On Jacobi fields in quaternion Kaehler manifolds with constant Q-sectional curvature

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Kosmanek [6] gave a characterization of Kaehler manifolds of constant holomorphic sectional curvature in relation with Jacobi fields. That is, the following property ($\mathscr{C}\mathscr{J}$) is satisfied if and only if the Kaehler manifold is of constant holomorphic sectional curvature:

($\mathscr{C}\mathscr{J}$) "*For a given geodesic* $\gamma(t)$ *in a Kaehler manifold* (J, g), *every Jacobi field* Y *along* γ *such that* $Y(0)=0$ *and* $\nabla_{\dot\gamma}Y(0)=J\dot\gamma(0)$, *is proportional to* $J\dot\gamma$, *where* $\dot\gamma(t)$ *denotes the tangent vector at* $\gamma(t)$".

The main purpose of this paper is to study the corresponding problem in quaternion Kaehler manifolds and characterize the manifolds of constant Q-sectional curvature, that is to prove Theorem 1.

On the other hand, Kashiwada [4] recently obtained analogous result for Sasakian manifolds (ϕ, ξ, g) with constant ϕ-holomorphic sectional curvature in terms of Jacobi field along geodesics orthogonal to ξ. From a point of view of submersion [8], the results for Kaehler manifolds and Sasakian manifolds are closely related and so are the relations between quaternion Kaehler manifolds and manifolds with Sasakian 3-structure $(\{\xi, \eta, \zeta\}, \tilde g)$. We apply Theorem 1 to study Jacobi fields in the manifolds with Sasakian 3-structure when each ϕ-, ψ- and θ-holomorphic sectional curvatures are constant on the distribution $\tilde D=\{\tilde X|\tilde g(\xi, \tilde X)=\tilde g(\eta, \tilde X)=\tilde g(\zeta, \tilde X)=0\}$.

§ 1. Quaternion Kaehlerian manifolds

Let M be a differentiable manifold of dimension n and assume that there is a 3-dimensional vector bundle V consisting of tensors of type (1.1) over M satisfying the condition:

"In any coordinate neighborhood U of M, there is a local base $\{F, G, H\}$ of V such that

$$(1.1)\quad \begin{aligned} &F^2=G^2=H^2=-I, \\ &GH=-HG=F, \qquad HF=-FH=G, \qquad FG=-GF=H, \end{aligned}$$

I denoting the identity tensor field of type (1.1) in M".

In an almost quaternion manifold (M, V), we take two intersecting coordinate neighborhoods U, U' and local basis $\{F, G, H\}$, $\{F', G', H'\}$ satisfying (1.1) in U and U', respectively, then they have relations in $U\cap U'$ as

$$
\begin{aligned}
F' &= s_{11}F+s_{12}G+s_{13}H, \\
G' &= s_{21}F+s_{22}G+s_{23}H, \\
H' &= s_{31}F+s_{32}G+s_{33}H,
\end{aligned}
\tag{1.2}
$$

where $s_{\alpha\beta}(\alpha, \beta=1, 2, 3)$ form an element $s_{UU'}=(s_{\alpha\beta})$ of the special orthogonal group SO(3) of dimension 3. In any almost quaternion manifold (M, V), there is a Riemannian metric g such that

$$
g(FX, Y)+g(X, FY)=0, \qquad g(GX, Y)+g(X, GY)=0,
$$
$$
g(HX, Y)+g(X, HY)=0
$$

hold for any local base $\{F, G, H\}$ and any vector fields X, Y. Assume that the Riemannian connection ∇ of (M, g) satisfies for any local base $\{F, G, H\}$

$$
\begin{aligned}
\nabla_X F &= \qquad\quad + r(X)G-q(X)H, \\
\nabla_X G &= -r(X)F \qquad\quad +p(X)H, \\
\nabla_X H &= q(X)F-p(X)G,
\end{aligned}
\tag{1.3}
$$

where p, q and r are certain 1-forms defined in U. Then (M, g, V) is called a *quaternion Kaehler manifold* (See [2]).

