Homotopy groups of s.s. complexes of embeddings

By Masahiro KURATA

(Received October 21, 1974)

§ 1. Introduction

Recently codimension 3 embeddings have been discussed in many papers. It is known that codimension 3 topological embeddings are approximated by PL embeddings and close (topological or PL) embeddings are isotopic (e.g. [2]).

In this paper we shall compare homotopy groups of s.s. complexes of topological embeddings with those of PL embeddings, and show that the local contractibility of spaces of topological embedding implies similar properties of PL embeddings with codimension $\geqq 3$.

Main results are followings.

THEOREM A. *Suppose that* M^m, Q^q *are* m-dim. *and* q-dim. *PL manifolds, and* $\mathcal{E}^{TOP}(M, Q)$ (*resp.* $\mathcal{E}^{PL}(M, Q)$) *is a Kan complex of locally flat topological* (*resp. PL*) *embeddings. Then*

1) *If* $q-m\leqq 2$ $q\geqq 5$, $\mathcal{E}^{PL}(M, Q)$ *is homotopy equivalent to* $\mathcal{E}^{TOP}(M, Q)$ *provided* $H^i(M, Z_2)=0$ *for* $i\leqq 3$.

2) *If* $q-m\geqq 3$ *or* $q\leqq 3$, $\mathcal{E}^{PL}(M, Q)$ *is homotopy equivlent to* $\mathcal{E}^{TOP}(M, Q)$.

THEOREM B. *When* $q-m\geqq 3$, $\mathcal{E}^{PL}(M, Q)$ *is locally* n-*connected for any* $n\geqq 0$.

This gives a partial answer to the question of Edward ([2]). The case when $n\leqq q-m-3$ is obtained by Lusk ([8]).

By rB^k and rS^{k-1} we denote the k-ball with radius $r\{(X_1, \cdots, X_k)\in R^k \mid |X_i|\leqq r\}$ and the $(k-1)$-sphere $\partial(rB^k)$, respectively. R^k is identified with the subspace $R^k\times 0\subset R^{k+1}$.

ASSUMPTIONS. In this paper we assume the followings: A map denoted by

$$f\colon M\times \Delta^k \longrightarrow Q\times \Delta^k$$

is level preserving i.e. $pr_2\circ f=pr_2$, f_t is given by $f(x, t)=(f_t(x), t)$, and $f(M\times \Delta^k)\subset \operatorname{int} Q\times \Delta^k$. It is valid to generalize the result in this paper for proper embeddings. Embeddings and immersions are to be locally flat. For manifolds M^m and Q^q, we assume $m<q$ because $m=q$ case is obtained

by Morlet ([9]). The topology of spaces of embeddings means the majorant topology (cf. [2], [3]).

DEFINITION. Topological (resp. PL) stiefel manifold $V^{TOP}{}_{q,m}$ (resp. $V^{PL}{}_{q,m}$) is a Kan complext whose k-simplex f is a topological (resp. PL) embedding

$$f:\ R^m\times\Delta^k\longrightarrow R^q\times\Delta^k$$

preserving 0-section. Note that $V^{TOP}{}_{q,q}=TOP_q$ (resp. $V^{PL}{}_{q,q}=PL_q$).

DEFINITION. Let M', Q be topological (resp. PL) manifolds, $N\subset M\subset M'$ be locally flat proper submanifolds and $\theta: N\rightarrow Q$ be a topological (resp. PL) immersion. Then, $\mathscr{I}^{TOP}{}_{M',\theta}(M, Q)$ (resp. $\mathscr{I}^{PL}{}_{M',\theta}(M, Q)$) is a Kan complex of M'-germs of topological (resp. PL) immersions of M in Q whose restriction on N is θ. $\mathscr{R}^{TOP}{}_{M',\theta}(M, Q)$ (resp. $\mathscr{R}^{PL}{}_{M',\theta}(M, Q)$) is a complex of its representations. When $M'=M$ and $N=\phi$, we omit the subscripts M' and θ respectively.

DEFINITION. $\tilde{\mathscr{I}}_{B^n\times R^{m-n}}(B^n, R^q)$ is a complex consisting of $B^n\times R^{m-n}$-germs of immersions of B^n in R^q which is PL on a neighbourhood of $S^{n-1}\times R^{m-n}$. $\tilde{\mathscr{R}}_{B^n\times R^{m-n}}(B^n, R^q)$ is a complex of its representations.

$\mathcal{E}^{TOP}(M, Q)$, $\mathcal{E}^{PL}(M, Q)$ and $\tilde{\mathcal{E}}_{B^n\times R^{m-n}}(B^n, R^q)$ are defined similarly for embeddings instead of immersions.

