Almost immersions

By Kazuaki KOBAYASHI

(Received February 12, 1975)

1. We work in the category of compact PL spaces and PL maps [Z]. Given spaces X and Y and a map $f: X\to Y$, we define $U_r(f)$ to be the set of points $x\in X$ such that $f^{-1}f(x)$ contains at least r points. $S_r(f)$ is the closure of $U_r(f)$ in X. We call $B(f)=S_2(f)-U_2(f)$ branch locus of f (i.e. for $x\in B(f)$ $f^{-1}f(x)=x$ but $f|U(x)$ is not embedding for any neighborhood $U(x)$ of x.) Conversely we consider whether $f|U_2(f)$ is a local embedding ($=$immersion) or not (i.e. for $x\in U_2(f)$ there is a neighborhood $U(x)$ of x such that $f|U(x)$ is an embedding). For this problem we obtain a following.

PROPOSITION. 1. *If M^m, Q^q are closed m- and q-*dim. *manifolds and if $f: M^m\to Q^q$ $(m<q)$ is a map, $f|U_2(f)$ is an immersion.*

Next we consider about almost immersions. Given a map $f: M\to Q$, we call f almost immersion if $B(f)=S_2(f)-U_2(f)=$one point and we call f special almost immersion if f is an almost immersion and if $S_2(f|lk(a, M))=\{a_1, \cdots, a_{2n}\}$ (finite set of even points) for a unique point $\{a\}=B(f)$ and $S_3(f|lk(a, M))=\phi$. According to Irwin [1] f is a simple immersion if $S_3(f)=\phi=B(f)$. Then we prove

THEOREM 1. *Let M^m, Q^q be $(2m-q-1)$-connected closed manifolds, $3m\leqq 2q-1$ and $q\geqq 6$. Then any PL map $f: M\to Q$ is homotopic to a special almost immersion g; $M\to Q$ with $S_3(g)=\phi$.*

THEOREM 2. *Let M^m be $(2m-q-1)$-connected closed manifold and Q^q be $(2\mathrm{m}-q)$-connected, $q\leqq m-3$. Then any PL map f; $M\to Q$ is homotopic to an almost immersion $g: M\to Q$ with $S_3(g)=\phi$.*

In this paper M, Q always mean m- and q-dim. closed manifold if otherwise is not stated. D^n, S^n are always n-dim. ball and sphere respectively. For a simplial complex K and a simplex $\varDelta^t$ in K, let

$$St(\varDelta^t, K)=\left\{\varDelta^p\in K\,\middle|\,\varDelta^p<\varDelta^q,\ \varDelta^q>\varDelta^t\right\},$$

$$Lk(\varDelta^t, K)=\left\{\varDelta^p\in St(\varDelta^t, K)\,\middle|\,\varDelta^p\cap\varDelta^t=\phi\right\}\ \text{be}$$

the star and the link of $\varDelta^t$ in K. For spaces or complexes X and Y, $X*Y$ denote a non-singular join of X with Y. We call $X*Y$ a linear cone on Y if X is one point. ∂M, $\mathrm{Int}\, M$ mean the boundary, the interior of the manifold M respectively. Sometime we use the notation $\mathring{M}$ instead of $\mathrm{Int}\,M$. Both $cl\,X$, $\overline{X}$ mean the closure of X in some space.

2. Let $f: M^m \to Q^q$ $(m<q)$ be a PL map. It is well known that we can move f into general position by homotopy (i.e. there is a PL map $f_0: M \to Q$ which is homotopic to f and $\dim f_0(\Delta^r)=r$ for any r-simplex of M and $\dim S_2(f) \leqq 2m-q$.) So unless otherwise is stated we suppose that any PL map f is in general position. Then there is an integer p such that $U_{p+1}(f)=\phi$ since M is compact.

