Metrics induced by capacities and boundary behaviors of quasiconformal mappings on open Riemann surfaces

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Introduction.

M. Nakai (cf. [4]) proved that every quasiconformal mapping between two open Riemann surfaces can be homeomorphically extended to their Royden compactifications. It is well-known (cf. [2]) that the Royden compactification is not metrizable. In this paper we shall study the homeomorphic extensibility of quasiconformal mappings between two open Riemann surfaces to their metrizable compactifications. To do so we shall introduce a new metric $d=d_R$ on an open Riemann surface R induced by the Kuramochi capacity on R. Our main results are the followings:

(i) Let R be an open Riemann surface. If each Kuramochi kernel $\tilde{g}_b$ with pole b on the Kuramochi boundary of R is unbounded, then the completion of R with respect to d is compact.

(ii) Let R_1 and R_2 be two open Riemann surfaces. If both R_1 and R_2 satisfy the assumption in (i), then every quasiconformal mapping from R_1 onto R_2 can be homeomorphically extended over their completions with respect to d.

1. Metrics induced by capacities.

Let R be an open Riemann surface. We say that a closed curve in R joining $a\in R$ and $b\in R$ means a continuous mapping γ: $z=z(t)$ of $[0, 1]$ into R such that $z(0)=a$ and $z(1)=b$. We write $\gamma=\{z(t);\ 0\leqq t\leqq 1\}$ for simplicity. We denote by $\Gamma_{a,b}=\Gamma_{a,b}(R)$ the family of all closed curves in R joining a and b.

A non-negative finite real-valued function Φ on the family of all compact subsets on R is said to be a capacity in the sense of G. Choquet if it satisfies the following properties:

(a) If $K_1\subset K_2$, then $\Phi(K_1)\leqq\Phi(K_2)$.

(b) $\Phi(K_1\cup K_2)+\Phi(K_1\cap K_2)\leqq\Phi(K_1)+\Phi(K_2)$.

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(c) Given a compact subset K of R and any $\varepsilon>0$, there is an open set G in R such that any compact subset K' with $K\subset K'\subset G$ implies $\Phi(K')<\Phi(K)+\varepsilon$.

DEFINITION 1. For $a, b\in R$, we set

$$\mu(a, b)=\inf\{\Phi(\gamma);\ \gamma\in\Gamma_{a,b}(R)\}.$$

PROPOSITION 1.

(i) *μ is a pseudometric on R, i.e.,*

(1) $0\leqq\mu(a, b)<\infty,\ a=b\Rightarrow\mu(a, b)=0$.

(2) $\mu(a, b)=\mu(b, a)$.

(3) $\mu(a, b)\leqq\mu(a, c)+\mu(c, b)$.

(ii) $\mu(a, b)=\inf\{\Phi(K)$; *where K runs over all continua in R containing both a and b*$\}$.

PROOF. The proof of (i) follows immediately from the properties (a), (b) and (c) of capacity.

(ii) Let a and b any points in R. We denote by $\nu(a, b)$ the right hand side in (i). Since any closed curve is a continuum, we obtain that $\nu(a, b)\leqq\mu(a, b)$. For any $\varepsilon>0$, there is a continuum E in R containing both a and b such that $\Phi(E)<\nu(a, b)+\varepsilon$. Since E is a connected compact set, we can find a domain G (=a connected open set) on R such that $E\subset G$ and any compact set F with $E\subset F\subset G$ implies $\Phi(E)\leqq\Phi(F)<\Phi(E)+\varepsilon$. Since G is connected and contains both a and b, there is a closed curve γ in G joining both a and b. If we set $F=E\cup\gamma$, then we obtain that

$$\mu(a, b)\leqq\Phi(\gamma)\leqq\Phi(E\cup\gamma)<\Phi(E)+\varepsilon<\nu(a, b)+2\varepsilon.$$

Since ε is arbitrary, we complete the proof.

Some special metrics induced by capacities

DEFINITION 2 ([1]). Let R be a hyperbolic Riemann surface. Let z be a fixed point in R. For a compact subset K of $R-\{z\}$, we denote by $\omega_z(K)$ the harmonic measure of ∂K with respect to $R-K$ at z. For $a, b\in R-\{z\}$, we set

$$\mu(a, b)=H_z(a, b)=\inf\{\omega_z(\gamma);\ \gamma\in\Gamma_{a,b}(R-\{z\}).$$

It is known [1] that this $\mu(a, b)$ is a metric on $R-\{z\}$. Furthermore the topology induced by μ is compatible with the original topology on $R-\{z\}$.