§2. Extension lemma for embeddings

Our purpose in this section is to prove that any element of $\pi_k(\tilde{\mathcal{E}}(B^m, R^q), \mathcal{E}^{PL}(B^m, R^q))$ can be extended to $\pi_k(\tilde{\mathcal{E}}(B^q, R^q), \mathcal{E}^{PL}(B^q, R^q))$.

LEMMA 1. *Suppose*

$$f:\ (S^{q-1}\cup B^m)\times\dot{\Delta}^k\longrightarrow R^q\times\dot{\Delta}^k$$

is a level preserving topological (resp. PL) embedding with $q>m$, and for any $t\in\dot{\Delta}^k$ f_t can be extended to a topological (resp. PL) embedding $B^q\rightarrow R^q$.

Then there is an extension of f

$$F:\ B^q\times\dot{\Delta}^k\longrightarrow R^q\times\dot{\Delta}^k$$

which is level preserving topological (resp. PL) embedding. And such extension is unique up to topological (resp. PL) isotopy.

PROOF. This follows from the Alexander trick (resp. lemma 16 of [11]), inductively on k.

LEMMA 2. *Suppose an embedding*

$$f:\ 2B^m\times\frac{1}{2}B^{q-m}\times\Delta^k\longrightarrow R^q\times\Delta^k$$

satisfies

$$f\Big|\Big((2B^m-\operatorname{int}B^m)\times\Big(\frac{1}{2}\Big)B^{q-m}\times\Delta^k\Big)=id.$$

$$f\Big(\operatorname{int}B^m\times\Big(\frac{1}{2}\Big)B^{q-u}\times\Delta^k\Big)\cap\Big((2B^m-\operatorname{int}B^m)\times 2B^{q-m}\times\Delta^k\Big)=\phi$$

f_t *is isotopic to a PL embedding rel.*

$(2B^m-\operatorname{int}B^m)\times\Big(\frac{1}{2}\Big)B^{q-m}$ *for any* $t\in\Delta^k$.

Let $U\subset R^q\times\Delta^k$ *be a neighbourhood of* $f\Big(B^m\times\Big(\frac{1}{2}\Big)B^{q-m}\times\Delta^k\Big)\cup(S^{m-1}\times B^{q-m}\times\Delta^k)$.

Then if $q\neq 4$, *there is a PL embedding*

$$F:\ (2B^q-\operatorname{int}B^q)\times\Delta^k\longrightarrow R^q\times\Delta^k$$

such that

$$F(S^{q-1}\times\Delta^k)\subset U$$
$$F\Big((2B^q-\operatorname{int}B^q)\times\Delta^k\Big)\cap f\Big(\operatorname{int}B^m\times\Big(\frac{1}{2}\Big)B^{q-m}\times\Delta^k\Big)=\phi$$
$$F(2B^m-\operatorname{int}B^m)\times 2B^{q-m}\times\Delta^k=id.$$

Moreover when a PL embedding

$$g:\ (2B^q-\operatorname{int}B^q)\times \mathrm{t}\longrightarrow R^q\times t$$

is given for some $t\in\Delta^k$ *such that*

$$g(S^{q-1}\times t)\subset U$$
$$g\Big|\Big((2B^m-\operatorname{int}B^m)\times 2B^{q-m}\times t\Big)=id.$$
$$g\Big((2B^q-\operatorname{int}B^q)\times t\Big)\cap f\Big(\operatorname{int}B^m\times\Big(\frac{1}{2}\Big)B^{q-m}\times t\Big)=\phi,$$

F can be chosen to be an extension of g.

PROOF. We prove by induction on k.

When $k=0$, it follows from the straightening theorem ([6]). In the case $k\geqq 1$. By the isotopy extension theorem ([3]), there is a topological embedding

$$G:\ 2B^q\times\Delta^k\longrightarrow R^q\times\Delta^k$$

such that

$$G\Big|\Big(2B^m\times\Big(\frac{1}{2}\Big)B^{q-m}\times\Delta^k\Big)=f$$

$$G\Big|\left((2B^m-\text{int}\,B^m)\times 2B^{q-m}\times\Delta^k\right)=id.$$
$$G(B^m\times 2B^{q-m}\times\Delta^k)\subset U.$$

