^{\alpha}(x^{h}, t)/\partial t)_{t=0}$ define a vector field $v=v^{\alpha}\partial/\partial y^{\alpha}$ along $f(M)$, which is called the *variation vector* of the variation F. For $f:(M, g)\to(N, \bar{g})$ we put

$$E(f, D)=\int_D \|df\|^2 d\sigma_g,$$

D being a compact domain with boundary ∂D in M, where $d\sigma_g$ the volume element of (M, g) and

$$\|df\|^2=A^{\beta}_j A^{\alpha}_i g^{ji}\bar{g}_{\beta\alpha}=g^{*}_{ji}g^{ji}. \tag{1.16}$$

On putting

$$\delta_F E(f, D)=\left[\frac{d}{dt}E(F_t, D)\right]_{t=0},$$

where $F_t(p)=F(p, t)$ for any $p\in M$, we can easily verify

$$\delta_F E(f, D)=2\int_D\left[(\nabla_j v^{\beta})A^{\alpha}_i g^{ji}\bar{g}_{\beta\alpha}\right]d\sigma_g,$$

where

$$\nabla_j v^{\alpha}=\partial_j v^{\alpha}+\left\{{\alpha \atop \gamma\beta}\right\}A^{\gamma}_j v^{\beta},$$

and hence, because of

$$(\nabla_j v^{\beta})A^{\alpha}_i g^{ji}\bar{g}_{\beta\alpha}=g^{ji}\nabla_j(v^{\beta}A^{\alpha}_i\bar{g}_{\beta\alpha})-v^{\beta}A^{\alpha}\bar{g}_{\beta\alpha},$$

we have

$$\delta_F E(f, D)=-2\int_D[v^{\beta}A^{\alpha}\bar{g}_{\beta\alpha}]d\sigma_g \tag{1.17}$$

or

$$(1.18)\qquad \delta_F E(f, D) = -2\int_D [v^j E^i g^*_{ji} + v^0 h]\, d\sigma_g,$$

where $v^\alpha = v^i A^\alpha_i + v^0 C^\alpha$, when the variation vector v vanishes along $f(\partial D)$. When $\delta_F E(f, D)=0$ for any variation F and f whose variation vector vanishes along $f(\partial D)$ and for any D, f is called a *harmonic mapping*. Thus, from (1.17), it follows that f is harmonic if and only if $T=0$ (i.e., $A^\alpha=0$) (See [1]). When $\delta_F E(f, D)=0$ for any D and for any variation F whose variation vector vanishes along $f(\partial D)$ and is tangent to $f(M)$ f is called a *relatively harmonic immersion*. Thus we have from (1.18)

PROPOSITION 1.3. *$f:(M, g)\to(N, \bar g)$ is relatively harmonic if and only if $E=0$, i.e., $E^h=0$.*

PROPOSITION 1.4. *$f:(M, g)\to(N, \bar g)$ is harmonic if and only if it is relatively harmonic (i.e. $E=0$) and relatively minimum (i.e. $h=0$) at the same time.*

§ 2. Laplacian of $\|df\|^2$

We now put in U

$$(2.1)\qquad \nabla_k A^\alpha_{ji} = \partial_k A^\alpha_{ji} + \left\{ {\alpha \atop \gamma\beta} \right\} A^\gamma_k A^\beta_{ji} - \left\{ {m \atop kj} \right\} A^\alpha_{mi} - \left\{ {m \atop ki} \right\} A^\alpha_{jm}.$$