Given a vector X at a point P of M, we denote by $Q(X)$ the 4-dimensional subspace spanned by X, FX, GX and HX, and call it a *Q-section* determined by X. It is easily shown that this definition is independent of the choice of local base. The orthogonal complemented subspace of $Q(X)$ in $T_P(M)$ will be denoted by $Q^{\perp}(X)$. If for any $Y, Z\in Q(X)$, the sectional curvature $\sigma(Y, Z)$ is a constant $k(X, P)$, then $k(X, P)$ is called the *Q-sectional curvature* at P. Moreover, suppose that the sectional curvature $k(X, P)$ is a constant $k(P)$ independent of X at each point P, then we say that the quaternion Kaehler manifold (M, V) is of *constant Q-sectional curvature.* In such a case it is known that the function $k(P)$ is constant in M, and if dim $M\geqq 8$ (Theorem 5 in [2]), the curvature tensor R satisfies

$$
\begin{aligned}
R(X, Y)Z = \frac{k}{4}\Big\{ & g(Y, Z)X-g(X, Z)Y+g(FY, Z)FX-g(FX, Z)FY \\
& -2g(FX, Y)FZ+g(GY, Z)GX-g(GX, Z)GY \\
& -2g(GX, Y)GZ+g(HY, Z)HX-g(HX, Z)HY \\
& -2g(HX, Y)HZ\Big\}.
\end{aligned}
\tag{1.4}
$$

§ 2. Lemmas

Let (M, g) be a quaternion Kaehler manifold of dimension $n=4m$ and $\{F, G, H\}$ be a local base of V in a coordinate neighborhood U in M. We can choose an orthonormal basis $\mathfrak{F}$ of the tangent space $T_P(M)$ at P as $\mathfrak{F}=\{e_1, \cdots, e_m, e_{\bar 1}, \cdots, e_{\bar m}, e_{\bar{\bar 1}}, \cdots, e_{\bar{\bar m}}, e_{\bar{\bar{\bar 1}}}, \cdots, e_{\bar{\bar{\bar m}}}\}$, where $e_{\bar i}=Fe_i$, $e_{\bar{\bar i}}=Ge_i$, $e_{\bar{\bar{\bar i}}}=He_i$, $(i=1, \cdots, m)$. Then we see that F, G and H have components as

$$(2.1)\qquad F: \begin{pmatrix} & -I_m & & \\ I_m & & & \\ & & & -I_m \\ & & I_m & \end{pmatrix}, \quad G: \begin{pmatrix} & & -I_m & \\ & & & I_m \\ I_m & & & \\ & -I_m & & \end{pmatrix}, \quad H: \begin{pmatrix} & & & -I_m \\ & & -I_m & \\ & I_m & & \\ I_m & & & \end{pmatrix}$$

with respect to $\mathfrak{F}$, where I_m being the identity (m, m)-matrix.

Putting

$$R_{\kappa\nu\mu\lambda}=g\Big(R(e_\kappa, e_\nu)\, e_\mu, e_\lambda\Big), \qquad \rho_{\lambda\mu}=-R_{\lambda\mu\lambda\mu}\,,^{*)}$$

we prove the following two lemmas for later use.

LEMMA 1. *The curvature tensor in a quaternion Kaehler manifold satisfies the following*;

$$(2.2)\qquad R_{\lambda\mu ij}=R_{\lambda\mu\bar i\bar j}=R_{\lambda\mu\bar{\bar i}\bar{\bar j}}=R_{\lambda\mu\bar{\bar{\bar i}}\bar{\bar{\bar j}}}\,,$$

$$(2.3)\qquad R_{\lambda\mu i\bar j}+R_{\lambda\mu\bar i j}=0\,, \qquad R_{\lambda\mu i\bar{\bar j}}+R_{\lambda\mu\bar{\bar i}j}=0\,, \qquad R_{\lambda\mu i\bar{\bar{\bar j}}}+R_{\lambda\mu\bar{\bar{\bar i}}j}=0\,,$$

$$(2.4)\qquad R_{\lambda\mu\bar i\bar{\bar j}}+R_{\lambda\mu\bar{\bar i}\bar j}=0\,, \qquad R_{\lambda\mu\bar{\bar i}\bar{\bar{\bar j}}}+R_{\lambda\mu\bar{\bar{\bar i}}\bar{\bar j}}=0\,, \qquad R_{\lambda\mu\bar{\bar{\bar i}}\bar j}+R_{\lambda\mu\bar i\bar{\bar{\bar j}}}=0\,,$$