Let $\{\Delta_i\}_{i=1,\cdots,N}$ be a partition of Δ^k with the following conditions: Δ_i is a PL k-ball, $(\bigcup_{j\leq i}\Delta_j)\cap\Delta_{i+1}$ is a face of Δ_{i+1}, and for any $t, t'\in\Delta_i$

$$G_t\left(B^m\times\left(\frac{1}{2}\right)B^{q-m}\right)\subset G_{t'}\left(B^m\times\text{int}\,(r_2B)^{q-m}\right)$$
$$G_t(B^m\times r_3B^{q-m})\subset G_{t'}\left(B^m\times\text{int}(2B^{q-m})\right)$$

where $\frac{1}{2}<r_2<1<r_3<2$ are fixed. We prove inductively that when F is defined on $(2B^q-\text{int}\,B^q)\times(\bigcup_{j\leq i-1}\Delta_j)$ with $F_t(S^q)\subset G_t(2B^q)$, we can extend F over $(2B^q-\text{int}\,B^q)\times(\bigcup_{j\leq i}\Delta_j)$. For convenience sake, we replace $(\Delta_i, (\bigcup_{j\leq i-1}\Delta_j)\cap\Delta_i)$ by $(\Delta^{k-1}\times I, \Delta^{k-1})$, where $\Delta^{k-1}=\Delta^{k-1}\times 0\subset\Delta^{k-1}\times I$. Thus

$$F:\ (2B^q-\text{int}\,B^q)\times\Delta^{k-1}\longrightarrow R^q\times\Delta^{k-1}.$$

For some $t_0\in\Delta^{k-1}$, G_{t_0} is approximated by a PL embedding

$$h_{t_0}:\ 2B^q\longrightarrow R^q$$

such that for some r_1 with $\frac{1}{2}<r_1<r_2<1$

$$h_{t_0}\left(2B^m\times\text{int}\,(r_1B^{q-m})\right)\supset G_{t_0}\left(2B^m\times\left(\frac{1}{2}\right)B^{q-m}\right)$$

for any $t\in\Delta^{k-1}\times I$,

$$h_{t_0}(B^m\times r_1B^{q-m})\subset U\cap V$$

where $V\subset R^q\times\Delta^{k-1}$ is an open set bounded by $F(((2S^{m-1}\times B^{q-m})\cup(2B^m\times S^{q-m-1}))\times\Delta^{k-1})$,

$$h_{t_0}\left(B^m\times\text{int}\,(r_2B^{q-m})\right)\supset G_t\left(B^m\times\left(\frac{1}{2}\right)B^{q-m}\right)$$
$$G_t(B^m\times\text{int}\,2B^{q-m})\supset h_{t_0}(B^m\times r_3B^{q-m})$$

for any $t\in\Delta^{k-1}\times I$, and

$$h_{t_0}\Big|\left((2B^m-\text{int}\,B^m)\times 2B^{q-m}\right)=id.$$

By the hypothesis of induction on k combining with the isotopy extension theorem ([11]), there is a PL extension of h_{t_0}

$$h:\ 2B^q\times\Delta^{k-1}\longrightarrow R^q\times\Delta^{k-1}$$

satisfying

$$G\left(2B^m \times \frac{1}{2}B^{q-m} \times \Delta^{k-1}\right) \subset h\left(2B^m \times \operatorname{int}(r_1 B^{q-m}) \times \Delta^{k-1}\right)$$
$$h(B^m \times r_1 B^{q-m} \times \Delta^{k-1}) \subset V \cap h_{t_0}\left(2B^m \times \operatorname{int}(r_2 B^{q-m})\right) \times \Delta^{k-1}$$
$$h_t(x) = h_{t_0}(x)$$

for $x \in ((2B^m - \operatorname{int} B^m) \times B^{q-m}) \cup B^m \times r_3 B^{q-m} - \operatorname{int}(r_2 B^{q-m}))$, and for some r_4 with $r_3 < r_4 < 2$

$$F(S^{q-1} \times \Delta^{k-1}) \subset h(B^m \times r_4 B^{q-m} \times \Delta^{k-1})$$
$$\subset G(B^m \times \operatorname{int} 2B^{q-m} \times \Delta^{k-1}).$$

For $\varepsilon > 0$, define

$$W = (1-\varepsilon)B^m \times \left(r_4 B^{q-m} - \operatorname{int}(r_1 B^{q-m})\right) \cup B^m \times \left(r_3 B^{q-m} - \operatorname{int}(r_2 B^{q-m})\right).$$