PROOF OF PROPOSTION. Let K, J be subdivisions of M, Q respectively such that $f: K \to J$ is simplicial. For $x \in U_2(f)$ there is an integer $k \geqq 2$ such that $x \in U_k(f)-U_{k+1}(f)$. Since $\dim(U_k(f)-U_{k+1}(f)) \leqq km-(k-1)q \equiv r$, there exists a t-simplex $\Delta^t \in K (t \leqq r)$ such that $x \in \mathrm{Int}\,\Delta^t$. Let $L = Lk(\Delta^t, K)$. We first prove that $f|L$ is an embedding. Let $\Delta_i^l, \Delta_j^l \in L (\Delta_i^l \neq \Delta_j^l)$ with $f(\Delta_i^l)=f(\Delta_j^l)$ ($\dim \Delta_i = \dim \Delta_j$ because f is non-degenerate). We may assume $l=2m-q$ since f is in general position. Then $f(\Delta_i^l * \Delta^t)=f(\Delta_j^l * \Delta^t)$ and $\dim(f(\Delta_i^l * \Delta^t) \cap f(\Delta_j^l * \Delta^t) - f((\Delta_i^l \cap \Delta_j^l) * \Delta^t) = l+t+1 > l = 2m-q$. This contradicts to $\dim S_2(f) \leqq 2m-q$. Hence $f|L$ is an embedding. Next we show $f|\Delta^t * L = f|St(\Delta^t, K)$ embedding. $f|\Delta^t * L = f|\bigcup_{\Delta_j \in L}(\Delta^t * \Delta_j)$ and $\Delta^t * \Delta_j$ is a simplex of K by the definition of L. Since f is non-degenerate and simplicial with respect to K, $f|\Delta^t * \Delta_j$ is an embedding. If $P \equiv f(\Delta^t * \Delta_i) \cap f(\Delta^t * \Delta_j) - f(\Delta^t * (\Delta_i \cap \Delta_j)) \neq \phi$ for some Δ_i, $\Delta_j \in L$, $f^{-1}(P) \cap \Delta_i \neq \phi$ and $f^{-1}(P) \cap \Delta_j \neq \phi$ and $f(f^{-1}(P) \cap \Delta_i) = f(f^{-1}(P) \cap \Delta_j)$ because $f|\Delta^t$ is an embedding. This contradicts to $f|L$ embedding. Hence $f|St(\Delta^t, K) = f|St(x, K)$ is an embedding and so $f|U_2(f)$ is an immersion.

COROLLARY 1. *$U_2(f) \supset U_{r+1}(f)$ and*

$$S_2(f)-U_2(f) \supset S_{r+1}(f)-U_{r+1}(f) \text{ for any } \quad r \geqq 1.$$

PROOF. First statement is obvious by definition. To show second statement it is sufficiently to show that $(U_2(f)-U_{r+1}(f)) \cap S_{r+1}(f)=\phi$ for any $r \geqq 1$. Let $P=(U_2(f)-U_{r+1}(f)) \cap S_{r+1}(f)$ and $P \neq \phi$. If $x_1 \in P, f^{-1}f(x_1)=\{x_1, \cdots, x_l\}$ $(2 \leqq l \leqq r)$ and for any neighborhood $U(x_1)$ of x_1, there exists $y_1 \in U(x_1)$ such that $f^{-1}f(y_1)=\{y_1, \cdots, y_p\}$, $p \geqq r+1$ (*).

On the other hand for the above x_i there is a neighborhood $\widetilde{V}(x_i)$ for which $f|\widetilde{V}(x_i)$ is an embedding by proposition. We choose a neighdorhood $W(f(x_1))$ of $f(x_1)$ in Q so that $W(f(x_1)) \cap f(M) \subset \bigcup_{i=1}^{l} f(\widetilde{V}(x_i))$ and put $V(x_i)= \widetilde{V}(x_i) \cap f^{-1}(W(f(x_1)))$. Then we can show $f^{-1}f(y_1)=\{y_1, \cdots, y_q\}$ $(q \leqq l)$ for any $y_1 \in V(x_1)$. Because $f^{-1}f(y_1) \cap \widetilde{V}(x_1)=y_1$ since $f|\widetilde{V}(x_1)$ is an embedding. By the same way

$$f^{-1}f(y_1)\cap \tilde{V}(x_i)=\begin{cases} y_i & \text{if } f(y_1)\in f(\tilde{V}(x_i)) \\ \phi & \text{if } f(y_1)\notin f(\tilde{V}(x_i)). \end{cases}$$

So $f^{-1}f(y_1)\subset f^{-1}(f(V(x_1))\cap f(M))=f^{-1}(W(f(x_1))\cap f(M))\subset \bigcup_{i=1}^{l} f(\tilde{V}(x_i))$.

