Parallel tensors in the space-time V

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§1. Introduction.

The space-time V was defined by the present authors and its properties have been studied in detail in a series of papers [1], [2], $\cdots$, [10][1]. The present paper is a continuation of these ones, and naturally, the same notations and terminologies will be used throughout the paper.

The object of the present paper is to deal with the problem of determining all parallel tensors in the space-time V. A parallel tensor of weight m is defined by the equation

$$\nabla_j w_{i_1\cdots i_n} = 0, \tag{1.1}$$

where ∇_j denotes covariant derivative. The problem for spherically symmetric space-times has been solved by one of the present authors. (Cf. Chap. IX of [11] and Chap. X of [12].) As was stated in §102 and §104 of [12], we succeeded to solve the problem for general n and for almost all kinds of spherically symmetric space-times.

As will be seen in due course, the problem for the V is completely solved only for $n \leqq 3$, contrary to the case of spherically symmetric space-times. We hope that some systematic method of dealing with the problem of parallel tensors for general n and for general Riemannian space will appear, and expect that the results obtained in the present paper will be of use in the investigation of the problem as presenting some remarkable examples.

§2. Preliminaries, 1.

The following results are obtained in §99 of [12]:

(i) When the $v_{i_1\cdots i_n}$ is a scalar v, (1.1) becomes

$$\nabla_j v = \partial_i v = 0, \tag{2.1}$$

and the general solution is given by $v=$const., as is well-known.

(ii) When the $v_{i_1\cdots i_n}$ is a relative scalar of weight m, the general solution is given by

$$v = (\sqrt{-g})^m \times \text{const.}, \tag{2.2}$$

1) Numbers in brackets refer to the references at the end of the paper.

where g is the determinant of the metric tensor g_{ij}. (Note that in [12], $\sqrt{g}$ was used in place of the above $\sqrt{-g}$, but this is of no importance.)

(iii) The general form of the relative tensor of weight m is given by multiplying that of the parallel ordinary tensor by $(\sqrt{-g})^m$.

These results are given in [12] with respect to spherically symmetric space-times. But it is evident that they are valid for any Riemannian space. Thus, considering the fact that the equation (1. 1) is independent of raising or lowering of the indices, we shall deal only with the ordinary covariant tensors exclusively in the following.

Now, in order to deal with the condition in another form, which is convenient for our present purpose, we take any orthonormal set of vectors $\overset{\alpha}{h}_i$ and $\underset{\alpha}{h}^i$ $(i, j, \cdots = 1, \cdots, 4\,;\ \alpha, \beta, \cdots = 1, \cdots, 4)$ satisfying

$$\sum_\alpha \overset{\alpha}{h}_i \overset{\alpha}{h}_j = g_{ij},\ \ \sum_\alpha \underset{\alpha}{h}^i \underset{\alpha}{h}^j = g^{ij}\,;\ \ \sum_\alpha \overset{\alpha}{h}_i \underset{\alpha}{h}^j = \delta_i^j,\ \ \overset{\alpha}{h}_i \underset{\beta}{h}^i = \delta_\beta^\alpha\,;\ \ \underset{\alpha}{h}^i = g^{ij} \overset{\alpha}{h}_j, \tag{2. 3}$$

and put

$$\gamma_{\alpha\beta\gamma} = (\nabla_i \overset{\alpha}{h}_j)\, \underset{\beta}{h}^j \underset{\gamma}{h}^i = -\gamma_{\beta\alpha\gamma}\,. \tag{2. 4}$$

$\gamma_{\alpha\beta\gamma}$ is nothing but the coefficient of rotation. Then (1. 1) is equivalent to the following equations for the scalar components $V_{\alpha_1\cdots\alpha_n} = \underset{\alpha_1}{h}^{i_1} \cdots \underset{\alpha_n}{h}^{i_n} v_{i_1\cdots i_n}$:

$$\underset{\beta}{h}^s \partial_s V_{\alpha_1\cdots\alpha_n} + \sum_\rho \{\gamma_{\rho\alpha_1\beta} V_{\rho\alpha_2\cdots\alpha_n} + \cdots + \gamma_{\rho\alpha_n\beta} V_{\alpha_1\cdots\alpha_{n-1}\rho}\} = 0\,. \tag{2. 5}$$