Then $(\nabla_k A^\alpha_{ji} X^k Y^j Z^i)\partial/\partial y^\alpha$ is the local expression of a vector field defined along $f(M)$. Taking accound of (1.4), (1.5) and (2.1), we obtain the following formula of Ricci-type:

$$(2.2)\qquad \nabla_k\nabla_j A^\alpha_i - \nabla_j\nabla_k A^\alpha_i = \bar R_{\delta\gamma\beta}{}^\alpha A^\delta_k A^\gamma_j A^\beta_i - R_{kji}{}^h A^\alpha_h,$$

where $\bar R_{\delta\gamma\beta}{}^\alpha$ and $R_{kji}{}^h$ are components of the curvature tensors of $\bar g$ and g, respectively. We are now going to compute the Laplacian $\Delta\|df\|^2$. We here have

$$(2.3)\qquad \frac{1}{2}\Delta\|df\|^2 = \frac{1}{2} g^{lk}\nabla_l\nabla_k (A^\beta_j A^\alpha_i g^{ji}\bar g_{\beta\alpha})$$
$$= g^{lk}(\nabla_l\nabla_k A^\beta_j) A^\alpha_i g^{ji}\bar g_{\beta\alpha} + \|B\|^2,$$

where

$$\|B\|^2 = A^\beta_{lk} A^\alpha_{ji} g^{lj} g^{ki} \bar g_{\beta\alpha} = {}'\|D\|^2 + \|H\|^2,$$
$$(2.4)\qquad {}'\|D\|^2 = D^m_{lk} D^h_{ji} g^{lj} g^{ki} g^*_{mh}, \qquad \|H\|^2 = H_{lk} H_{ji} g^{lj} g^{ki}.$$

Thus, using (2.2) and putting

$$\bar R_{\delta\gamma\beta\alpha} = \bar R_{\delta\gamma\beta}{}^\lambda \bar g_{\lambda\alpha},$$

we obtain from (2. 3)

$$\frac{1}{2}\Delta\|df\|^2=(\nabla_j A^\beta)A_i^\alpha g^{ji}\bar{g}_{\beta\alpha}+\|B\|^2$$
$$+\bar{R}_{\delta\gamma\beta\alpha}A_l^\delta A_i^\gamma A_k^\beta A_j^\alpha g^{lk}g^{ji}+R_i^h g^*_{hj}g^{ij},$$

where $\nabla_j A^\alpha$ was defined by (1. 15), and then

$$\frac{1}{2}\Delta\|df\|^2-\delta S=-\|T\|^2+\|B\|^2$$
$$+\bar{R}_{\delta\gamma\beta\alpha}A_l^\delta A_i^\gamma A_k^\beta A_j^\alpha g^{lk}g^{ji}+R_j^h g^*_{hi}g^{ji},$$

where $R_k^h=R_{kji}{}^h g^{ji}$ are components of the Ricci tensor of g and

(2. 5) $$\delta S=g^{ji}\nabla_j(A_i^\beta A^\alpha\bar{g}_{\beta\alpha}),\qquad \|T\|^2=A^\beta A^\alpha\bar{g}_{\beta\alpha}.$$
$$=g^{ji}\nabla_j(E^h g^*_{hi}),$$

Substituting (2. 5) into the equation above, we have

(2. 6) $$\frac{1}{2}\Delta\|df\|^2-\delta S=-'\|E\|^2-h^2+\|H\|^2+'\|D\|^2$$
$$+\bar{R}_{\delta\gamma\beta\alpha}A_l^\delta A_i^\gamma A_k^\beta A_j^\alpha g^{lk}g^{ji}+R_j^h g^*_{hi}g^{ji},$$

where we have used

$$\|T\|^2='\|E\|^2+h^2,\qquad '\|E\|^2=g^*_{ji}E^jE^i.$$

Since $E=0$ implies $\delta S=0$ as a consequence of (2. 5), we have by using (2. 6)

LEMMA 2. 1. *For a relatively harmonic immersion* $f:(M, g)\to(N, \bar{g})$, *we have*

(2. 7) $$\frac{1}{2}\Delta\|df\|^2='\|D\|^2+\|H\|^2-h^2$$
$$+\bar{R}_{\delta\gamma\beta\alpha}A_l^\delta A_i^\gamma A_k^\beta A_j^\alpha g^{lk}g^{ji}+R_j^h g^*_{hi}g^{ji}.$$