$$(2.5)\qquad \rho_{ij}=\rho_{\bar i\bar j}=\rho_{\bar{\bar i}\bar{\bar j}}=\rho_{\bar{\bar{\bar i}}\bar{\bar{\bar j}}}$$

$$(2.6)\qquad \rho_{i\bar j}=\rho_{\bar i j}\,, \qquad \rho_{i\bar{\bar j}}=\rho_{\bar{\bar i}j}\,, \qquad \rho_{i\bar{\bar{\bar j}}}=\rho_{\bar{\bar{\bar i}}j}\,.$$

$$(2.7)\qquad \rho_{\bar i\bar{\bar j}}=\rho_{\bar{\bar i}\bar j}\,, \qquad \rho_{\bar{\bar i}\bar{\bar{\bar j}}}=\rho_{\bar{\bar{\bar i}}\bar{\bar j}}\,, \qquad \rho_{\bar{\bar{\bar i}}\bar j}=\rho_{\bar i\bar{\bar{\bar j}}}\,,$$

$$(2.8)\qquad \rho_{ij}+\rho_{i\bar j}=-R_{i\bar i j\bar j}\,, \qquad \rho_{ij}+\rho_{i\bar{\bar j}}=-R_{i\bar{\bar i}j\bar{\bar j}}\,, \qquad \rho_{ij}+\rho_{i\bar{\bar{\bar j}}}=-R_{i\bar{\bar{\bar i}}j\bar{\bar{\bar j}}}\,,$$

$$\rho_{\bar i\bar j}+\rho_{\bar i\bar{\bar j}}=-R_{\bar i\bar{\bar i}\bar j\bar{\bar j}}\,, \qquad \rho_{\bar{\bar i}\bar{\bar j}}+\rho_{\bar{\bar i}\bar{\bar{\bar j}}}=-R_{\bar{\bar i}\bar{\bar{\bar i}}\bar{\bar j}\bar{\bar{\bar j}}}\,, \qquad \rho_{\bar{\bar{\bar i}}\bar{\bar{\bar j}}}+\rho_{\bar{\bar{\bar i}}\bar j}=-R_{\bar{\bar{\bar i}}\bar i\bar{\bar{\bar j}}\bar j}\,.$$

Proof. From the identity obtained by (5.9) in [2], we have

$$\begin{aligned} R_{\lambda\mu ih} &= R_{\lambda\mu\bar i\bar h}-4a(G_{\lambda\mu}G_{ih}+H_{\lambda\mu}H_{ih}) \\ &= R_{\lambda\mu\bar{\bar i}\bar{\bar h}}-4a(H_{\lambda\mu}H_{ih}+F_{\lambda\mu}F_{ih}) \\ &= R_{\lambda\mu\bar{\bar{\bar i}}\bar{\bar{\bar h}}}-4a(F_{\lambda\mu}F_{ih}+G_{\lambda\mu}G_{ih})\,, \end{aligned}$$

$4m(m+2)\,a$ being a constant equal to the scalar curvature in M. Here,

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*) Latin indices i, j, k run over the range $\{1, \cdots, m\}$, and Greek indices λ, μ, ν, κ run over the range $\{1, \cdots, 4m\}$.

F, G, H have components as (2.1) with respect to $\mathfrak{F}$, hence we see $F_{ih}=G_{ih}=H_{ih}=0$, which give (2.2). Similarly from (2.2) in [2], we have

$$R_{\lambda\mu\bar{i}h}+R_{\lambda\mu i\bar{h}}=-4a(G_{\lambda\mu}H_{ih}+H_{\lambda\mu}G_{ih})=0\,,$$

$$R_{\lambda\mu\bar{\bar{i}}h}+R_{\lambda\mu i\bar{\bar{h}}}=-4a(H_{\lambda\mu}F_{ih}+F_{\lambda\mu}H_{ih})=0\,,$$

$$R_{\lambda\mu\bar{\bar{\bar{i}}}h}+R_{\lambda\mu i\bar{\bar{\bar{h}}}}=-4a(F_{\lambda\mu}G_{ih}+G_{\lambda\mu}F_{ih})=0\,,$$