Let ε and $\delta < 0$ be small such that

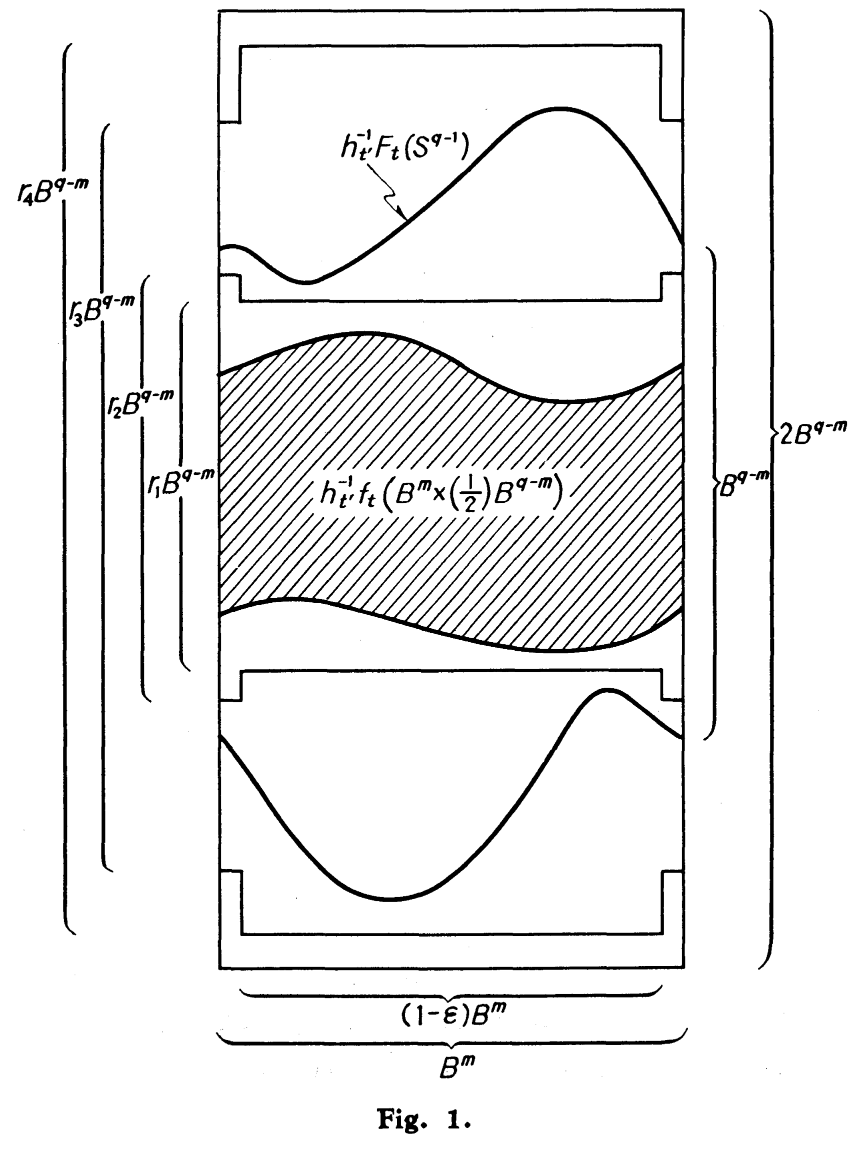

Fig. 1.

$$h_t(W)\cap f_{(t,t')}\left(B^m\times\frac{1}{2}B^{q-m}\right)=\phi \quad \text{for} \quad (t,t')\in\Delta^{k-1}\times[0,\delta]$$

$$F(S^{q-1}\times\Delta^{k-1})\subset h\Big((W\cup(S^{m-1}\times B^{q-m}))\times\Delta^{k-1}\Big)$$

(Figure 1).

Let

$$\phi_t:\ 2B^q\longrightarrow 2B^q$$

be a PL isotopy as follows.

$\phi_0=id.$

$\phi_t(x)=x$ for $x\in(2B^m-\text{int}\,B^m)\times 2B^{q-m}$

$\phi_t(W)\subset W$

$\phi_t(W)\subset B^m\times(r_3B^{q-m}-r_2B^{q-m})$ for $\delta\leqq t\leqq 1.$

Define

$$\phi:\ 2B^q\times\Delta^{k-1}\times I\longrightarrow 2B^q\times\Delta^{k-1}\times I$$

by $\phi(x,t,t')=(\phi_{t'}(x),t,t')$.

Then $(h\times 1)\circ\phi\circ(h^{-1}\times 1)\circ(F\times 1)$ is the desired extension of F.

PROPOSITION. *Let*

$$f:\ 2B^m\times\Delta^k\longrightarrow R^q\times\Delta^k$$

be an embedding with $f|2B^m-\text{int}\,B^m)\times\Delta^k)=id.$ *Then there is a PL embedding*

$$F:\ (2B^q-\text{int}\,B^q)\times\Delta^k\longrightarrow R^q\times\Delta^k$$

with

$$F\Big((2B^q-\text{int}\,B^q)\times\Delta^k\Big)\cap f(\text{int}\,B^m\times\Delta^k)=\phi.$$

PROOF. This follows from Lemma 2.