Next, putting

(2. 8) $$L_{ji}=H_{ji}-\frac{1}{n}hg_{ji},\qquad \|L\|^2=L_{lk}L_{ji}g^{lj}g^{ki},$$

we obtain

(2. 9) $$\|L\|^2=\|H\|^2-\frac{1}{n}h^2.$$

Thus, substituting (2. 9) into (2. 7), we have

LEMMA 2. 2 *For a relatively harmonic immersion* $f:(M, g)\to(N, \bar{g})$, *we have*

(2. 10) $$\frac{1}{2}\Delta\|df\|^2 = '\|D\|^2 + \|L\|^2 - \frac{n-1}{n}h^2 + \bar{R}_{\delta\gamma\beta\alpha} A_l^\delta A_i^\gamma A_k^\beta A_j^\alpha g^{lk} g^{ji} + R_j{}^h g^*_{hi} g^{ji}.$$

§ 3. Some inequalities

We shall, in this section, give some inequalities for later use, assumeing that M is compact. At each point of (M, g), we take n orthonormal vectors $e_{(1)}, \cdots, e_{(n)}$ such that

(3. 1) $$g^*_{ji} = \lambda_1 e_{(1)j} e_{(1)i} + \cdots + \lambda_n e_{(n)j} e_{(n)i},$$

where $e^h_{(s)}$ are components of $e_{(s)}$ and $e_{(s)i} = e^h_{(s)} g_{hi}$, and hence we have $\lambda_1, \cdots, \lambda_n > 0$ because (g^*_{ji}) is positive definite.

On putting $\bar{e}_{(s)} = (df)e_{(s)}$, $\bar{e}_{(1)}, \cdots, \bar{e}_{(n)}$ are linearly independent and tangent to $f(M)$. Denoting by $\bar{e}^\alpha_{(s)}$ components of $\bar{e}_{(s)}$, we obtain

(3. 2) $$\bar{e}^\alpha_{(s)} = A_i^\alpha e^i_{(s)}$$

and hence

(3. 3) $$\|\bar{e}_{(s)}\|_2 = \bar{g}_{\beta\alpha} \bar{e}^\beta_{(s)} \bar{e}^\alpha_{(s)} = \lambda_s, \quad \langle \bar{e}_{(r)}, \bar{e}_{(s)} \rangle = \bar{g}_{\beta\alpha} \bar{e}^\beta_{(r)} \bar{e}^\alpha_{(s)} = 0, \quad (r \neq s)$$

because of (3. 1). On the other hand, we have

$$A_l^\gamma A_k^\beta g^{lk} = A_l^\gamma A_k^\beta \sum_s e^l_{(s)} e^k_{(s)} = \bar{e}^\gamma_{(s)} \bar{e}^\beta_{(s)}$$

because of (3. 2). Thus, using the equation above, we find

$$\bar{R}_{\delta\gamma\beta\alpha} A_l^\delta A_i^\gamma A_k^\beta A_j^\alpha g^{lk} g^{ji} = \sum_{r \neq s} \bar{R}_{\delta\gamma\beta\alpha} \bar{e}^\delta_{(r)} \bar{e}^\gamma_{(s)} \bar{e}^\beta_{(r)} \bar{e}^\alpha_{(s)},$$

from which, using (3. 3),

(3. 4) $$\bar{R}_{\delta\gamma\beta\alpha} A_l^\delta A_i^\gamma A_k^\beta A_j^\alpha g^{lk} g^{ji} = -\sum_{r \neq s} \bar{\sigma}(\bar{e}_{(r)}, \bar{e}_{(s)}) \lambda_r \lambda_s,$$

where $\bar{\sigma}(\bar{X}, \bar{Y})$ denotes the sectional curvature of $(N, \bar{g})$.

We now consider the following condition:

(C) *There exists a constant* c *such that*

$$c \geqq \bar{\sigma}(\bar{X}, \bar{Y})$$

at any point $p \in N$ *for any two linearly independent vectors* $\bar{X}$ *and* $\bar{Y}$ *at* p.