Thus our present problem is to solve (2. 5) for the given V. Here it should be noted that the components of $\overset{\alpha}{h}_i$, and hence those of $\underset{\alpha}{h}^i$ and $\gamma_{\alpha\beta\gamma}$, are not necessarily real, by virtue of the relation (2. 3). We use (2. 3), however, in order to deal with x^2, x^3 and x^4 (i.e. y, z and t) symmetrically. Further, as is easily seen, the results to be obtained are independent of the special choice of the coordinate system and that of the ennuple $\overset{\alpha}{h}_i$.

§3. Preliminaries, 2.

Now we give some fundamental properties of V which will play important roles in the present paper. V is a space-time whose metric can be brought into the form

$$ds^2 = -dx^2 - Bdy^2 - Cdz^2 + Ddt^2, \qquad \big((x^i) \equiv (x, y, z, t)\big), \tag{3. 1}$$

where B, C and D are positive-valued functions of x. A characteristic system (abbreviated to c. s.) is defined as a set of 4 unit vectors and some scalars satisfying some tensor equations. All intrinsic properties of V can be expressed in terms of c. s. Among these scalars, the most fundamental ones are λ_2, λ_3 and λ_4. They are given by

$$(3.2) \qquad \lambda_2 = -\beta/2,\ \lambda_3 = -\gamma/2,\ \lambda_4 = -\delta/2,\ (\beta \equiv B'/B,\ \gamma \equiv C'/C,\ \delta \equiv D'/D),$$

in the coordinate system of (3. 1), which is called standard for the c. s. Here a prime denotes the derivative with respect to x. (Cf. §3 of [1].)

Now the following theorem is proved in [2] by using the coordinate system of (3. 1) (Prop. 5. 6):

PROPOSITION 3. 1. (I) *When $\beta\gamma\delta \neq 0$, the V admits no parallel vector field (hereafter 'field' will be omitted).* (II) *When one of* (II$_2$) ($\beta=0$, $\gamma\delta\neq0$), (II$_3$) ($\gamma=0$, $\delta\beta\neq0$) *and* (II$_4$) ($\delta=0$, $\beta\gamma\neq0$) *holds, we have one and only one parallel vector v_i to within a constant multiplier. It is $c\delta_i^2$, $c\delta_i^3$ (space-like), and $c\delta_i^4$ (time-like) respectively. Here c is a non-vanishing arbitrary constant.* (III) *When one of* (III$_2$) ($\gamma=\delta=0$, $\beta\neq0$), (III$_3$) ($\delta=\beta=0$, $\gamma\neq0$) *and* (III$_4$) ($\beta=\gamma=0$, $\delta\neq0$) *holds and the V is non-flat, we have two linearly independent (with constant coefficients) parallel vectors. They are $c\delta_i^3+c'\delta_i^4$, $c\delta_i^4+c'\delta_i^2$ (space-like, null, time-like) and $c\delta_i^2+c'\delta_i^3$ (space-like) respectively. Here c and c' are arbitrary constants which do not satisfy $c=c'=0$. When the V is flat, we have evidently 4 linearly independent parallel vectors.* (IV) *When $\beta=\gamma=\delta=0$ holds, the V is flat.*

Further it is evident from the results in [2] that the condition that the V in the case of (III$_a$), ($a=2, 3, 4$), be flat is given by $k_2 \equiv -B'/2\sqrt{B} = \text{const.} \neq 0$ or $k_3 \equiv -C'/2\sqrt{C} = \text{const.} \neq 0$ or $k_4 \equiv iD'/2\sqrt{D} = \text{const.} \neq 0$ respectively. (Again, we use the purely imaginary quantity k_4 for convenience' sake.) The condition $k_a=0$ is equivalent to $\beta=0$ or $\gamma=0$ or $\delta=0$ respectively.