Hence $f^{-1}f(y_1)=\{y_1, \cdots, y_q\}$ $(q\leqq l)$. This contradicts to (*).

Hence $P=\phi$.

Corollary 2. *Let $f: M^m\to Q^q$ $(q\geqq m+3)$ be a PL map which is in general position and let $V_r(f)=U_r(f)-U_{r+1}(f)$. If $V_r(f)\neq\phi$ for some $r\geqq 2$, for any $s(2\leqq s\leqq r)$ $V_s(f)\neq\phi$.*

Proof. By Proposition $f|U_2(f)$ is an immersion. For $x\in V_r(f)$ let $f^{-1}f(x_1)=\{x_1, \cdots, x_r\}$. Then $f|St(x_i, K): St(x_i, K)\to St(f(x_1), J)$ is a proper locally flat embedding for each i and some triangulations K, J of M, Q respectively. Let $U(x_i)=St(x_i, K)$ and $U(f(x_1))=St(f(x_1), J)$. Then by [A–Z, Th. 1.] we can ambient isotope $f(U(x_i))$ transversal to $\bigcup_{j\neq i} f(U(x_j))$ by small ambient isotopy of $U(f(x_1))$. Hence we can consider $U(f(x_i))\supset \bigcup_i f(U(x_i))$ as like as the m-dim. planes in q-dim. euclidean space R^q. So let $U(f(x_1))=R^q$ and $f(U(x_i))=E_i^m$ $(1\leqq i\leqq r)$. Then there are $(q-1)$-dim, hyperplanes $E_{i,1}^{q-1}, \cdots, E_{i,q-m}^{q-1}$ such that $E_i^m=E_{i,1}^{q-1}\cap\cdots\cap E_{i,q-m}^{q-1}$. For any k, $l(k\neq l)$ $E_{i,k}^{q-1}$ and $E_{i,l}^{q-1}$ are neither parallel nor coincide. Furthermore for any $E_{i,k}^{q-1}$ and $E_{j,l}^{q-1}$ (not $i=j$, $k=l$) they are neither parallel nor coincide. Because for $i=j$, $k\neq l$, it is obvious. If $i\neq j$ and if $E_{i,k}^{q-1}=E_{j,l}^{q-1}$,

$$E_i^m\cap E_j^m=E_{i,1}^{q-1}\cap\cdots\cap E_{i,q-m}^{q-1}\cap E_{j,1}^{q-1}\cap\cdots\cap \hat{E}_{j,l}^{q-1}\cap\cdots\cap E_{j,q-m}^{q-1}$$

and so $\dim(E_i^m\cap E_j^m)=q-\{(q-m)+(q-m-1)\}=2m-q+1$. It contradicts E_i^m and E_j^m in general position. And if $E_{i,k}^{q-1}$ is parallel to $E_{j,l}^{q-1}$, $E_{i,k}^{q-1}\cap E_{j,l}^{q-1}=\phi$ and $E_i^m\cap E_j^m\subset E_{i,k}^{q-1}\cap E_{j,l}^{q-1}=\phi$. It contradicts to $E_i^m\cap E_j^m\neq\phi$. So taking $E_{i_1}^m\cap\cdots\cap E_{i_s}^m$ $(i_j\neq i_k$ if $j\neq k$, $2\leqq s\leqq r)$,

$$E_{i_1,1}^{q-1}\cap\cdots\cap E_{i_1,q-m}^{q-1}\cap\cdots\cap E_{i_s,1}^{q-1}\cap\cdots\cap E_{i_s,q-m}^{q-1}\neq\phi.$$

Hence $V_s(f)\neq\phi$ for $2\leqq s\leqq r$.

3. Lemma 1. *Let A^m, B^q be balls and let $f: \partial A\to\partial B$ be a PL map such that $S_2(f)=a_1\cup a_2$. Then there is a special almost immersion $g: A\to B$ which is an extension of f.*

Proof. Let g be a linear cone extension of f. Then $S_2(g)=(a*a_1)\cup(a*a_2)$ where a is the center of A and $S_2(g)-\{a\}=U_2(g)-U_3(g)$, $a\in B(g)$. And it is obvious that $S_2(g|lk(a, A))$ contains only two points. So $g: A\to B$ is a sepecial almost immersion which is an extension of f.