Under the condition (C), since $\lambda_s > 0$, we have from (3. 4)

(3. 5) $$\bar{R}_{\delta\gamma\beta\alpha} A_l^\delta A_i^\gamma A_k^\beta A_j^\alpha g^{lk} g^{ji} \geqq -c \sum_{r \neq s} \lambda_r \lambda_s.$$

On putting

(3.6) $$\tilde{\lambda}=\frac{1}{n}\sum_{s}\lambda_s,$$

we easily obtain

(3.7) $$\sum_{r\neq s}\lambda_r\lambda_s=-\sum_{s}(\lambda_s-\tilde{\lambda})^2+n(n-1)\tilde{\lambda}^2,$$

where we can easily verify

(3.8) $$n\tilde{\lambda}=g^*_{ji}g^{ji}=\|df\|^2=\text{Trace } g^*.$$

Thus, substituting (3.7) into (3.5), we have

(3.9) $$\bar{R}_{\delta\gamma\beta\alpha}A^\delta_l A^\gamma_i A^\beta_k A^\alpha_j g^{lk}g^{ji}\geqq c\sum_{s}(\lambda_s-\tilde{\lambda})^2-n(n-1)c\tilde{\lambda}^2$$

when the condition (C) is satisfied.

We take n orthonormal vectors $e_{(s)}$ satisfying (3.1) at each point of (M, g). Then we have

(3.10) $$h=H_{ji}g^{ji}=\sum_{s}H_{ji}e^j_{(s)}e^i_{(s)}.$$

Next let $\bar{a}_1(p), \cdots, \bar{a}_n(p)$ be eigenvalues of H with respect to g^* at $p\in M$. Then we can put

(3.11) $$A=\underset{p\in M}{\text{Max}}\,\text{Max}\left\{|\bar{a}_1(p)|, \cdots, |\bar{a}_n(p)|\right\}\geqq 0,$$

provided that M is compact. Then for any vector field $X=X^h\partial/\partial x^h$, we get

$$|H_{ji}X^jX^i|\leqq A(g^*_{ji}X^jX^i),$$

from which, using (3.10),

(3.12) $$|h|\leqq nA\tilde{\lambda}, \text{ i.e.,}\qquad h^2\leqq n^2A^2\tilde{\lambda}^2,$$

when M is compact.

In the last step, using (3.1), we have

(3.13) $$R_j{}^h g^*_{hi}g^{ji}=\lambda_1(R_{ji}e^j_{(1)}e^i_{(1)})+\cdots+\lambda_n(R_{ji}e^j_{(n)}e^i_{(n)}),$$

where $R_{ji}=R_j{}^h g_{hi}$, and hence

(3.14) $$\tilde{\lambda}r\leqq R_j{}^h g^*_{hi}g^{ji},$$

where we have put

(3.15) $$\frac{r}{n}=\text{Min } R_{ji}A^jA^i,$$

$A=A^h\partial/\partial x^h$ running over the unit sphere bundle over (M, g), provided that M is compact.

Using (3.9), (3.12) and (3.14) and taking account of Lemma 2.2, we have

LEMMA 3.1. *Assume that the conditions (C) is satisfied for a relatively harmonic immersion* $f:(M, g)\to(N, \bar{g})$ *and that M is compact. Then we have*

$$\frac{1}{2}\Delta\|df\|^2 \geqq '\|D\|^2+\|L\|^2+c\sum_s(\lambda_s-\bar{\lambda})^2 -n(n-1)(A^2+c)\bar{\lambda}^2+r\bar{\lambda}. \tag{3.16}$$

We now put, assuming that M is compact,

$$A'=\frac{1}{n}\mathrm{Max}\,|h(p)|. \tag{3.17}$$

Then, substituting (3.9), (3.14) and (3.17) into (2.10) given in Lemma 2.2, we have

LEMMA 3.2. *Assume that the condition (C) is satisfied for a relatively harmonic immersion* $f:(M, g)\to(N, \bar{g})$ *and that M is compact. Then we have*

$$\frac{1}{2}\Delta\|df\|^2 \geqq '\|D\|^2+\|L\|^2+c\sum_s(\lambda_s-\bar{\lambda})^2 -n(n-1)c\bar{\lambda}^2+r\bar{\lambda}-n(n-1)A'^2. \tag{3.18}$$

§ 4. Theorems.