DEFINITION 3. Let R be a hyperbolic Riemann surface and let $C=C_R$ be the Green capacity on R. For $a, b\in R$, let

$$\mu(a, b)=\rho(a, b)=\rho_R(a, b)=\inf\{C(\gamma);\ \gamma\in\Gamma_{a,b}(R)\}.$$

DEFINITION 4. Let R be an open Riemann surface. Let K_0 be a closed disk in R and let $R_0=R-K_0$. Let $\widetilde{C}$ be the Kuramochi capacity on R_0 (cf. [2]). For $a, b\in R_0$, let

$$\mu(a, b)=d(a, b)\doteq d_{R_0}(a, b)=\inf\{\widetilde{C}(\gamma)\,;\ \gamma\in\Gamma_{a,b}(R_0)\}.$$

By elementary properties of Green capacity and Kuramochi capacity, we obtain the following lemma.

LEMMA 1. (i) *Let R be a hyperbolic Riemann surface and G be a domain on R. Then*

$$\rho_R(a, b)\leqq\rho_G(a, b) \quad \textit{for} \quad a, b\in G.$$

(ii) *Let R be an open Riemann surface. Let K_0 be a closed disk in R and let $R_0=R-K_0$. Then*

$$d_{R_0}(a, b)\leqq\rho_{R_0}(a, b) \quad \textit{for} \quad a, b\in R_0.$$

(iii) *Let R be an open Riemann surface. Let K_0, K_0' be closed disks in R with $K_0\subset K_0'$. Set $R_0=R-K_0$ and $R_0'=R-K_0'$. Then*

$$d_{R_0}(a, b)\leqq d_{R_0'}(a, b) \quad \textit{for} \quad a, b\in R_0'.$$

2. The metric ρ.

LEMMA 2. *Let $U=\{|z|<1\}$. Then $\rho_U(a, b)=C_U\left(\left[0, \left|\dfrac{b-a}{1-\bar{a}b}\right|\right]\right)$ for $a, b\in U$.*

PROOF. We may assume that $a\neq b$.

(i) Suppose $a=0$. Let γ be any curve in $\Gamma_{0,b}(U)$. Then it follows from Hilfssatz 19.1 in [3] that $C_U([0, |b|])\leqq C_U(\gamma)$. Hence we see that $\rho_U(0, b)=C_U([0, |b|])$.

(ii) Suppose a is arbitrary. Then we can find a linear transformation $w=T(z)$ of U onto itself such that $T(0)=0$ and $T(b)=\left|\dfrac{b-a}{1-\bar{a}b}\right|$. Since ρ_U is invariant under conformal mappings, it follows from (i) that

$$\rho_U(a, b)=\rho_U\big(T(a), T(b)\big)=\rho_U\left(0, \left|\frac{b-a}{1-\bar{a}b}\right|\right)=C_U\left(\left[0, \left|\frac{b-a}{1-\bar{a}b}\right|\right]\right).$$

COROLLARY (Schwartz lemma). *Every analytic mapping f: $U\to U$ is distancedecreasing, i.e., satisfies*

$$\rho_U(f(a), f(b))\leqq\rho_U(a, b) \qquad (a, b\in U)$$

and the equality is valid if and only if f is an automorphism of U.

PROPOSITION 2. (i) *ρ_U is a metric on U.*

(ii) *The metric ρ_U is compatible with the topology on U.*

PROOF. By properties of capacity and Lemma 2, we have (i). Furthermore it follows from Lemma 2 that (ii) is valid.

For $a, b\in U$, we set

$$\sigma(a, b)=\frac{1}{2}\log\frac{1+r}{1-r}$$

where $r=\left|\frac{b-a}{1-\bar{a}b}\right|$. The function $\sigma(a, b)$ is called a non-euclidean metric on U.

PROPOSITION 3. *For a fixed $a\in U$, we have*

$$\lim_{b\to\xi}\frac{\rho_U(a, b)}{\sigma(a, b)}=\frac{4}{\pi^2}$$

uniformly in $\xi\in\partial U$.