The V in the case of (I) is characterized by the condition that it admit a c. s. satisfying $\lambda_2\lambda_3\lambda_4\neq0$. Similarly, the V in the case of (II$_a$) is characterized by the existence of a c. s. satisfying $\lambda_2=0$, $\lambda_3\lambda_4\neq0$) or ($\lambda_3=0$, $\lambda_4\lambda_2\neq0$) or ($\lambda_4=0$, $\lambda_2\lambda_3\neq0$) respectively, and the V in the case of (III$_a$) admitting 2 independent parallel vectors is characterized by the conditions that it is non-flat and admits a c. s. satisfying ($\lambda_2\neq0$, $\lambda_3=\lambda_4=0$) or ($\lambda_3\neq0$, $\lambda_4=\lambda_2=0$) or ($\lambda_4\neq0$, $\lambda_2=\lambda_3=0$) respectively. We denote these V's by V(I), V(II$_2$), $\cdots$, V(III$_2$), $\cdots$ respectively.

On the other hand, it is shown in [12] that in $S(B)$, i. e. the flat space-time, the problem of parallel tensors is completely solved for arbitrary n as follows:

PROPOSITION 3. 2. *In the space-time $S(B)$, a parallel tensor of n-th order is given by $(\overset{1}{\lambda}, \overset{2}{\lambda}, \overset{3}{\lambda}, \overset{4}{\lambda})_n$, where $\overset{\alpha}{\lambda}=\overset{\alpha}{\lambda}_i$, $(\alpha=1, \cdots, 4)$, are 4 parallel unit* *vectors mutually orthogonal and the notation* $(\overset{1}{\lambda}, \cdots)_n$ *denotes any linear combination (with constant coefficients) of tensors of n-th order made from* $\overset{\alpha}{\lambda}$*'s by outer products.*

Evidently g_{ij} is parallel. Further if we put $\eta_{ijlm}=\sqrt{-g}\,\varepsilon_{ijlm}$, where ε_{ijlm} is anti-symmetric with respect to each pair of indices and $\varepsilon_{1234}=1$, it is also parallel. (If the group of coordinate transformations contains the transformations whose Jacobians are negative, η_{ijlm} is defined by $\rho\sqrt{-g}\,\varepsilon_{ijlm}$, where ρ is a pseudo-scalar satisfying $|\rho|=1$.) These g_{ij} and η_{ijlm} are contained in the expression $(\overset{1}{\lambda}, \cdots)_n$ implicitly.

§4. Equations to be solved.

We use the coordinate system of (3. 1) and take the orthonormal ennuple defined by

$$\overset{1}{h}_i = i\delta_i^1, \quad \overset{2}{h}_i = i\sqrt{B}\,\delta_i^2, \quad \overset{3}{h}_i = i\sqrt{C}\,\delta_i^3, \quad \overset{4}{h}_i = \sqrt{D}\,\delta_i^4. \tag{4. 1}$$

Then the non-vanishing components of $\gamma_{\alpha\beta\gamma}$ are given by

$$\gamma_{122} = -\gamma_{212} = -i\beta/2, \quad \gamma_{133} = -\gamma_{313} = -i\gamma/2, \quad \gamma_{144} = -\gamma_{414} = -i\delta/2, \tag{4. 2}$$

and the equation (2. 5) becomes

$$\partial_x V_{\alpha_1\cdots\alpha_n} = 0, \quad \text{i.e. } V_{\alpha_1\cdots\alpha_n} = \textit{function of } (y, z, t), \tag{4. 3a}$$