LEMMA 2. *Let A^m, B^q be balls and $f\colon \partial A\to\partial B$ be a simple immersion whose $S_2(f)$ consists of two connected components C_1, C_2 which are PL subspaces. Suppose $3m\leqq 2q-1$, $q\geqq 6$ then there is a special almost immersion $g: A\to B$ which is an extension of f.*

PROOF. Choose a linear cone X_1^{2m-q} on C_1 in ∂A. Then we may assume $X_1\cap C_2=\phi$ (if necessary, by putting X_1, C_2 into general position with respect to each other). And remove a second derived neighborhood $N_1=N(X_1, \partial A'')$ of X_1 and next put a linear cone X_2^{2m-q} on C_2 in $\partial A-\operatorname{Int} N_1\cong D^{m-1}$. Choose a linear cone Y^{2m-q+1} on $f(X_1\cup X_2)$ in ∂B. Since by putting Y in general position with respect to $f(\partial A)$

$$\dim(Y\cap f(\partial A-(X_1\cup X_2)))\leqq 2m-q+1+(m-1)-(q-1)\leqq 0,$$

let $Y\cap f(\partial A-(X_1\cup X_2))=\{u_1, \cdots, u_r\}$. Then we may assume $u_i\notin f(S_2(f))$. Let $v_i=f^{-1}(u_i)$. Joining a point $c_1\in C_1$ with v_i by simple polygonal arc α_i in ∂A so that

(1) $\mathring{\alpha}_i\cap\mathring{\alpha}_j=\phi \qquad (i\neq j)$

(2) $\alpha_i\cap X_1=c_1$

(3) $\alpha_i\cap X_2=\phi$,

(it is possible because $q>m+2$).

Joining $f(c_1)$ with u_i by simple polygonal arc β_i on Y so that

(4) $\mathring{\beta}_i\cap\mathring{\beta}_j=\phi \qquad (i\neq j)$

(5) $\beta_i\cap f(\partial A)=f(c_1)\cup u_i$.

Since $\dim\partial B=q-1\geqq 5$, by *Embedding Theorem* [Z, Chap. 8] there exist *PL* embeddings $h: D^2\to\partial B$ $(i=1, 2, \cdots, r)$ satisfying

(6) $h_i(\mathring{D}^2)\cap h_j(\mathring{D}^2)=\phi \qquad (i\neq j)$

(7) $h_i(\partial D^2)=f(\alpha_i)\cup\beta_i$

(8) $h_i(D^2)\cap f(\partial A-\alpha_i)=\phi$.

let $\tilde{X}_1=X_1\cup\bigcup_i\alpha_i$, $\tilde{Y}=Y\cup\bigcup_i h_i(D^2)$, then $\tilde{X}_1\searrow 0$ in ∂A, $\tilde{Y}\searrow 0$ in ∂B and $f^{-1}(\tilde{Y}\cap f(\partial A))=\tilde{X}_1\cup X_2$. So the second derived neighborhood $N=N(\tilde{Y}, \partial B'')$, $\tilde{N}_1=N(\tilde{X}_1, \partial A'')$, $N_2=N(X_2, \partial A'')$ of these are $(m-1)$- and $(q-1)$-balls respectively and $f|\tilde{N}_1, f|N_2$ are proper embeddings into N. $(\partial N: f(\partial N_1), f(\partial N_2))\cong(S^{q-2}: S^{m-2}, S^{m-2})$ is a link. Let $\tilde{W}_1=N(\tilde{X}_1, A'')$, $W_2=N(X_2, A'')$, $W=N(\tilde{Y}, B'')$ be second derived neighborhoods of $\tilde{X}_1$, X_2, $\tilde{Y}$ respectively, then they are all balls. Taking $a_1\in\partial\tilde{W}_1-N_1$, $a_2\in\partial W_2-N_2$, $a\in\partial W-N$, then we may consder $\tilde{W}_1\cong a_1*\tilde{N}_1$, $W_2\cong a_2*N_2$, $W\cong a*N$. So we define a map $\tilde{f}\colon \partial A\cup\tilde{W}_1\cup W_2\to\partial B\cup W$ by $\tilde{f}=f$ on ∂A, $\tilde{f}(a_1)=\tilde{f}(a_2)=a$ and by extending linearly on $\widetilde{W}_1$, W_2. Then $S_2(\tilde{f})=(a_1*C_1)\cup(a_2*C_2)$ and $\tilde{f}$ is a simple immersion. Let $A_0=\overline{A-(\widetilde{W}_1\cup W_2)}\cong D^m$, $B_0=\overline{B-W}=D^q$, then $\tilde{f}|\partial A_0$: $\partial A_0\to\partial B_0$ is a simple immersion and $S_2(\tilde{f}|\partial A_0)=a_1\cup a_2$. So by LEMMA 1, there is a special almost immersion $\tilde{g}: A_0\to B_0$ such that $\tilde{g}|\partial A=\tilde{f}|\partial A_0$.