PROOF. For a fixed $a\in U$ and an arbitrary $b\in U$, let $r=\left|\frac{b-a}{1-\bar{a}b}\right|$. If b tends to a boundary point $\xi\in\partial U$, then r tends to 1. Thus it follows from [3] that

$$\rho_U(a, b)=\frac{2}{\pi^2}\log\frac{8}{1-r}+O(1) \quad \text{as} \quad r\to 1.$$

Hence we have

$$\rho_U(a, b)=\frac{2}{\pi^2}\log\frac{1}{1-r}+O(1) \quad \text{as} \quad r\to 1.$$

Since $\sigma(a, b)=\frac{1}{2}\log\frac{1}{1-r}+O(1)$, we complete the proof.

COROLLARY. *For t $(0<t<1)$, we set $D_t=\{|a|<t\}$. Then, for any t_1 $(0<t_1<1)$ and $c>\frac{4}{\pi^2}$, there exists t_2 $(t_1<t_2<1)$ such that*

$$c^{-1}\sigma(a, b)\leqq\rho_U(a, b)\leqq c\,\sigma(a, b)$$

for any $a\in D_{t_1}$ and $b\in D-\bar{D}_{t_2}$.

LEMMA 3. *Let R be an open Riemann surface and K_0 be a closed disk on R. Let U be an open disk on R with $U\cap K_0=\emptyset$. Let K be a compact subset of U such that $U-K$ is connected. Let $\{K_n\}_{n=1}^{\infty}$ be a family of compact subsets of K. Then the following properties are equivalent each other.*

(a) *$C_U(K_n)\to 0$ as $n\to\infty$.*

(b) *$C_G(K_n)\to 0$ as $n\to\infty$ for any hyperbolic domain G on R with $U\subset G$.*

(c) *$\tilde{C}(K_n)\to 0$ as $n\to\infty$.*

PROOF. Let u_n (resp. v_n) be the harmonic measure of ∂K_n with respect to $U-K_n$ (resp. $G-K_n$). Then it is easy to see that $C_U(K_n)$ (resp. $C_G(K_n)$, $\widetilde{C}(K_n)$)$\to 0$ as $n\to\infty$ if and only if u_n (resp. v_n, $1_{\tilde{K}_n}$)[1] converges to zero locally uniformly in $U-K$ (resp. $G-K$, R_0-K) as $n\to\infty$. Since $C_G(K_n)\leqq C_U(K_n)$, $\widetilde{C}(K_n)\leqq C_{R_0}(K_n)$ and $v_n\leqq 1_{\tilde{K}_n}$ in $U-K$ $(n=1, 2, \cdots)$, we complete the proof.

THEOREM 1. *Suppose R is a hyperbolic Riemann surface. Then*

(i) *ρ_R is a metric on R.*

(ii) *The metric ρ_R is compatible with the topology on R.*

PROOF. We set $\rho=\rho_R$ in the following. Since ρ is a pseudometric, it is sufficient to prove that $\rho(a, b)=0$ implies $a=b$. If this were not the case, then we could find a and b in R such that $\rho(a, b)=0$ and $a\neq b$. Then we can find a parametric disk U on R with center at a such that $b\in U$. Since $a\neq b$, $\rho_U(a, b)>0$. On the other hand, since $\rho(a, b)=0$, there is a subfamily $\{\gamma_n\}_{n=1}^{\infty}$ of $\Gamma_{a,b}$ such that $C_R(\gamma_n)\to 0$ as $n\to\infty$. Let $\zeta=\phi(z)$ be a local parameter on U such that $\phi(a)=0$ and $\phi(U)=\{|\zeta|<1\}$. For each n, we can find a subcurve γ_n' of γ_n on $\phi^{-1}(|\zeta|\leqq|\phi(b)|)$ which connects a and a point of $\phi^{-1}(|\phi(b)|)$. Then $0<\rho_U(a, b)\leqq C_U(\gamma_n')$. Since $C_R(\gamma_n')\leqq C_R(\gamma_n)\to 0$ as $n\to\infty$, it follows from Lemma 3 that $C_U(\gamma_n')\to 0$ as $n\to\infty$. This is a contradiction. Thus $\rho(a, b)=0$ implies $a=b$.