First we shall give some remarks. The condition $'\|D\|^2=0$ implies $D=0$, which means that $f:(M, g)\to(N, \bar{g})$ is relatively affine. The condition $\sum(\lambda_s-\bar{\lambda})^2=0$ implies $g^*=\rho^2 g$. Thus, if $'\|D\|^2=0$ and $\sum(\lambda_s-\bar{\lambda})=0$, then $f:(M, g)\to(N, \bar{g})$ is relatively homothetic. The condition $\|L\|^2=0$ implies $L_{ji}=0$, i.e., $H_{ji}=\frac{h}{n}g_{ji}$. When the condition $H_{ji}=\frac{h}{n}g_{ji}$ is satisfied, $f:(M, g)\to(N, \bar{g})$ is said to be *relatively umbilic.* If $f:(M, g)\to(N, \bar{g})$ is relatively homothetic and relatively umbilic at the same time, then the isometric immersion $f:(M, g^*)\to(N, \bar{g})$ is umbilic, i.e., $H=\frac{\bar{h}}{n}g^*$. Taking account of remarks given above and Lemma 3.1, we now have

THEOREM 4.1. *Let* $f:(M, g)\to(N, \bar{g})$ *be a relatively harmonic immersion of a Riemannian space* (M, g) *of dimension n into another* $(N, \bar{g})$ *of dimension* $n+1$ *and M be compact. Then,*

(1) $f:(M, g)\to(N, \bar{g})$ *is relatively homothetic and the isometric immersion* $f:(M, g^*)\to(N, \bar{g})$ *is umbilic or totally geodesic if the following condition* (A_1) *is satisfied:*

(A$_1$) $$\text{Trace } g^* \leqq \frac{r}{(n-1)(A^2+c)}$$

when there is a constant $c>0$ *such that* $c \geqq \bar{\sigma}$, $\bar{\sigma}$ *being the sectional curvature of* $(N, \bar{g})$, *and* (M, g) *has positive definite Ricci tensor*;

(2) $f:(M, g) \to (N, \bar{g})$ *is relatively affine and relatively umbilic if the following condition* (A$_2$) *is satisfied*:

(A$_2$) $$\text{Trace } g^* \leqq \frac{r}{(n-1)A^2}$$

when $\bar{\sigma} \leqq 0$, $A>0$ *and* (M, g) *has positive definite Ricci tensor*; *where* A *and* r *are defined respectively by* (3.11) *and* (3.15). *In case* (2), Trace g^* *is necessarily constant.*

Assume now that $\bar{\sigma} \leqq c=0$ and $A=0$ in Lemma 3.1. Then, M being compact, the condition $r \geqq 0$ implies $D=0$, $L=0$ and $r=0$. Thus, using (2.10), (3.4) and (3.14), we find

$$\sum_{r \neq s} \bar{\sigma}(\bar{e}_{(r)}, \bar{e}_{(s)}) \lambda_r \lambda_s = R_j{}^h g^*_{hi} g^{ji} \geqq 0 .$$

On the other hand, since $\bar{\sigma} \leqq 0$, we obtain

$$\sum_{r \neq s} \bar{\sigma}(\bar{e}_{(r)}, \bar{e}_{(s)}) \lambda_r \lambda_s \leqq 0 .$$

Therefore, we have

(4.1) $$R_j{}^h g^*_{hi} g^{ji} = 0$$

when $r \geqq 0$. The condition $r \geqq 0$ is satisfied if and only if the Ricci tensor of (M, g) is positive semi-definite. Then, using (3.13) and (4.1), we have $R_{ji} e^j_{(1)} e^i_{(1)} = \cdots = R_{ji} e^j_{(n)} e^i_{(n)} = 0$, which means that the Ricci tensor of (M, g) vanishes. Summing up, we have