$$\begin{aligned}\partial_b V_{\alpha_1\cdots\alpha_n} = k_b \Big\{ &(\delta^b_{\alpha_1} V_{1\alpha_2\cdots\alpha_n} + \delta^b_{\alpha_2} V_{\alpha_1 1\alpha_3\cdots\alpha_n} + \cdots + \delta^b_{\alpha_n} V_{\alpha_1\cdots\alpha_{n-1}1}) \\ &-(\delta^1_{\alpha_1} V_{b\,\alpha_2\cdots\alpha_n} + \delta^1_{\alpha_2} V_{\alpha_1 b\cdots\alpha_n} + \cdots + \delta^1_{\alpha_n} V_{\alpha_1\cdots\alpha_{n-1}b})\Big\},\end{aligned} \tag{4. 3b}$$

where $(b=2, 3, 4)$ and $\partial_a = \partial/\partial x^a$. (4. 3) gives the equations to be solved.

The integration of (4. 3a) is evident. When n is large, however, it is very laborious to integrate (4. 3b), which is composed of 3×4^n equations. To execute the calculations easily as far as possible, we further classify V(I)'s and V(II)'s as follows: V(I) is $V(\mathrm{I})_a$ when all of k_2, k_3 and k_4 are non-constant, $V(\mathrm{I})_b$ when one of k_a's is a non-zero constant and the remaining two are non-constant, $V(\mathrm{I})_c$ when two of k_a's are non-zero constants and the remaining one is non-constant, and $V(\mathrm{I})_d$ when all of k_a's are non-zero constants. Similarly, V(II) is $V(\mathrm{II})_a$ when the two surviving k_a's are non-constants, $V(\mathrm{II})_b$ when one is non-constant and the other is a non-zero constant, and $V(\mathrm{II})_c$ when both of the two are non-zero constants. In general, the solution of (4. 3b) is very easy when $k_b \neq$ const. holds, compared with the case of $k_b=$const. $(\neq 0)$.

When $n=1$, (4. 3b) becomes

$$\partial_y V_1 = -k_2 V_2,\quad \partial_y V_2 = k_2 V_1,\quad \partial_y V_3 = \partial_y V_4 = 0, \text{ and cycl.}, \tag{4.4}$$

where *cycl.* means the expressions obtained by the cyclic changes of (2, 3, 4) and (y, z, t). Then it is easy to obtain Prop. 3. 1.

§ 5. When $n=2$.

When $n=2$, (4. 3b) becomes

$$\left\{\begin{aligned} &\partial_y V_{11}+k_2(V_{12}+V_{21})=\partial_y V_{22}-k_2(V_{12}+V_{21}) \\ &\qquad =\partial_y V_{12}-k_2(V_{11}-V_{22})=\partial_y V_{21}-k_2(V_{11}-V_{22})=0, \\ &\partial_y V_{13}+k_2 V_{23}=\partial_y V_{31}+k_2 V_{32}=\cdots=\partial_y V_{24}-k_2 V_{14}=\partial_y V_{42}-k_2 V_{41}=0, \\ &\partial_y V_{33}=\partial_y V_{34}=\partial_y V_{43}=\partial_y V_{44}=0, \qquad \text{and cycl.} \end{aligned}\right. \tag{5.1}$$

By solving (5. 1), we can easily obtain: (*I*) When the V is V(I), we have $V_{\alpha\beta}=e\delta_{\alpha\beta}$, from which we have

$$v_{ij}=c g_{ij}. \tag{5.2}$$

Here e and c are arbitrary constants. (*II*) Next we consider the case in which the V is V(II). If we take $V(\mathrm{II}_2)$, for example, we have $V_{\alpha\beta}=e_1\delta_{\alpha\beta}+e_2\delta^2_\alpha\delta^2_\beta$, i. e.