(ii) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in R which tends to $a\in R$ as $n\to\infty$. Let U be a parametric disk in R with center at a. Then there is an n_0 such that $a_n\in U$ if $n\geqq n_0$. Since $0\leqq\rho_R(a, a_n)\leqq\rho_U(a, a_n)\to 0$ as $n\to\infty$ by Proposition 1, we see that $\rho_R(a, a_n)\to 0$ as $n\to\infty$. Conversely suppose a is a point in R and $\{a_n\}_{n=1}^{\infty}$ is a sequence in R such that $\rho(a, a_n)\to 0$ as $n\to\infty$. Then there are a parametric disk U with center at a which corresponds to a unit disk $\{|\zeta|<1\}$ and a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ such that each a_{n_k} does not belong to U. Let $\zeta=\phi(z)$ be a local parameter on U such that $\phi(a)=0$ and $\phi(U)=\{|\zeta|<1\}$. Since $\rho(a, a_n)\to 0$ as $n\to\infty$, we can find $\gamma_n\in\Gamma_{a,a_n}$ $(n=1, 2, \cdots)$ such that $C_R(\gamma_n)\to 0$ as $n\to\infty$. For each n_k, we can find a subcurve γ_{n_k}' of γ_{n_k} which connects a and a point b_{n_k} of $\phi^{-1}(|\zeta|=1/2)$ on $\phi^{-1}(|\zeta|\leqq 1/2)$. Since $C_R(\gamma_{n_k}')\leqq C_R(\gamma_{n_k})\to 0$ as $k\to\infty$, it follows from Lemma 3 that $C_U(\gamma_{n_k}')\to 0$ as $k\to\infty$. On the other hand, we obtain that $0<\rho_{\phi(U)}(0, 1/2)=\rho_U(a, b_{n_k})\leqq C_U(\gamma_{n_k}')$. This is a contradiction. Therefore $a_n\to a$ as $n\to\infty$.

1) See p. 163 in [2] for the definition.

3. The metric d and quasiconformal mappings.

Let R be an open Riemann surface. Let K_0 be a closed disk in R and let $R_0=R-K_0$. Let $\tilde{C}$ be the Kuramochi capacity on R_0 (cf. [2]).

THEOREM 2. (i) *d is a metric on R_0.*

(ii) *The metric d is compatible with the topology on R_0.*

PROOF. (i) Suppose $d(a, b)=0$ for some two distinct points a and b of R_0. Then there exists a parametric disk U with center at a such that $b\in U$. Let $\zeta=\psi(z)$ be a local parameter on U such that $\psi(a)=0$ and $\psi(U)=\{|\zeta|<1\}$. Since $d(a, b)=0$, there is a sequence $\{\gamma_n\}_{n=1}^{\infty}$ in $\Gamma_{a,b}(R_0)$ such that $\tilde{C}(\gamma_n)\to 0$ as $n\to\infty$. For each n, we can find a subcurve γ'_n of γ_n which connects a and a point b_n of $\psi^{-1}(|\zeta|\leqq|\psi(b)|)$ on $\psi^{-1}(|\zeta|\leqq|\psi(b)|)$. Since $\tilde{C}(\gamma'_n)\to 0$ as $n\to\infty$, it follows from Lemma 3 that $0<\rho_{\psi(U)}(0, \psi(b))=\rho_U(a, b_n)\leqq C_U(\gamma'_n)\to 0$ as $n\to\infty$. This is a contradiction. Thus d is a metric.

The proof of (ii) can be proved by an analogous argument to the one of (ii) in Theorem 1.

We refer to [4] for the definition and properties of quasiconformal mapping. Let R_1, R_2 be open Riemann surfaces and ϕ be a quasiconformal mapping of R_1 onto R_2. We denote by $BCD(R_i)$ the family of all bounded continuous Dirichlet functions on $R_i (i=1, 2)$ (cf. [2]). For $f\in BCD(R_i)$, we denote by $\|f\|_{R_i}^2$ the Dirichlet integral of f on $R_i (i=1, 2)$. As for quasiconformal mappings and Dirichlet functions, the following theorem is well-known.

THEOREM 3 (cf. [4]). *Let ϕ be a quasiconformal mapping of R_1 onto R_2. Then*

$$1/K(\phi)\ \|f\|_{R_1}^2\leqq\|f\circ\phi^{-1}\|_{R_2}^2\leqq K(\phi)\ \|f\|_{R_1}^2$$

for each $f\in BCD(R_1)$, where $K(\phi)$ is the maximum dilatation of ϕ.

Let K_0 be a closed disk in R_1 and $K'_0=\phi(K_0)$. Then the following theorem follows from Theorem 3 and Satz 17.6 in [2].