LEMMA 3. *If* $q\neq 4$,

$$\pi_k\Big(\tilde{\mathcal{E}}(B^q,R^q),\ \mathcal{E}^{PL}(B^q,R^q)\Big)\longrightarrow\pi_k\Big(\tilde{\mathcal{E}}(B^m,R^q),\ \mathcal{E}^{PL}(B^m,R^q)\Big)$$

is epimorphic for $k\geqq 1$.

PROOF. Let

$$f:\ 2B^m\times\Delta^k\longrightarrow R^q\times\Delta^k$$

be an embedding which represents an element of $\pi_k(\tilde{\mathcal{E}}(B^m,R^q),\ \mathcal{E}^{PL}(B^m,R^q))$ i.e. f is PL on $(2B^m-\text{int}\,B^m)\times\Delta^k\cup 2B^m\times\dot{\Delta}^k$. One can assume that $f|(2B^m-\text{int}\,B^m)\times\Delta^k=id$ up to isotopy in $\tilde{\mathcal{E}}(B^m,R^q)$.

By Proposition, there is an extension of f

$$\left((2B^q-\operatorname{int} B^q)\cup 2B^m\right)\times \Delta^k \longrightarrow R^q\times \Delta^k$$

which is PL on $(2B^q-\operatorname{int} B^q)\times \Delta^k$. By the isotopy extension theorem ([3]), above embedding can be extended to

$$F\colon\ 2B^q\times \Delta^k \longrightarrow R^q\times \Delta^k.$$

Thus F satisfies $F|2B^m\times \Delta^k=f$ and $F|2B^q-\operatorname{int} B^q)\times \Delta^k$ is PL.

On the other hand, by Lemma 1 (PL case) there is a PL embedding

$$G\colon\ 2B^q\times \dot{\Delta}^k \longrightarrow R^q\times \dot{\Delta}^k$$

such that $G=F$ on $((2B^q-\operatorname{int} B^q)\times 2B^m)\times \dot{\Delta}^k$.

Lemma 1 (topological case) implies that $F|(2B^q\times \dot{\Delta}^k)$ is isotopic to G rel. $((2B^q-\operatorname{int} B^q)\cup B^m)\times \dot{\Delta}^k$. Therefore we may assume that F is PL on $2B^q\times \dot{\Delta}^k$. This completes the proof.

§3. Homotopy groups $\pi_k(V^{TOP}{}_{q,m},\ V^{PL}{}_{q,m})$

In this section we compute the homotopy groups $\pi_k(V^{TOP}{}_{q,m},\ V^{PL}{}_{q,m})$. The topological and PL immersion theorems ([4], [7]) imply the following.

LEMMA 4.

1) $\pi_0\left(\tilde{\mathscr{I}}_{B^k\times R^{m-k}}(B^k, R^q)\right)\cong \pi_0\left(\tilde{\mathscr{R}}_{B^k\times R^{m-k}}(B^k, R^q)\right)$

$$\cong \pi_k(V^{TOP}{}_{q,m},\ V^{PL}{}_{q,m}).$$

2) *for* $m<q$,

$$\pi_k\left(\tilde{\mathscr{I}}_{B^n\times R^{m-n}}(B^n, R^q),\ \mathscr{I}^{PL}{}_{B^n\times R^{m-n}}(B^n, R^q)\right)$$

$$\cong \pi_k\left(\tilde{\mathscr{R}}_{B^n\times R^{m-n}}(B^n, R^q),\ \mathscr{R}^{PL}{}_{B^n\times R^{m-n}}(B^n, R^q)\right)$$

$$\cong \pi_{k+n}(V^{TOP}{}_{q,m},\ V^{PL}{}_{q,m}).$$

LEMMA 5. *If* $q\neq 4$,

$$\pi_k\left(\tilde{\mathscr{E}}(B^n\times R^{m-n}, R^q),\ \mathscr{E}^{PL}(B^n\times R^{m-n}, R^q)\right)$$

$$\longrightarrow \pi_k\left(\tilde{\mathscr{I}}(B^n\times R^{m-n}, R^q),\ \mathscr{I}^{PL}(B^n\times R^{m-n}, R^q)\right)$$

is an isomorphism where the map is induced by the natural inclusion.