THEOREM 4.2. *If, in Theorem* 4.1, *the following condition* A_3 *is satisfied, then* $f:(M, g^*) \to (N, \bar{g})$ *is relatively affine and* (M, g) *has vanishing Ricci tensor*:

(A$_3$) $\bar{\sigma} \leqq 0$, $f:(M, g^*) \to (N, \bar{g})$ *is totally geodesic and* (M, g) *has positive semidefinite Ricci tensor. In this case,* Trace g^* *is necessarily constant.*

If in case (1) of Theorem 4.1 $(N, \bar{g})$ is a sphere $(S^{n+1}, \bar{g}_0)$ with constant curvature c, then (M, g) is necessarily a sphere (S^n, g_0) with constant curvature. If in case (2) of Theorem 4.1 $(N, \bar{g})$ is a Euclidean space $(E^{n+1}, \bar{g}_0)$, then (M, g) becomes a sphere (S^n, g_0) of constant curvature and $f:(M, g) \to (N, \bar{g})$ is a relatively homothetic immersion, because in this case (M, g) is an irreducible Riemannian space.

If in Theorem 4.2 $(N, \bar{g})$ is a flat torus, then (M, g) is necessarily a flat torus.

Taking account of Lemma 3.2, we have

THEOREM 4.3. *Let* $f:(M, g)\to(N, \bar{g})$ *be a relatively harmonic immersion of a Riemannian space* (M, g) *of dimension* n *into another* $(N, \bar{g})$ *of dimension* $n+1$ *and* M *be compact. Then,*

(1) $f:(M, g)\to(N, \bar{g})$ *is relatively homothetic and the isometric immersion* $f:(M, g^*)\to(N, \bar{g})$ *is umbilic if the following condition* (B_1) *is satisfied:*

$$(\mathrm{B}_1)\qquad 0<n\alpha\leqq \mathrm{Trace}\ g^*\leqq n\beta,$$

α *and* β *being roots of the quadratic equation* $n(n-1)ct^2-rt+n(n-1)A'^2=0$, *when there is a constant* c *such that*

$$\frac{r^2}{4n^2(n-1)^2A'^2}\geqq c>0, \qquad c\geqq\bar{\sigma},$$

$\bar{\sigma}$ *being the sectional curvature of* $(N, \bar{g})$, *where* $A'>0$ *and* (M, g) *has positive definite Ricci tensor;*

(2) $f:(M, g)\to(N, \bar{g})$ *is relatively homothetic and* $f:(M, g^*)\to(N, \bar{g})$ *is totally geodesic, if the following condition* (B_2) *is satisfied:*

$$(\mathrm{B}_2)\qquad \mathrm{Trace}\ g^*\leqq\frac{r}{(n-1)c}$$

when there is a constant $c>0$ *such that* $c\geqq\bar{\sigma}$, *where* $f:(M, g)\to(N, \bar{g})$ *is relatively minimum, (i.e.,* $h=0$*) and* (M, g) *has positive definite Ricci tensor;*

(3) $f:(M, g)\to(N, \bar{g})$ *is relatively affine and relatively umbilic if the following condition* (B_3) *is satisfied:*

$$(\mathrm{B}_3)\qquad \mathrm{Trace}\ g^*\geqq\frac{n^2(n-1)A'^2}{r}$$

when $\bar{\sigma}\leqq 0$, $A'>0$ *and* (M, g) *has positive definite Ricci tensor; where* A' *and* r *are defined respectively by* (3.17) *and* (3.15). *In each case,* Trace g^* *is necessarily constant.*

We can easily prove the following Theorem 4.4 in the same way as taken in the proof of Theorem 4.2.