THEOREM 4. $1/K(\phi)\tilde{C}(E)\leqq\tilde{C}'(\phi(E))\leqq K(\phi)\tilde{C}(E)$ *for any compact subset E of R_1-K_0, where $\tilde{C}'$ is the Kuramochi capacity on $R_2-K'_0$.*

PROOF. Let E be an arbitrary compact subset of R_1-K_0. Let F be any regular compact subset[2] of R_1-K_0 such that $E\subset F$. We set

2) We say that a compact set is regular if its relative boundary consists of a finite number of analytic arcs.

$$f=\tilde{p}^{\chi^{F'}} \text{ on } R_1-K_0 \text{ and } =0 \text{ on } K_0.$$ [3]

Then $f\in BCD(R_1)$ and $f\circ\phi^{-1}\in BCD(R_2)$. Since $f\circ\phi^{-1}=1$ on $\phi(F)(\supset\phi(E))$ and $=0$ on K_0', we see that

$$\|\tilde{p}^{\chi^{\phi(F')}}\|_{R_2}^2 \leqq \|f\circ\phi^{-1}\|_{R_2}^2 \leqq K(\phi)\,\|f\|_{R_1}^2 \leqq K(\phi)\,\|\tilde{p}^{\chi^{F'}}\|_{R_2}^2.$$

Hence we obtain that $\tilde{C}'(\phi(E))\leqq\tilde{C}'(\phi(F))\leqq K(\phi)\,\tilde{C}(F)$. Since F is arbitrary, we obtain that $\tilde{C}'(\phi(E))\leqq K(\phi)\,\tilde{C}(E)$. On the other hand, since ϕ^{-1} is a quasiconformal mapping and $K(\phi^{-1})=K(\phi)$, we see that $\tilde{C}(E)\leqq K(\phi)\,\tilde{C}'(\phi(E))$. This completes the proof.

COROLLARY. *For any Borel subset A of R_1-K_0, we have*

$$1/K(\phi)\,\tilde{C}(A)\leqq \tilde{C}'(\phi(A))\leqq K(\phi)\,\tilde{C}(A).$$

We denote by d_1 (resp. d_2) the metric defined in Definition 4 with respect to R_1-K_0 (resp. R_2-K_0'). By the aid of Theorem 4, we obtain the following theorem.

THEOREM 5. $1/K(\phi)\,d_1(a,b)\leqq d_2(\phi(a),\phi(b))\leqq K(\phi)\,d_1(a,b)$ *for any* $a, b\in R_1-K_0$.

COROLLARY. *Let R be an open Riemann surface and let K_0 be a closed disk in R. If ϕ is a quasiconformal mapping of R onto itself such that $\phi(K_0)=K_0$, then*

$$1/K(\phi)\,d(a,b)\leqq d(\phi(a),\phi(b))\leqq K(\phi)\,d(a,b)$$

for any $a, b\in R-K_0$.

By a discussion similar to that in the proof of Theorem 4, we have the following.

PROPOSITION 4. *Let R_1 and R_2 be hyperbolic Riemann surfaces. If ϕ: $R_1\to R_2$ is an onto quasiconformal mapping, then*

$$1/K(\phi)\,\rho_{R_1}(a,b)\leqq \rho_{R_2}(\phi(a),\phi(b))\leqq K(\phi)\,\rho_{R_1}(a,b)$$

for any $a, b\in R_1$.

COROLLARY. *Let R be a hyperbolic Riemann surface. If ϕ is a quasiconformal mapping of R onto itself, then*

$$1/K(\phi)\,\rho_R(a,b)\leqq \rho_R(\phi(a),\phi(b))\leqq K(\rho)\,\rho_R(a,b)$$

for any $a, b\in R$.

DEFINITION 5. Let A be a non-empty subset of R_0. We define the diameter $\delta(A)$ of A with respect to d by $\sup_{a,b\in A} d(a,b)$. Furthermore we set $\delta(\emptyset)=0$.

3) See p. 185 in [2] for the definition of χ^F.

LEMMA 4. *If F is a connected closed snbset of R_0, then $\delta(F)\leqq\tilde{C}(F)$.*

4. d-completions of open Riemann surfaces.