which imply (2.3). Besides

$$R_{\lambda\mu\bar{i}\bar{\bar{j}}}+R_{\lambda\mu\bar{\bar{i}}\bar{j}}=4a(F_{\lambda\mu}G_{ij}-G_{\lambda\mu}F_{ij})+8ag_{ij}\,,$$

$$R_{\lambda\mu\bar{\bar{i}}\bar{\bar{\bar{j}}}}+R_{\lambda\mu\bar{\bar{\bar{i}}}\bar{\bar{j}}}=4a(G_{\lambda\mu}H_{ij}-H_{\lambda\mu}G_{ij})+8ag_{ij}\,,$$

$$R_{\lambda\mu\bar{\bar{\bar{i}}}\bar{j}}+R_{\lambda\mu\bar{i}\bar{\bar{\bar{j}}}}=4a(H_{\lambda\mu}F_{ij}-F_{\lambda\mu}H_{ij})+8ag_{ij}\,.$$

Since $F_{ij}=G_{ij}=H_{ij}=0$ and $g_{ij}=\delta_{ij}$, we have (2.4). Then (2.5), (2.6) and (2.7) can be deduced from (2.2), (2.3) and (2.4).

Next from Bianchi identity we have

$$R_{i\bar{i}j\bar{j}}=R_{ij\bar{i}\bar{j}}-R_{i\bar{j}\bar{i}j}=R_{ijij}+R_{i\bar{j}i\bar{j}}=-\rho_{ij}-\rho_{i\bar{j}}$$

by virtue of $(2.2)_1$, $(2.3)_1$. The others are followed by similar way (Q.E.D).

We next prove

LEMMA 2. *A quaternion Kaehler manifold* ($\dim M\geqq 8$) *is of constant Q-sectional curvature k, if and only if the curvature tensor R satisfies*

(2.9) $$g\big(R(X,Y)X,Z\big)=0\,,\qquad Y\in Q(X),\quad Z\in Q^{\perp}(X)$$

or equivalently

(2.10) $$R(X,Y)X=-kY\,,\qquad Y\in Q(X)$$

for every vector field X.

The necessity is obvious from (1.4). We shall show the sufficiency. If (2.9) is satisfied for every X, we have

$$g\Big(R\big(X+tY,F(X+tY)\big)(X+tY),Z\Big)=0$$

for any X, Y, Z and $t\in R$ such that $Z\in Q^{\perp}(X+tY)$. Then we have

(2.11) $$\begin{aligned} & t^3 g\big(R(Y,FY)Y,Z\big)+t^2 g\big(R(X,FY)Y+R(Y,FX)Y+R(Y,FY)X,Z\big)\\ & +tg\big(R(Y,FX)X+R(X,FY)X+R(X,FX)Y,Z\big)\\ & +g\big(R(X,FX)X,Z\big)=0\,. \end{aligned}$$

If we put $X=e_i$, $Y=e_j$ and $Z=F(tX-Y)=te_{\bar{i}}-e_{\bar{j}}$ ($Z\in Q^{\perp}(X+tY)$), then we have

$$t^4 R_{j\bar{j}j\bar{i}} + t^3(-R_{j\bar{j}j\bar{j}} + R_{i\bar{j}j\bar{i}} + R_{j\bar{j}i\bar{i}} + R_{j\bar{i}j\bar{i}}) + t^2(R_{i\bar{i}j\bar{i}} - R_{j\bar{j}i\bar{j}})$$
$$+ t(-R_{j\bar{i}i\bar{j}} - R_{i\bar{j}i\bar{j}} - R_{i\bar{i}j\bar{j}} + R_{i\bar{i}i\bar{i}}) + R_{i\bar{i}i\bar{j}} = 0\,,$$

from $(2.2)_2$. Thus we have

$$t^4 R_{j\bar{j}j\bar{i}} + t^3(-2\rho_{i\bar{j}} + R_{i\bar{i}j\bar{j}} + \rho_{j\bar{j}}) + t^2(R_{i\bar{i}j\bar{i}} - R_{j\bar{j}i\bar{j}})$$
$$+ t(2\rho_{j\bar{i}} - R_{i\bar{i}j\bar{j}} - \rho_{i\bar{i}}) + R_{i\bar{i}i\bar{j}} = 0\,.$$