We define a map $g: A\to B$ by

$$g=\begin{cases}\tilde{f} & \text{on} \quad \partial A\cup\widetilde{W}_1\cup W_2\\ \tilde{g} & \text{on} \quad A_0.\end{cases}$$

It is clear that g is a special almost immersion.

REMARK. 1. Even if $S_2(f)$ has as connected component more than 2, it is clear that we may deal separately with each pair of connected components which is identified under f. So LEMMA 1 and LEMMA 2 may be carried out well clear of even components more than 2.

REMARK. 2. Let A^m, B^q balls and f: $\partial A\to\partial B$ be an immersion, then there is an almost immersion $g: A\to B$ using cone extension such that $g|\partial A=f$.

PROOF OF THEOREM 1. Since f is in general position, $\dim S_2(f)\leqq 2m-q$ and $S_3(f)=\phi$. And the branch locus $B(f)=S_2(f)-U_2(f)$ is contained in the $(2m-q-1)$-skeletion of some triangulation of M. Using *Engulfing Theorem* ([Z. Chap. 7]) there is a collapsible subpolyhedron X^{2m-q} in M which contains $B(f)$. Since $\dim(X\cap(S_2(f)-B(f)))\leqq 3m-2q<0$, $f|X$ is an embedding and so $f(X)\searrow 0$ in Q. Now put $A^m=N(X, M'')$, $B^q=N(f(X), Q'')$ then A, B are m-and q-balls and by the map f: $M\to Q$, $f(M-A)\subset Q-B, f(\partial A)\subset\partial B$ and $f(\mathrm{Int}\, A)\subset\mathrm{Int}\, B$. Furthermore $B(f)\subset\mathrm{Int}\, A$. Since $f|M-\mathrm{Int}\, A$ is a simple immersion, by LEMMA 2 and REMARK 1 there is a special almost immersion

$$\tilde{g}: A\to B \quad \text{such that} \quad \tilde{g}|\partial A=f.$$

So we define f: $M\to Q$ by

$$g=\begin{cases}f & \text{on} \quad M-\mathrm{Int}\, A\\ \tilde{g} & \text{on} \quad A.\end{cases}$$

Then g is a special almost immersion which is homotopic to f.

REMARK. Let f: $M^m\to Q^q$ be a PL map with $S_3(f)=\phi$. Then a connected component $\widetilde{C}$ of $f(U_2(f))$ is a connected manifolds. We denote $f^{-1}(\widetilde{C})=C_1\cup C_2$ where $C_i (i=1, 2)$ are subset of $f^{-1}(\widetilde{C})$ satisfping the following conditions,

1. ${}^{V}c_1\in C_1 \quad {}^{\exists}c_2\in C_2 \ni f(c_1)=f(c_2)$
2. ${}^{V}d_2\in C_2 \quad {}^{\exists}d_1\in C_1 \ni f(d_1)=f(d_2)$
3. $C_i\,(i=1, 2)$ are connected and $C_1\cap C_2=\phi$.

Then we may consider following cases.

CASE 1. If C_1 is a closed subset in $f^{-1}(\widetilde{C})$ or if C_1 is an open subset in $f^{-1}(\widetilde{C})$, $f^{-1}(\widetilde{C})$ is not connected.

CASE 2. If C_1 is neither closed nor open in $f^{-1}(\widetilde{C})$, $f^{-1}(\widetilde{C})$ is connected. Since f is a PL map, $\partial C_i\equiv Cl(C_i)-\mathrm{Int}(C_i)$ is contained in the $(2m-q-1)$-skeleton of some triangulation of M. So if CASE 2 happens to THEOREM 1, we take X^{2m-q} so that it contains not only $B(f)$ but also ∂C_i. Then LEMMA 2 is available.