Let R be an open Rienann surface. Let K_0 be a cloded disk in R and let $R_0=R-K_0$. We denote by R_N^* (resp. R_D^*) the Kuramochi compactification (resp. the Royden compactification) of R. The Kuramochi boundary (resp. the Royden boundary) of R is denoted by Δ_N (resp. Δ_D) (cf. [2]). For a subset A of R, we denote by $\bar{A}^N$ (resp. $\bar{A}^D$) the closure of A in R_N^* (resp. R_D^*). Let R_1^* and R_2^* be two compactifications of R. If there is a continuous mapping π of R_1^* onto R_2^* whose restriction to R is the identity and $\pi^{-1}(R)=R$, then π is called a canonical mapping of R_1^* onto R_2^* and R_2^* is called a quotient space of R_1^*. It is known (cf. [2]) that R_N^* is a quotient space of R_D^*.

We denote by Δ_B the set of all Kuramochi boundary points b such that the Kuramochi kernel $\tilde{g}_b$ with pole b is bounded. The set $\Delta_1\cap\Delta_B$ is denoted by Δ_S (cf. [2]). By definition, we see that $\Delta_S\subset\Delta_B$.

DEFINITION 6. We denote by U_{HM} the class of all open Riemann surfaces with $\Delta_B\neq\emptyset$.

It is known that $O_G\cap U_{HM}=\emptyset$.

THEOREM 6. *Suppose R is an open Riemann surface with $R\notin U_{HM}$. If D is an open disk in R with $D\supset K_0$, then $R-D$ is totally bounded with respect to d.*

PROOF: (i) For $b\in\Delta_1$, let $F_n(b)=\{z\in R_0\,;\ \tilde{g}_b(z)\geqq n\}$ $(n=1, 2, \cdots)$. First we shall prove that $\bigcup_{b\in\Delta_1}\overline{F_n(b)}^D$ $(n=1, 2, \cdots)$ is a neighborhood of Δ_D in R_D^*. Let n be fixed. If this were not the case, then we could find $\xi\in\Delta_D$ such that $U(\xi)-\bigcup_{b\in\Delta_1}\overline{F_n(b)}^D\neq\emptyset$ for any neighborhood $U(\xi)$ of ξ in R_D^*. Let π be the canonical mapping of R_D^* onto R_N^* and let $\pi(\xi)=b_0$. Let $\{V_j(b_0)\}_{j=1}^{\infty}$ be a sequence of open neighborhoods of b_0 in R_N^* such that $V_j(b_0)\supset V_{j+1}(b_0)$ and $\bigcap_{j=1}^{\infty}\overline{V_j(b_0)}^N=\{b_0\}$. Set $U_j=\pi^{-1}(V_j(b_0))$ for each j. Let b be any fixed point of Δ_1. Since $U_j-\overline{F_n(b)}^D\neq\emptyset$ for each j, there is $z_j\in\overline{(U_j-\overline{F_n(b)}^D)}\cap R_0$ $(=V_j(b_0)\cap R_0-F_n(b))$ for each j. Then $\tilde{g}_b(z_j)\leqq n$ for each j. Since $a\rightarrow\tilde{g}_b(a)$ is lower semicontinuous on $R_N^*-K_0$, we obtain that $\tilde{g}_b(b_0)\leqq n$. Since b is arbitrary, it follows from the symmetry of the Kuramochi kernel and a domination principle (cf. Folgesatz 17.2 in [2]) that $\tilde{g}_{b_0}\leqq n$ on R_0. Thus $\Delta_B\neq\emptyset$ and this contradicts $R\notin U_{HM}$. Hence we see that $\bigcup_{b\in\Delta_1}\overline{F_n(b)}^D$ is a neighborhood of Δ_D in R_D^* for each n.

(ii) For each n and $b\in\Delta_1$, we can find an open subset $G_n(b)$ of R_D^* with $\overline{F_{n+1}(b)}^D\subset G_n(b)\subset\overline{F_n(b)}^D$. By (i), we see that $\bigcup_{b\in\Delta_1} G_n(b)$ is an open covering of Δ_D. Hence we can find $b_1,\cdots,b_k\in\Delta_1$ such that $\bigcup_{j=1}^{k} G_n(b_j)$ is a neighborhood of Δ_D in R_D^*. It is easy to see that $\bigcup_{j=1}^{k}\overline{F_n(b_j)}^N$ is a neighborhood of Δ_N in R_N^*. Since $\delta(F_n(b_j))\leqq\widetilde{C}(F_n(b_j))=1/n$[4], we complete the proof.