$$v_{ij}=c_1 g_{ij}+c_2 \overset{2}{v}_i \overset{2}{v}_j. \tag{5.3}$$

Here e's and c's are arbitrary constants and $\overset{2}{v}_i$ is the parallel unit vector whose components are given by $\overset{2}{v}_i=\delta^2_i$ in the coordinate system in which $B=1$ holds (cf. Prop. 3.1). (Note that the scalar components of $\overset{2}{v}_i$ are given by δ^2_α.) (*III*) Lastly we consider V(III). If we take $V(\mathrm{III}_2)$, for example, we can obtain $V_{\alpha\beta}=e_1\delta_{\alpha\beta}+e_2\delta^3_\alpha\delta^3_\beta+e_2'\delta^4_\alpha\delta^4_\beta+e_3\delta^3_\alpha\delta^4_\beta+e_3'\delta^4_\alpha\delta^3_\beta+e_4\varepsilon_{\alpha\beta\gamma\delta}\delta^\gamma_3\delta^\delta_4$, i. e.

$$v_{ij}=c_1 g_{ij}+c_2 \overset{3}{v}_i \overset{3}{v}_j+c_2' \overset{4}{v}_i \overset{4}{v}_j+c_3 \overset{3}{v}_i \overset{4}{v}_j+c_3' \overset{4}{v}_i \overset{3}{v}_j+c_4 \eta_{ijlm} \overset{3}{v}{}^l \overset{4}{v}{}^m, \tag{5.4}$$

where e's and c's are arbitrary constants, and $\overset{3}{v}_i$ and $\overset{4}{v}_i$ are the independent parallel vectors given in Prop. 3.1 in the case of (III_2).

§ 6. When $n=3$.

In this case (4. 3b) is composed of 3×4^3 equations:

$$\left\{\begin{aligned} &\partial_y V_{111}+k_2(V_{211}+V_{121}+V_{112})=\partial_y V_{112}+k_2(V_{212}+V_{122}-V_{111})=0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\partial_y V_{441}+k_2 V_{442}=\partial_y V_{442}-k_2 V_{441}=\partial_y V_{443}=\partial_y V_{444}=0, \qquad \text{and cycl.} \end{aligned}\right. \tag{6.1}$$

We denote by (E_2) the equation (4. 3b) with $b=2$. Then we can show that, when $k_2 \neq \text{const.}$, (E_2) is equivalent to

$$(6.2)\quad \begin{cases} V_{\alpha\beta\gamma} = V_{\alpha\beta\gamma}(z, t)\,; \; V_{abc} = 0, \quad V_{12\rho} + V_{21\rho} \overset{P}{=} 0, \\ V_{11\rho} \overset{P}{=} V_{22\rho}, \quad V_{1\rho\sigma} \overset{P}{=} V_{2\rho\sigma} \overset{P}{=} 0, \qquad (a, b, c = 1, 2\,;\; \rho, \sigma = 3, 4), \end{cases}$$

where $\overset{P}{=}$ means that the $n!$ (in this case, $3!$) equations obtained by the permutation of n indices hold.

On the other hand, when $k_2 = \text{const.}$ $(\neq 0)$, (E_2) is equivalent to

$$(6.3)\quad \begin{cases} V_{11\rho} \overset{P}{=} p_{11\rho} \sin 2k_2 y + q_{11\rho} \cos 2k_2 y + r_{11\rho}, \\ V_{22\rho} \overset{P}{=} -p_{11\rho} \sin 2k_2 y - q_{11\rho} \cos 2k_2 y + r_{11\rho}, \\ V_{12\rho} \overset{P}{=} -p_{11\rho} \cos 2k_2 y + q_{11\rho} \sin 2k_2 y + r_{12\rho}, \\ V_{21\rho} \overset{P}{=} -p_{11\rho} \cos 2k_2 y + q_{11\rho} \sin 2k_2 y - r_{12\rho}\,; \end{cases}$$

$$(6.4)\quad \begin{cases} V_{\rho\sigma 1} \overset{P}{=} a_{\rho\sigma 1} \sin k_2 y + b_{\rho\sigma 1} \cos k_2 y, \quad V_{\rho\sigma 2} \overset{P}{=} -a_{\rho\sigma 1} \cos k_2 y + b_{\rho\sigma 1} \sin k_2 y\,; \\ V_{\rho\sigma\tau} = V_{\rho\sigma\tau}(z, t)\,; \end{cases}$$