Hence we have

(2. 12) $$\rho_{i\bar{i}} = 2\rho_{i\bar{j}} - R_{i\bar{i}j\bar{j}} = \rho_{j\bar{j}}\,.$$

Taking account of (2.8), we have

(2. 13) $$\rho_{i\bar{i}} = 3\rho_{ij} + \rho_{i\bar{j}}\,.$$

If we substitute $Y=e_{\bar{j}}$ $(Z=te_{\bar{i}}+e_j)$ instead of e_j $(Z=te_{\bar{i}}-e_{\bar{j}})$, we have

$$\rho_{i\bar{i}} = 3\rho_{i\bar{j}} + \rho_{ij}\,,$$

which, together with (3.5), induces

(2. 14) $$\rho_{ij} = \rho_{i\bar{j}} \quad \text{and} \quad \rho_{j\bar{j}} = \rho_{i\bar{i}} = 4\rho_{ij}\,.$$

Similarly we obtain

(2. 15) $$\rho_{ij} = \rho_{i\bar{j}} = \rho_{i\bar{\bar{j}}} = \rho_{i\bar{\bar{\bar{j}}}} \quad \text{and} \quad \rho_{i\bar{\bar{i}}} = \rho_{i\bar{\bar{\bar{i}}}} = 4\rho_{ij}\,.$$

Next we put $X=e_i$, $Y=e_{\bar{j}}$, $Z=-H(tX-Y)=te_{\bar{\bar{i}}}-e_{\bar{\bar{\bar{j}}}}$ $(Z\in Q^{\perp}(tX+Y))$. Then we have

$$t^4 R_{j\bar{\bar{\bar{j}}}j\bar{\bar{i}}} + t^3(-2\rho_{i\bar{\bar{\bar{j}}}} + R_{i\bar{\bar{i}}j\bar{\bar{\bar{j}}}} + \rho_{j\bar{\bar{\bar{j}}}}) + t^2(R_{i\bar{\bar{i}}j\bar{\bar{i}}} - R_{j\bar{\bar{\bar{j}}}i\bar{\bar{\bar{j}}}})$$
$$+ t(2\rho_{j\bar{\bar{i}}} - R_{i\bar{\bar{i}}j\bar{\bar{\bar{j}}}} - \rho_{i\bar{\bar{i}}}) + R_{i\bar{\bar{i}}i\bar{\bar{\bar{j}}}} = 0\,.$$

That is, we have

$$\rho_{i\bar{\bar{i}}} = 2\rho_{i\bar{\bar{\bar{j}}}} - R_{i\bar{\bar{i}}j\bar{\bar{\bar{j}}}} = \rho_{j\bar{\bar{\bar{j}}}}$$

and

$$\rho_{i\bar{\bar{i}}} = 3\rho_{i\bar{\bar{\bar{j}}}} + \rho_{i\bar{j}}\,.$$

Replacing $e_{\bar{j}}$ $(Z=te_{\bar{\bar{i}}}-e_{\bar{\bar{\bar{j}}}})$ with $e_{\bar{\bar{\bar{j}}}}$ $(Z=te_{\bar{\bar{i}}}-e_{\bar{j}})$, we get

$$\rho_{i\bar{\bar{i}}} = 3\rho_{i\bar{j}} + \rho_{i\bar{\bar{\bar{j}}}}\,.$$

Hence we get

(2. 16) $$\rho_{ij} = \rho_{i\bar{j}} = \rho_{i\bar{\bar{\bar{j}}}}\,, \qquad \rho_{i\bar{\bar{i}}} = 4\rho_{ij}$$

by virtue of Lemma 1. Similarly we have

$$\rho_{\bar{i}\bar{\bar{i}}} = \rho_{\bar{\bar{i}}i} = 4\rho_{ij}\,, \tag{2.17}$$
$$\rho_{\bar{i}\bar{\bar{j}}} = \rho_{\bar{\bar{i}}\bar{j}} = \rho_{ij}\,, \qquad \rho_{\bar{i}\bar{j}} = \rho_{\bar{i}\bar{\bar{\bar{j}}}} = \rho_{ij}\,.$$

Summing up the equalties (2.14)~(2.17) obtained above, we can conclude that a quaternion Kaehler manifold satisfying (2.12) is of constant Q-sectional curvature. As this result, we have (2.10) by virtue of (1.4).