THEOREM 4.4. *If, in Theorem* 4.3, *the following condition* (B_4) *is satisfied, then* $f:(M, g)\to(N, \bar{g})$ *is relatively affine,* $f:(M, g^*)\to(N, \bar{g})$ *is totally geodesic and* (M, g) *has vanishing Ricci tensor:*

(B_4) $\bar{\sigma}\leqq 0$, $f:(M, g)\to(N, \bar{g})$ *is relatively minimum (i.e.,* $h=0$*),* (M, g) *has positive semi-definite Ricci tensor. In this case,* Trace g^* *is necessarily constant.*

In the last step, we assume that $f:(M, g)\to(N, \bar{g})$ is relatively harmonic and $f:(M, g^*)\to(N, \bar{g})$ is umbilical (or totally geodesic), i.e., $H=ag^*$ with

$a\neq 0$ (or $H=0$). Suppose moreover that $(N, \bar{g})$ is of constant curvature $\bar{c}$. Then, in the present case, we have

$$\begin{aligned} (4.2)\qquad \|H\|^2-h^2 &= a^2\Big(\sum_s \lambda_s^2-\big(\sum_s \lambda_s\big)^2\Big) \\ &= -a^2\sum_{r\neq s}\lambda_r\lambda_s = a^2\sum_s(\lambda_s-\bar{\lambda})^2-n(n-1)a^2\bar{\lambda}^2, \end{aligned}$$

$$\begin{aligned} (4.3)\qquad \bar{R}_{\delta\gamma\beta\alpha}A_l^\delta A_i^\gamma A_k^\beta A_j^\alpha g^{lk}g^{ji} &= -\bar{c}\sum_{r\neq s}\lambda_r\lambda_s \\ &= \bar{c}\sum_s(\lambda_s-\bar{\lambda})^2-n(n-1)\bar{c}\bar{\lambda}^2 \end{aligned}$$

because of (3.7), where we have used

$$\|H\|^2-h^2 = H_{lk}H_{ji}g^{lk}g^{ji}-(H_{ji}g^{ji})^2,$$

$$\bar{R}_{\delta\gamma\beta\alpha} = \bar{c}(\bar{g}_{\delta\beta}\bar{g}_{\gamma\alpha}-\bar{g}_{\gamma\beta}\bar{g}_{\delta\alpha}).$$

Substituting (3.15), (4.2) and (4.3) into (2.7), we have in the present case

$$\begin{aligned} (4.4)\qquad \frac{1}{2}\Delta\|df\|^2 \geqq\ &'\|D\|^2+(\bar{c}+a^2)\sum_s(\lambda_s-\bar{\lambda})^2 \\ &-\frac{n-1}{n}(\bar{c}+a^2)(\mathrm{Trace}\ g^*)^2+\frac{r}{n}(\mathrm{Trace}\ g^*). \end{aligned}$$

Taking account of (4.4), we have

THEOREM 4.5. *Let* $(N, \bar{g})$ *be a Riemannian space of dimension* $n+1$ *with constant curvature* $\bar{c}$ *and* (M, g) *a compact Riemannian space of dimension* n. *Assume that* $f:(M, g)\to(N, \bar{g})$ *is a relatively harmonic immersion and* $f:(M, g)\to(N, \bar{g})$ *is an umbilic (or totally geodesic) immersion, i.e.,* $H=ag^*$ *with* $a\neq 0$ *(or* $H=0$*). Then,*

(1) $f:(M, g)\to(N, \bar{g})$ *is relatively homothetic if the following condition* (D_1) *is satisfied*:

$$(\mathrm{D}_1)\qquad \mathrm{Trace}\ g^* \leqq \frac{r}{(n-1)(\bar{c}+a^2)}$$

when $\bar{c}+a^2>0$ *and* (M, g) *has positive definite Ricci tensor*;

(2) $f:(M, g)\to(N, \bar{g})$ *is relatively affine and* (M, g) *has vanishing Ricci tensor if the following condition* (D_2) *is satisfied*:

(D_2) $\bar{c}+a^2=0$, (M, g) *has positive semi-definite Ricci tensor. Where* r *is defined by* (3.15.) *In the case* (2), Trace g^* *is necessarily constant.*

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