PROOF. First we prove that the map is an epimorphism. Let

$$f\colon\ B^n\times R^{m-n}\times \Delta^k \longrightarrow R^q\times \Delta^k$$

represents an element of $\pi_k(\tilde{\mathscr{I}}(B^n\times R^{m-n}, R^q),\ \mathscr{I}^{PL}(B^n\times R^{m-n}, R^q))$. There is a family of neighbourhoods $\{\Delta_i, f_i, V_i, \varphi_i\}_{i=1,\dots,l}$ induced by f ([7]), i.e. Δ_i satisfies the followings:

$\{\Delta_i\}$ is a partition of Δ^k

Δ_i is a PL k-ball

$(\bigcup_{j\leq i-1}\Delta_j)\cap\Delta_i$ is a face of Δ_i,

f_i is a topological embedding

$$f\colon\ B^n\times R^{m-n}\times\Delta^k\longrightarrow V_i\times\Delta^k$$

where V_i is a q-dim. PL manifolds, and

$$\varphi_i\colon\ V_i\longrightarrow R^q$$

is a PL immersion with the following commutative diagram

$$\begin{array}{ccc} & \overset{f_i}{\nearrow} & V_i\times\Delta_i \\ & & \downarrow \varphi_i\times 1 \\ B^n\times R^{m-n}\times\Delta_i & \xrightarrow[f\mid B^n\times R^{m-n}\times\Delta_i]{} & R^q\times\Delta_i \end{array}$$

For induced neighbourhoods above, there are neighbourhoods $V_{ij}\subset V_i$ with

$$f_i\left(B^n\times R^{m-n}\times(\Delta_i\cap\Delta_j)\right)\subset V_{ij}\times(\Delta_i\cap\Delta_j)$$

and PL homeomorphisms

$$\Phi_{ij}\colon\ V_{ij}\longrightarrow V_{ji}$$

commuting the following diagrams

$$\begin{array}{ccc} & \overset{f_i\mid(\Delta_i\cap\Delta_j)}{\nearrow} & V_{ij}\times(\Delta_i\cap\Delta_j) \\ & & \swarrow \varphi_i\times 1 \\ B^n\times R^{m-n}\times(\Delta_i\cap\Delta_j) & \xrightarrow{f\mid(\Delta_i\cap\Delta_j)} & R^q\times(\Delta_i\cap\Delta_j) \\ & & \nwarrow \varphi_j\times I \quad \searrow \Phi_{ij}\times 1 \\ & \underset{f_j\mid(\Delta_i\cap\Delta_j)}{\searrow} & V_{ji}\times(\Delta_i\cap\Delta_j) \end{array}$$

Let $\{\Delta_{lj}\}_j$ be a subdivision of Δ_l such that $(\Delta_{lj},\Delta_{lj}\cap\Delta_j)$ is PL homeomorphic to $(\Delta^{k-1}\times I,\Delta^{k-1}\times 0)$ and $\Delta_{lj}\cap\Delta_j=\Delta_{lj}\cap(\bigcup_{i\leq l-1}\Delta_i)$. Then there is a PL ambient isotopy

$$H_t\colon\ V_l\times\Delta_l\longrightarrow V_l\times\Delta_l,\ t\in I$$

with $H_t=$identity on $V_l\times((\bigcup_{i\leq l-1}\Delta_i)\cap\Delta_l)$ and $H_1(f_l(B^n\times R^{m-n})\times\Delta_{lj})\subset V_{lj}\times\Delta_{lj}$. The regular homotopy h_t from f defined by

$$h_t(x,t')=\begin{cases}(\varphi_l\times 1)\circ H_t\circ f_l(x,t') & \text{if } t'\in\Delta_l\\ f(x,t') & \text{otherwise}\end{cases}$$

is in $\pi_k(\tilde{\mathscr{I}}_{B^n\times R^{m-n}}(B^n, R^q), \mathscr{I}^{PL}{}_{B^n\times R^{m-n}}(B^n, R^q))$. And $\{\Delta'_i, f'_i, \varphi'_i\}_{i=1,\cdots,l-1}$ defined as follows is a family of induced neighbourhood by h_1:

$$\Delta'_i = \Delta_i \cup \Delta_{li}$$
$$\varphi'_i = \varphi_i$$
$$f'_i|(B^n\times R^{m-n}\times \Delta_i) = f_i$$
$$f'_i|(B^n\times R^{m-n}\times \Delta_{li}) = (1\times \Phi_{li})\circ H_1\circ f_0|(B^n\times R^{m-1}\times \Delta_{li}).$$