§ 3. The property $(\mathscr{Q}\mathscr{J})$

Let γ be a geodesic in a quaternion Kaehler manifold and Y be a Jacobi field along γ. Then Y satisfies

$$Y'' + R(Y, \dot{\gamma})\,\dot{\gamma} = 0\,,$$

where Y' denotes the covariant differentiation along γ. Along γ, we can define an almost complex structure J which is parallel along γ. In fact, for a local base $\{F, G, H\}$ in U, we may put in $U \cap \gamma$

$$J = aF + bG + cH\,, \qquad a^2 + b^2 + c^2 = 1\,, \tag{3.1}$$

which a, b, c satisfy

$$\begin{cases} a' - br(\dot{\gamma}) + cq(\dot{\gamma}) = 0\,, \\ b' - cp(\dot{\gamma}) + ar(\dot{\gamma}) = 0\,, \qquad a(0)^2 + b(0)^2 + c(0)^2 = 1\,, \\ c' - aq(\dot{\gamma}) + bp(\dot{\gamma}) = 0\,, \end{cases} \tag{3.2}$$

p, q, r being local 1-forms defined in (1.3).

Assume that M is of constant Q-sectional curvature k, then the curvature tensor is in the form (1.4). So we have

$$R(\dot{\gamma}, J\dot{\gamma})\,\dot{\gamma} = -kJ\dot{\gamma}\,,$$

when t is an affine parameter. Hence we see that

$$Y(t) = (\sin\sqrt{k}\,t)\,J\dot{\gamma}(t) \qquad \left(\text{resp. } tJ\dot{\gamma},\ (\sinh\sqrt{-k}\,t)\,J\dot{\gamma}\right)$$

are Jacobi fields along γ, when $k>0$ (resp. $k=0$, $k<0$). Moreover the following property is satisfied: "*Every Jacobi field Y with initial conditions $Y(0)=0$ and $Y'(0) = J\dot{\gamma}(0)$ is proportional to $J\dot{\gamma}$.*" We call such a property $(\mathscr{Q}\mathscr{J})$ following Kosmanek.

Conversely we assume that $(\mathscr{Q}\mathscr{J})$ is satisfied. If we denote by J_γ the set of Jocobi fields Y along γ which are orthogonal to γ and satisfy $Y(0)=0$, $Y'(0) \in Q(\dot{\gamma}(0))$, then $\dim J_\gamma = 3$. In fact, we define a quaternion structure $\{J_1, J_2, J_3\}$ by

$$J_\alpha = s_{\alpha 1}F + s_{\alpha 2}G + s_{\alpha 3}H \qquad (\alpha = 1, 2, 3)$$

where $s_{\alpha\beta}$ form an element of SO(3) and for a fixed α, $s_{\alpha\beta}$ satisfy (3.2). Then J_α are all parallel along γ and $\{J_1\dot{\gamma}, J_2\dot{\gamma}, J_3\dot{\gamma}\}$ are linearly independent.

Taking account of this fact, for every geodesic γ and any vector $w_0 \in Q(\dot{\gamma}(0))$, there exists a Jacobi field Y which is contained in $Q(\dot{\gamma})$ at every point $\gamma(t)$ and endowed the initial conditions $Y(0)=0$, $Y'(0)=w_0$. Then, on account of (1.3), Y' and Y'' are also contained in $Q(\dot{\gamma})$. Hence

$$g\big(R(\dot{\gamma}, Y)\dot{\gamma}, Z\big)=0 \tag{3.3}$$

for any vector field $Z \in Q(\dot{\gamma})$. At the point $\gamma(0)$, $Y(0)$ being taken arbitrarily, we conclude that such a quaternion Kaehler manifold is of constant Q-sectional curvature by Lemma 2. Thus we have

THEOREM 1. *Let M be a quaternion Kaehler manifold. The property* $(\mathscr{Q}\mathscr{J})$ *is satisfied if and only if M is of constant Q-sectional curvature. And* dim $J_\gamma=3$, *for every geodesic* γ.