$$(6.5)\quad \begin{cases} \partial_y V_{111} + k_2(V_{211} + V_{121} + V_{112}) = \partial_y V_{222} - k_2(V_{122} + V_{212} + V_{221}) = 0, \\ \partial_y V_{211} + k_2(-V_{111} + V_{221} + V_{212}) \overset{P}{=} \partial_y V_{122} + k_2(V_{222} - V_{121} - V_{112}) \overset{P}{=} 0. \end{cases}$$

Here $\rho, \sigma, \tau = 3, 4$, and p's, q's, r's, a's and b's are functions of (z, t).

Using these equations and dealing with the cases $V(\mathrm{I})_a$, $V(\mathrm{I})_b, \cdots$, $V(\mathrm{II})_a$, $V(\mathrm{II})_b, \cdots$, $V(\mathrm{III})$ separately, we can arrive at the following conclusion (again, the calculations are omitted for brevity's sake):

PROPOSITION 6.1. *When the V is* $V(\mathrm{I})$, *we have*

$$(6.6)\qquad v_{ijk} = 0.$$

When the V is $V(\mathrm{II})$, *we have*

$$(6.7)\qquad v_{ijk} = c_1 \mu_i g_{jk} + c_2 \mu_j g_{ik} + c_3 \mu_k g_{ij} + c_4 \mu_i \mu_j \mu_k + c_5 \eta_{ijkl} \mu^l,$$

where c's are arbitrary constants and μ_i *is the parallel unit vector uniquely determined (i.e.* $\overset{2}{v}_i$ *or* $\overset{3}{v}_i$ *or* $\overset{4}{v}_i$ *in* $V(\mathrm{II}_2)$ *or* $V(\mathrm{II}_3)$ *or* $V(\mathrm{II}_4)$ *respectively. When the V is* $V(\mathrm{III})$, *we have*

$$(6.8)\quad \begin{cases} v_{ijk} = (c_1 \mu_i + c_1' \nu_i) g_{jk} + (c_2 \mu_j + c_2' \nu_j) g_{ik} + (c_3 \mu_k + c_3' \nu_k) g_{ij} \\ \qquad + c_4 \mu_i \mu_j \mu_k + c_4' \nu_i \nu_j \nu_k + (c_5 \nu_i \mu_j \mu_k + c_5' \mu_i \nu_j \nu_k) + \cdots \\ \qquad + \left\{ (c_8 \mu_i + c_8' \nu_i) \eta_{jkpq} + \cdots + (c_{10} \mu_k + c_{10}' \nu_k) \eta_{ijpq} \right\} \mu^p \nu^q, \end{cases}$$

where c's and c′'s are arbitrary constants and μ_i *and* ν_i *are mutually orthogonal parallel unit vectors stated in Prop.* 3.1 *(i.e.* $(\overset{3}{v}_i, \overset{4}{v}_i)$ *or* $(\overset{4}{v}_i, \overset{2}{v}_i)$ *or*

$(\overset{2}{v}_i, \overset{3}{v}_i)$ *in* $V(\mathrm{II}_2)$ *or* $V(\mathrm{II}_3)$ *or* $V(\mathrm{II}_4)$ *respectively). The number of arbitrary constants is* 20.

Arranging the results in Prop. 3.1, §5 and Prop. 6.1, we have

PROPOSITION 6.2. *When the non-flat* V *is* $V(\mathrm{I})$ *or* $V(\mathrm{II})$ *or* $V(\mathrm{III})$, *we have*

$$v_{i_1\cdots i_n} = (g, \eta)_n \qquad or \tag{6.9}$$

$$= (g, \eta, \mu)_n \qquad or \tag{6.10}$$

$$= (g, \eta, \mu, \nu)_n \tag{6.11}$$

respectively, if we assume $1 \leqq n \leqq 3$.