Thus we can decrease the number of $\{\Delta_i\}$. Finally there is a regular homotopy in $\pi_k(\tilde{\mathscr{I}}_{B^n\times R^{m-n}}(B^n, R^q), \mathscr{I}^{PL}{}_{B^n\times R^{m-n}}(B^n, R^q))$ from f to g such that g can be lifted to an embedding g'

$$\begin{array}{ccc} & \overset{g'}{\nearrow} & R^q\times \Delta^k \\ & & \big\downarrow \varphi\times 1 \\ B^n\times R^{m-n}\times \Delta^k & \xrightarrow[g]{} & R^q\times \Delta^k \end{array}$$

where φ is a PL immersion. Let

$$g_t:\ R^q \longrightarrow R^q$$

be a PL ambient isotopy which shrinks the image of g' so small that $(\varphi\times 1)\circ(g_1\times 1)\circ g'$ is an embedding. Then $(\varphi\times 1)\circ(g_t\times 1)\circ g'$ is a regular homotopy in $\pi_k(\tilde{\mathscr{I}}_{B^n\times R^{m-n}}(B^n, R^q), \mathscr{I}^{PL}{}_{B^n\times R^{m-n}}(B^n, R^q))$ which takes g to an embedding. This proves the map is epimorphic. Similar argument implies that the map is monomorphic.

Now we are ready to determine the relative homotopy groups of topological and PL stiefel manifolds.

THEOREM 1. *For* $k\geqq 0$,

1) *if* $q-m\leqq 2$, $q\geqq 5$

$$\pi_k(V^{TOP}{}_{q,m},\ V^{PL}{}_{q,m}) = \begin{cases} 0 & \text{for } k\neq 3 \\ Z_2 & k=3 \end{cases}$$

2) *if* $q-m\geqq 3$ *or* $q\leqq 3$ $V^{PL}{}_{q,m}$ *is homotopy equivalent to* $V^{TOP}{}_{q,m}$.

PROOF. In the case $q-m\leqq 2$ $q\geqq 5$, and $k\leqq m$, the existance and the uniqueness of normal bundles of codim$\leqq 2$ embeddings, both in topological and PL case ([1], [6], [10], [11]), implies that

$$\begin{aligned} \pi_0\left(\tilde{\mathscr{I}}_{B^k\times R^{m-k}}(B^k, R_q)\right) &\cong \pi_0\left(\tilde{\mathscr{I}}_{B^k\times R^{q-k}}(B^k, R^q)\right) \\ &\cong \pi_k(TOP_q, PL_q) \\ &\cong \begin{cases} 0 & k\neq 3 \\ Z_2 & k=3. \end{cases} \end{aligned}$$

By lemma 4. 1), the required result follows.

In the case when $q-m\geqq 3$ or $q\leqq 3$, and $k\leqq m$. A topological embedding is isotopic to PL embedding (e.g. [2]), then

$$\pi_k(V^{TOP}{}_{q,m}, V^{PL}{}_{q,m})=\pi_0\left(\tilde{\mathscr{I}}_{B^R\times R^{m-k}}(B^k, R^q)\right)$$
$$=0$$

In the case $q\neq 4$, $k>m$. It follows from lemma 1 that

$$\pi_k\left(\tilde{\mathcal{E}}(B^q, R^q), \mathcal{E}^{PL}(B^q, R^q)\right)=0$$

for $k\geqq 1$. Combining with the above, lemma 3, 4.2) and 5 implies

$$\pi_k(V^{TOP}{}_{q,m}, V^{PL}{}_{q,m})=0.$$

Finally, a k-isotopy of embeddings of B^1 is approximated by PL embeddings, therefore

$$\pi_k(V^{TOP}{}_{4,1}, V^{PL}{}_{4,1})=0$$

for $k\geqq 0$.

§4. Proof of main theorems

Lemma 4, 5 and Theorem 1 imply the following straightening theorem for k-isotopies of handles.

THEOREM 2. *For any* $k\geqq 0$

1) *if* $q-m\leqq 2$ $q\geqq 5$,

$$\pi_k\left(\tilde{\mathcal{E}}(B^n\times R^{m-n}, R^q), \mathcal{E}^{PL}(B^n\times R^{m-n}, R^q)\right)=\begin{cases}0 & k+n\neq 3\\ Z_2 & k+n=3\end{cases}$$

2) *if* $q-m\geqq 3$ *or* $q\leqq 3$

$$\pi_k\left(\tilde{\mathcal{E}}(B^n\times R^{m-n}, R^q), \mathcal{E}^{PL}(B^n\times R^{m-n}, R^q)\right)=0.$$

The above implies Theorem A. Moreover we obtain th ε-version of Theorem A. 2):

THEOREM 3. *Suppose that*

$$f\colon M^m\times\Delta^k\longrightarrow Q^q\times\Delta^k$$

is an embedding with $f|(M\times\dot{\Delta}^k)$ *PL and* $q-m\geqq 3$ *or* $q\leqq 3$.