§ 4. Sasakian 3-structure

In this section we consider a corresponding property in the manifold $(\tilde{M}, \tilde{g})$ with Sasakian 3-structure $\{\xi, \eta, \zeta\}$. That is, ξ, η and ζ are mutually orthogonal Killing vector fields of unit length and the contact structures ϕ, ψ and θ defined by

$$\tilde{\nabla}\xi=\phi, \qquad \tilde{\nabla}\eta=\psi, \qquad \tilde{\nabla}\zeta=\theta$$

are all Sasakian structures, where $\tilde{\nabla}$ denotes the Riemannian connection of $(\tilde{M}, \tilde{g})$.

Let $\tilde{P}$ be a point of $\tilde{M}$. We can find a sufficiently small coordinate neighborhood $\tilde{U}$ of $\tilde{P}$, in which the distribution $\tilde{D}$ spanned by ξ, η and ζ is regular. Then $\tilde{U}$ is a Riemannian manifold with the induced regular Sasakian 3-structure and we have a local fibering

$$\pi : \tilde{U} \longrightarrow \tilde{U}/\tilde{D}=U. \tag{*}$$

Since $\tilde{U}$ admits Sasakian 3-structure, U is a quaternion Kaehler manifold (cf. Ishihara [1], Tanno [7]).

We call a vector $\tilde{X}$ *vertical* when it is tangent to fibres and *horizontal* when it is orthogonal to fibres. An arbitrary geodesic $\tilde{\gamma}$ in $\tilde{U}$ needs not project to a geodesic, but it is known that if $\tilde{\gamma}$ is horizontal, then $\pi\circ\tilde{\gamma}$ is to be a geodesic and their affine parameters can be taken in common. (See [8]).

Then the following lemma is already known.

LEMMA 3. (*O'Neill* [8]) *Let* $\pi : \tilde{U} \to U$ *be a submersion and* $\tilde{\gamma}$ *be a horizontal* *geodesic in* $\widetilde{U}$. *Given a Jacobi field* Y *on* $\pi\circ\tilde{\gamma}$ *and a vertical vector* $\widetilde{U}$ *at* $\tilde{\gamma}(0)$, *there exists a unique Jacobi field* $\widetilde{Y}$ *on* $\tilde{\gamma}$ *such that* $\pi_*(\widetilde{Y})=Y$, $D(\widetilde{Y})=0$ *and* $\widetilde{Y}(0)=\widetilde{U}$. *Where* $D(\widetilde{Y})$ *is a (vertical) derived vector field from* $\widetilde{Y}$.

REMARK. We can not define the derived vector field without preparations for theory of submersions. The definition of D and its local expression were given in O'Neill [8] and Ishihara-Konishi [3, p. 48]. However the components with respect to ξ, η and ζ are given by

$$\tilde{g}\Big(D(\widetilde{Y}), \xi\Big)=\frac{d}{dt}\tilde{g}(\widetilde{Y}, \xi)-2\tilde{g}(\phi\dot{\tilde{\gamma}}, \widetilde{Y})\,\tilde{g}(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}}),$$

$$\tilde{g}\Big(D(\widetilde{Y}), \eta\Big)=\frac{d}{dt}\tilde{g}(\widetilde{Y}, \eta)-2\tilde{g}(\psi\dot{\tilde{\gamma}}, \widetilde{Y})\,\tilde{g}(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}}),$$

$$\tilde{g}\Big(D(\widetilde{Y}), \zeta\Big)=\frac{d}{dt}\tilde{g}(\widetilde{Y}, \zeta)-2\tilde{g}(\theta\dot{\tilde{\gamma}}, \widetilde{Y})\,\tilde{g}(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}}).$$

Let $\tilde{\gamma}$ be a horizontal geodesic in $\widetilde{U}$ and γ be its projection. We define tensor fields J_1, J_2, J_3 along γ by