PROOF OF THEOREM 2. Since $f: M\to Q$ is simplicial and in general position with respect to some triangulations of M and Q, $\dim B(f)\leqq 2m-q-1$ and by *Engulfing Theorem* ([Z Chap. 7]) there is a collapsible polyhedron X_0 in M which contains $B(f)$. And there is a collapsible polyhedron Y_0^{2m-q+1} in Q which contains $f(X_0)$. Put $W_0=(f^{-1}(Y_0)-X_0)$ then

$$\dim W_0\leqq 2m-q-2 \quad \text{and so} \quad \dim W_0\leqq \dim B(f).$$

We prove the theorem by induction as follows. Inductive hypothesis $\phi(i)$: There exist collapsible subspaces X_i and Y_i such that $B(f)\subset X_i\subset M$, $f(X_i)\subset Y_i\subset Q$ and $W_i=Cl(f^{-1}(Y_i)-X_i)$ has dimension $\leqq 2m-q-i-2$. The case $i=0$ has been proved above. We assume $\phi(j-1)$, $j\geqq 1$ and prove $\phi(j)$. $\dim W_{j-1}\leqq 2m-q-j-1$. By *Engulfing Therem* there is a subspace $\widetilde{X}_j^{2m-q-j}$ in M such that $W_{j-1}\subset \widetilde{X}_j^{2m-q-j}$ and $X_j=X_{j-1}\cup \widetilde{X}_j\searrow 0$. And there is a subspace $\widetilde{Y}_j^{2m-q-j+1}$ in Q such that $f(\widetilde{X}_j)\subset \widetilde{Y}_j$ and $Y_j=Y_{j-1}\cup \widetilde{Y}_j\searrow 0$. Then $f(X_j)\subset Y_j$ and we put Y_j rel. $f(\widetilde{X}_j)\cup Y_{j-1}$, into general pasition with respect to $f(M)$. Now $(Y_j\cap f(M))-f(X_j)=(\widetilde{Y}_j\cap f(M))-f(X_j)$ and $\dim W_j=\dim((Y_j\cap f(M))-f(X_j))\leqq 2m-q-j+1+m-q\leqq 2m-q-j-2$.
$\phi(2m-q-1)$ tell us that $W_{2m-q-1}=\phi$ and so $B(f)\subset X_{2m-q-1}\searrow 0$, $f(X_{2m-q-1})\subset Y_{2m-q-1}\searrow 0$ and $f^{-1}(Y_{2m-q-1})=X_{2m-q-1}$. We put $X=X_{2m-q-1}$, $Y=Y_{2m-q-1}$ and let $A=N(X, M'')$, $B=N(Y, Q'')$ (m-and q-balls respectively). Then $f(M-A)\subset Q-B$, $f(\partial A)\subset\partial B$, $f(A)\subset B$ and $B(f)\subset\mathrm{Int}\,A$. Since $f|\overline{M-A}$ is an immersion, by REMARK 2 we can extend $f|\partial A$ to an almost immersion $\tilde{f}: A\to B$ and we obtain a required almost immersion

$g: M\to Q$ by defining

$$g=\begin{cases} f & \text{on} \quad M-A \\ \tilde{f} & \text{on} \quad A. \end{cases}$$

In particular taking $X_0 \supset B(f) \cup S_3(f)$ because $3m-2q<2m-q-1$, g is an almost immersion with $S_3(g)=\phi$.

EXAMPLE. It is well known that n-dim. complex projective space $P^n(C)$ is immersible in R^{4n-1} but not R^{4n-2} for $n=2^r$. On the other hand THEOREM 1 and THEOEM 2 tell us that $P^n(C)$ is special almost immersible in R^{4n-2} for $n=2^r$, $r \geqq 2$.

References

[A-Z]· ARMSTRONG, M. A. and ZEEMAN, E. C.: Transversality for piecewise linear manifolds. Topology Vol. 6 (1967), pp. 433-466.

[I] IRWIN, M. C.: Polyhedral immersions, Proc. Camb. Phil. Soc. (1966), Vol. 62. pp. 45-50.

[z] ZEEMAN, E. C.: Seminar on combinatorial topology (mimeographed note), Institute des Hautes Etudes Scientifiques, Paris (1963).

Department of Mathematics
Hokkaido University