Here the notation $(g, \eta, \mu)_n$, for example, means a tensor of n-th order obtained as a linear combination (with constant coefficients) of the tensors of n-th order each of which is composed of g_{ij}, η_{ijlm} and μ_i by the process of outer products and contractions. Examples are given in (5.3) $(\mu_i = \overset{2}{v}_i)$ and (6.7).

Conversely, it is evident that (6.9), (6.10) and (6.11) give parallel tensors in each V's, since g_{ij} and η_{ijlm} are parallel in all V's, μ_i is parallel in $V(\mathrm{II})$, and μ_i and ν_i are parallel in $V(\mathrm{III})$.

Our conjecture is that Prop. 6.2 holds for all $n \geqq 1$. At the present stage of the investigation, this is only a conjecture.

§7. When $n=4$.

As is stated at the end of the last section, we can not prove Prop. 6.2 even for the case of $n=4$ at the present stage. We have proved only (6.9) for $V(\mathrm{I})_a$, $V(\mathrm{I})_b$ and $V(\mathrm{I})_c$ assuming $n=4$. The proof for $V(\mathrm{I})_d$ is laborious, so we stop here, only noting the conditions corresponding to (6.2) and {(6.3), (6.4), (6.5)}.

When $k_2 \neq$ const., (E_2), which is composed of 4^4 equations, is equivalent to

$$\left\{\begin{array}{l} V_{\alpha\beta\gamma\delta} = V_{\alpha\beta\gamma\delta}(z, t)\,; \quad V_{\alpha\rho\sigma\tau} \overset{P}{=} 0, \quad V_{\rho abc} \overset{P}{=} 0, \\ V_{11\rho\sigma} \overset{P}{=} V_{22\rho\sigma}, \quad V_{12\rho\sigma} + V_{21\rho\sigma} \overset{P}{=} 0, \\ V_{1111} = V_{2222} = p+q+r, \quad (V_{1122} = V_{2211} = p, \quad V_{1212} = V_{2121} = q, \\ V_{1221} = V_{2112} = r), \quad V_{2111} = -V_{1222} = b_1, \quad V_{1211} = -V_{2122} = b_2, \\ V_{1121} = -V_{2212} = b_3, \quad V_{1112} = -V_{2221} = b_4, \\ b_1 + b_2 + b_3 + b_4 = 0, \end{array}\right. \tag{7.1}$$

where $(a, b, c=1, 2\,;\ \rho, \sigma, \tau=3, 4)$ and $p, q, r, b_1, \cdots, b_4$ are functions of (z, t).

When $k_2=\text{const.}\ (\neq 0)$, we have from (E_2)

$$(7.2)\quad \begin{cases} V_{\rho\sigma\tau 1} \overset{P}{=} p_{\rho\sigma\tau} \sin k_2 y + q_{\rho\sigma\tau} \cos k_2 y, \quad V_{\rho\sigma\tau 2} \overset{P}{=} -p_{\rho\sigma\tau} \cos k_2 y + q_{\rho\sigma\tau} \sin k_2 y\,, \\ V_{\rho\sigma 11} - V_{\rho\sigma 22} \overset{P}{=} r_{\rho\sigma} \sin 2k_2 y + s_{\rho\sigma} \cos 2k_2 y\,, \\ V_{\rho\sigma 12} + V_{\rho\sigma 21} \overset{P}{=} -r_{\rho\sigma} \cos 2k_2 y + s_{\rho\sigma} \sin 2k_2 y\,, \end{cases}$$

$$(7.3)\quad \begin{cases} V_{2111} + V_{1211} + V_{1121} + V_{1112} = -a \cos 2k_2 y + b \sin 2k_2 y \\ \qquad\qquad\qquad\qquad\qquad\qquad -m \cos 4k_2 y + n \sin 4k_2 y\,, \\ V_{1222} + V_{2122} + V_{2212} + V_{2221} = -a \cos 2k_2 y + b \sin 2k_2 y \\ \qquad\qquad\qquad\qquad\qquad\qquad +m \cos 4k_2 y - n \sin 4k_2 y\,, \end{cases}$$