$$J_1X=\pi_*\phi X^L, \qquad J_2X=\pi_*\psi X^L, \qquad J_3X=\pi_*\theta X^L,$$

X being a vector field along γ and X^L the lift of X to $\tilde{\gamma}$. Then we see that J_α $(\alpha=1, 2, 3)$ are all almost complex structures which are parallel along γ. (See Ishihara [1]). Then, as a result of Theorem 1, if U is of constant Q-sectional curvature k (k is necessarily positive in this case), then $(\sin\sqrt{k}\,t)$ $J_\alpha\dot{\gamma}$ are Jacobi fields along γ. Taking account of Lemma 3 and Remark, $\widetilde{Y}_1=(\sin\sqrt{k}\,t)\,\phi\dot{\tilde{\gamma}}-(\cos\sqrt{k}\,t)\,\xi$ is seen to be a Jacobi field along $\tilde{\gamma}$, since its derived vector field $D(\widetilde{Y}_1)$ vanishes. Similarly $\widetilde{Y}_2=(\sin\sqrt{k}\,t)\,\psi\dot{\tilde{\gamma}}-(\cos\sqrt{k}\,t)\,\eta$ and $\widetilde{Y}_3=(\sin\sqrt{k}\,t)\,\theta\dot{\tilde{\gamma}}-(\cos\sqrt{k}\,t)\,\zeta$ are Jacobi fields.

On the other hand the Ricci curvature tensors $\tilde{S}$ of $\widetilde{U}$ and S of U are related by

$$\tilde{S}(X^L, Y^L)=S(X, Y)-6g(X, Y), \qquad \tilde{S}(X^L, \widetilde{V})=0,$$
$$\tilde{S}(\widetilde{V}, \widetilde{W})=(n-1)\,\tilde{g}(\widetilde{V}, \widetilde{W}),$$

where X, Y are vector fields in U and $\widetilde{V}$, $\widetilde{W}$ are vertical vector fields (cf. [3] and [7]). If U of constant Q-sectional curvature k, from (1.4)

$$S(X, Y)=k'(n+5)\,g(X, Y), \qquad k'=k/4$$

and hence

$$\tilde{S}(X^L, Y^L)=\Big\{(n+5)\,k'-6\Big\}\,g(X, Y).$$

However $\widetilde{U}$ being an Einstein manifold (See Kashiwada [5]),

$$(n+5)k'-6=n-1,$$

thus k' is necessarily equal to 1.

LEMMA 4. *In the fibering* (*), *if* U *is of constant Q-sectional curvature* k, *then* $\tilde{U}$ *is of constant curvature* 1.

PROOF) Co-Gauss equation of the curvature tensor $\tilde{R}$ of $\tilde{U}$ being

$$\begin{aligned}\tilde{R}(X^L, Y^L)Z^L = \{&R(X, Y)Z - g(FY, Z)FX + g(FX, Z)FY \\ &+ 2g(FX, Y)FZ - g(GY, Z)GX + g(GX, Z)GY \\ &+ 2g(GX, Y)GZ - g(HY, Z)HX + g(HX, Z)HY \\ &+ 2g(HX, Y)HZ\}^L\end{aligned}$$

for arbitrary local base $\{F, G, H\}$ (See [3]), then we have by (1.4) and Lemma 4

$$\tilde{R}(X^L, Y^L)Z^L = \tilde{g}(Y^L, Z^L)X^L - \tilde{g}(X^L, Z^L)Y^L. \tag{4.1}$$

Since $\tilde{U}$ have Sasakian 3-structure, the sectional curvature of the section containing at least one of ξ, η, ζ is equal to 1. Together with (4.1), we have a conclusion.

Thus we have

THEOREM 2. *Let* $\tilde{M}$ *be a Riemannian manifold with Sasakian 3-structure* $\{\xi, \eta, \zeta\}$. *If in the local fibering* $\tilde{U} \to \tilde{U}/\tilde{D}$, $\tilde{U}/\tilde{D}$ *is of constant Q-sectional curvature* k (*in such a case* k *is necessarily equal to* 1), *then for every horizontal geodesic* $\tilde{\gamma}$, $(\sin 2t)\phi\dot{\tilde{\gamma}} - (\cos 2t)\xi$, $(\sin 2t)\psi\dot{\tilde{\gamma}} - (\cos 2t)\eta$ *and* $(\sin 2t)\theta\dot{\tilde{\gamma}} - (\cos 2t)\zeta$ *are Jacobi fields along* $\tilde{\gamma}$, *when* t *is the arc-length. The converse is also true.*

REMARK. Taking account of a result in [4], we see that under the assumptions in Theorem 2, for not necessarily horizontal geodesic $\tilde{\gamma}$ but perpendicular to one of the structures, say ξ, $(\sin 2t)\phi\dot{\tilde{\gamma}} - (\cos 2t)\xi$ is a Jacobi field along $\tilde{\gamma}$.

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