Then there is an ambient isotopy rel. $Q\times\dot{\Delta}^k$

$$g_t\colon Q\times\Delta^k\longrightarrow Q\times\Delta^k \qquad t\in I$$

such that g_t *is within* ε *of the identity for given* $\varepsilon: Q\to(0,\infty)$, *and* $g_1 f$ *is a PL embedding.*

Consider a k-isotopy of PL embedding with $q-m\geqq 3$ or $q\leqq 3$

$$f:\ M\times\Delta^{k}\longrightarrow Q\times\Delta^{k}$$

such that $f|(M\times\dot{\Delta}^{k})=f_0\times id_{\dot{\Delta}^k}$, and f_t is snfflcient close to f_0. By the covering isotopy theorem ([3]), f is covered by an ambient isotopy

$$g:\ Q\times\Delta^{k}\longrightarrow Q\times\Delta^{k}$$

which

$$g\circ(f_0\times 1)=f$$

and g is close enough to the identity. Since spaces of topological homeomorphisms are locally contractible (relative case of [3]), we may assume

$$g|Q\times\dot{\Delta}^{k}=identity.$$

Again by Edward-Kirby [3] there is an isotopy, sufficiently close to the identity,

$$G:\ Q\times\Delta^{k}\times I\longrightarrow Q\times\Delta^{k}\times I$$

from g to the identity rel. $Q\times\dot{\Delta}^k$. By theorem 3, $G\circ(f\times 1)$ is approximated by PL embeddings rel. $Q\times(\dot{\Delta}^k\times I\cup\Delta^k\times\dot{I})$. Thus we have proved the local connectivity of s. s. complexs of codimension 3 PL embeddings as follows.

THEOREM 4. *On the assumption that* $q-m\geqq 3$ *or* $q\leqq 3$, *and* $k\geqq 0$. $\varepsilon:\ M\to(0,\infty)$ *is given.*
Suppose

$$f:\ M\times\Delta^{k}\longrightarrow Q\times\Delta^{k}.$$

is a PL embedding such that $f|M\times\dot{\Delta}^k=f_0\times id_{\dot{\Delta}^k}$ *and* f_t *is within* δ *of* f_0 (δ *depends on* ε *and* f_0) *for* $t\in\Delta^k$. *Then there is a PL ambient isotopy rel.* $Q\times(\dot{\Delta}^k\times I\cup\Delta^k\times\dot{I})$

$$g:\ Q\times\Delta^{k}\times I\longrightarrow Q\times\Delta^{k}\times I$$

within ε *of the identity, which takes* f *to* $f_0\times id_{\Delta^k}$.

Similar argument implies the low codimension case:

THEOREM 5. *On the assumption that* $q-m\leqq 2$, $q\geqq 5$ *and* $H^{2-k}(M, Z_2)=0$, *theorem* 4 *is true.*

References

[1] M. BROWN: Locally flat imbeddings of topological Manifolds, Ann. of Math. 75 (1962), 331-341.

[2] R. E. EDWARD: The equivalence of close piecewise linear embeddings, to appear.

[3] R. E. EDWARD and R. C. KIRBY: Deformations of spaces of impeddings, Ann. of Math. 93 (1971), 63-88.

[4] A. HAEFLIGER and V. POENARU: Classification des immersion combinatoire, Publs. Math. I.H.E.S. 23 (1964), 209-214.

[5] R. C. KIRBY and L. C. SIEBENMANN: Notices Ammer. Math. Soc. 16 (1969), 433, 695.

[6] R. C. KIRBY and L. C. SIEBENMANN: Codimension two locally flat imbeddings, Notices Amer. Math. Soc. 18 (1971), 983.

[7] M. KURATA: Immersions of topological manifolds, J. Fac. Sci. Hokkaido Univ. 21 (1971), 237-247.

[8] E. L. LUSK: Homotopy groups of spaces of imbeddings, Notices of Amer. Math. Soc. 17 (1970), 527.

[9] M. C. MORLET: Lissage des homéomorphismes, C.R. Acad. Sc. Paris, 268 (1969), 1323-1325.

[10] C. T. C. WALL: Locally flat *PL* submanifolds with codimension two, Proc. Camb. Phil. Soc. 63 (1967), 5-8.

[11] E. C. ZEEMAN: Seminar on Combinatorial Topology, I.H.E.S. (1963).

Department of Mathematics
Hokkaido University