$$(7.4)\quad \begin{cases} V_{1111} = (a/2) \sin 2k_2 y + (b/2) \cos 2k_2 y + (m/4) \sin 4k_2 y + (n/4) \cos 4k_2 y + f, \\ V_{2222} = -(a/2) \sin 2k_2 y - (b/2) \cos 2k_2 y + (m/4) \sin 4k_2 y + (n/4) \cos 4k_2 y + f, \\ V_{1122} + V_{2211} + V_{1212} + V_{2121} + V_{1221} + V_{2112} \\ \qquad = -(3/2)(m \sin 4k_2 y + n \cos 4k_2 y) + 2f, \end{cases}$$

$$(7.5)\quad \begin{cases} 3V^{\alpha}_{\rho 1} + X^{\alpha}_{\rho 2} \overset{P}{=} u_\rho \sin k_2 y + v_\rho \cos k_2 y\,, \\ 3V^{\alpha}_{\rho 2} + X^{\alpha}_{\rho 2} \overset{P}{=} -u_\rho \cos k_2 y + v_\rho \sin k_2 y\,, \end{cases}$$

where $(a, b, c=1, 2; \rho, \sigma, \tau=3, 4)$, r's, s's, a, b, m, n, f, u's and v's are arbitrary functions of (z, t). Further, for example, $V^1_{\rho 1} = V_{\rho 111}$, $V^1_{\rho 2} = V_{\rho 222}$, $X^1_{\rho 1} = V_{\rho 211} + V_{\rho 121} + V_{\rho 112}$ and $X^2_{\rho 2} = V_{1\rho 22} + V_{2\rho 12} + V_{2\rho 21}$, and the upper index α indicates the position of ρ.

Further it should be noted that the condition $\{(7.2), \cdots, (7.5)\}$ is not sufficient for (E_2) with $k_2=\text{const.} \neq 0$. Some other equations are necessary, but we omit them for brevity's sake.

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References

[1] H. TAKENO and S. KITAMURA: On some special kind of space-times, I, Tensor, N.S., **24** (1972), 266–272.

[2] H. TAKENO and S. KITAMURA: On some special kind of space-times, II, Hokkaido Math. J., **1** (1972), 43–62.

[3] H. TAKENO and S. KITAMURA: On the characteristic system of the space-time V, Tensor, N.S., **25** (1972), 63–75.

[4] H. TAKENO: Discrimination of the space-time V, I, Hokkaido Math. J., **2** (1973), 1–23.

[5] H. TAKENO: Discrimination of the space-time V, II, Hokkaido Math. J., **2** (1973), 26-39.

[6] H. TAKENO: On the recurrency of the space-time V, Tensor, N.S., **27** (1973), 309-318.

[7] H. TAKENO and S. KITAMURA: Group of motions in the space-time V, I, Tensor, N.S., **27** (1973), 197-207.

[8] H. TAKENO and S. KITAMURA: Group of motions in the space-time V, II, Tensor, N.S., **27** (1973), 265-275.

[9] H. TAKENO and S. KITAMURA: Group of motions in the space-time V, III, Tensor, N.S., **28** (1974), 25-33.

[10] H. TAKENO and S. KITAMURA: On the imbedding of the space-time V, Math Japon., **18** (1973), 137-148.

[11] H. TAKENO: The theory of spherically symmetric space-times, Sci. Rep. Res. Inst. Theor. Phys., Hiroshima Univ., No. **3** (1963).

[12] H. TAKENO: The theory of spherically symmetric space-times, (Revised edition), Sci. Rep. Res. Inst. Theor. Phys., Hiroshima Univ., No. **5** (1966).

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