COROLLARY. *If R is an open Riemann surface with $R\notin U_{HM}$, then the completion of $R-D$ with respect to d is compact.*

Let R be an open Riemann surface and K_0 be a closed disk in R. Let D be an open disk in R with $D\supset K_0$. Since the completion of $R-D$ with respect to d does not depend on the choice of K_0 (Lemma 1), we denote it by $(R-D)_d^*$. Furthermore we set $R_d^*=(R-D)_d^*\cup D$. If R does not belong to U_{HM}, then it follows from the above corollary that R_d^* is a compactification of R. We note that if $R=\{|z|<1\}$, then R_d^* is homeomorphic to $\{|z|\leqq 1\}$.

THEOREM 7. *If R is an open Riemann surface with $R\notin U_{HM}$, then R_d^* is a quotient space of R_D^*.*

PROOF. Let E_1 and E_2 be regular closed subsets of R with $\bar{E}_1^D\cap\bar{E}_2^D\neq\emptyset$. Let ξ be a point of $\bar{E}_1^D\cap\bar{E}_2^D$. It follows from the proof of Theorem 6 that, for any n, there exists a, $b\in\Delta_1$ such that $\overline{F_n(b)}^D$ is a neighborhood of ξ in R_D^*. Then $F_n(b)\cap E_i\neq\emptyset$ $(i=1, 2)$. Let z_i be a point of $F_n(b)\cap E_i$ $(i=1, 2)$. Then $0\leqq d(z_1, z_2)\leqq\widetilde{C}(F_n(b))=1/n$. Since n is arbitrary, we obtain that $\inf\{d(z_1, z_2); z_1\in E_1, z_2\in E_2\}=0$, that is, $\bar{E}_1^d\cap\bar{E}_2^d\neq\emptyset$ ($\bar{E}_i^d$ is the closure of E_i in R_d^*). Thus it can be seen that R_d^* is a quotient space of R_D^*.

THEOREM 8. *If $R\in O_G$, then R_d^* is homeomorphic to the Kerékjártó-Stoilow's compactification R_{KS}^* of R.*

PROOF. First we note that R_d^* is compact by the Corollary of Theorem 6. Let e be any point of $\Delta_{KS}=R_{KS}^*-R$. Let $\{G_n\}_{n=1}^\infty$ be a determining sequence of e. Then each G_n is a domain on R with compact relative boundary ∂G_n in R and $G_{n+1}\cup\partial G_{n+1}\subset G_n$ $(n=1, 2, \cdots)$, $\bigcap_{n=1}^{\infty} G_n=\emptyset$. Since $\delta(G_n\cup\partial G_n)\leqq\widetilde{C}(G_n\cup\partial G_n)\to 0$ as $n\to\infty$, $\bigcap_{n=1}^{\infty}\overline{G_n\cup\partial G_n}^d$ is a single point in R_d^*, where $\overline{G_n\cup\partial G_n}^d$ is the closure of $G_n\cup\partial G_n$ in R_d^*. Thus we denote it by $\pi(e)$. For each $z\in R$ we set $\pi(z)=z$. Then we can show that π is a continuous mapping of R_{KS}^* onto R_d^*. Let e_1 and e_2 be any points of Δ_{KS} with

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4) By the aid of Folgesatz 17. 22 in [2] we can prove the equality.

$e_1 \neq e_2$. Then we can find two domains Ω_1 and Ω_2 on R such that

(1) $\partial\Omega_1$ and $\partial\Omega_2$ are compact in R,

(2) the closure $\bar{\Omega}_i^{KS}$ of Ω_i in R_{KS}^* is a neighborhood of e_i in R_{KS}^* $(i=1, 2)$,

(3) $\bar{\Omega}_1^{KS} \cap \bar{\Omega}_2^{KS} = \emptyset$.

Since $0 < d(\partial\Omega_1, \partial\Omega_2) = \inf\{d(z_1, z_2); z_1 \in \partial\Omega_1, z_2 \in \partial\Omega_2\} \leqq d(a, b)$ for any $a \in \Omega_1$ and $b \in \Omega_2$, we have $\pi(e_1) \neq \pi(e_2)$. Hence we see that π is a homeomorphism of R_{KS}^* onto R_d^*.

Combining Theorem 5 and Theorem 6, we obtain the following theorem.

THEOREM 9. *Let R_1 and R_2 be two open Riemann surfaces which do not belong to U_{HM}. If ϕ is a quasiconformal mapping of R_1 onto R_2, then ϕ can be homeomorphically extended to a mapping from $(R_1)_d^*$ onto $(R_2)_d^*$.*

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