Characterization of the p-conformally flat Riemannian manifold

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§1. Introduction.

Recently, Bang-yen Chen and Kentaro Yano [1] proved the following:

THEOREM 1. *In order that a Riemannian manifold M of dimension $n>3$ is conformally flat, it is necessary and sufficient that there exists a (unique) quadratic form Q on M such that the sectional curvature $K(\sigma)$ with respect to a plane σ is the trace of the restriction of Q to σ, i.e. $K(\sigma)=\text{trace } Q/\sigma$, the metric being also restricted to σ.*

The object of this paper is to give the generalization of this theorem, *ipso facto*, the characterization of higher order conformally flatness.

We have the following:

THEOREM 2. *In order that a Riemannian manifold M of dimension $n\geqq 4p$ is p-conformally flat, it is necessary and sufficient that there exists a (unique) quadratic form Q, which satisfies the generalized first Bianchi identity as double form of type $(2p-1, 2p-1)$, on the bundle $\Lambda^{2p-1}(M)$ of $(2p-1)$-vectors of M such that the $2p$-th sectional curvature $K_{2p}(\sigma)$ with respect to an $2p$-plane σ is the trace of the restriction of Q to $\Lambda^{2p-1}(\sigma)$, i.e. $K_{2p}(\sigma)=\text{trace } Q/\Lambda^{2p-1}(\sigma)$.*

§2. Preliminaries.

Let M be an n-dimensional Riemannian manifold with the Riemannian metric g, let $\mathfrak{F}(M)$ be the algebra of functions on M and let $\mathfrak{X}(M)$ be the Lie algebra of vector fields on M. In what follows we write $g=\langle\,,\rangle$, where it is convenient.

For p an integer between 1 and n, let $\Lambda^p(M)$ denote the bundle of p-vectors of M and let $\Lambda^p(m)$ be the fiber over $m\in M$. $\Lambda^p(M)$ is a Riemannian vector bundle, with the inner product on the fiber $\Lambda^p(m)$ over m related to the inner product on the tangent space M_m of M at m by

$$(2.1)\qquad \langle X_1\wedge X_2\wedge\cdots\wedge X_p,\ Y_1\wedge Y_2\wedge\cdots\wedge Y_p\rangle=\det\left[\langle X_i, Y_j\rangle\right],\ (X_i, Y_j\in M_m).$$

We define a double form of type (p, q) on M to be an $\mathfrak{F}(M)$-multilinear map

$$\omega: \quad \mathfrak{X}(M)^p \times \mathfrak{X}(M)^q \longrightarrow \mathfrak{X}(M),$$

which is skew-symmetric in the first p-variables and also in the last q-variables. We shall use the notation

$$\omega(X_1, \cdots, X_p)(Y_1, \cdots, Y_q)$$

to denote the value of ω on the vector fields $X_1, \cdots, X_p, Y_1, \cdots, Y_q \in \mathfrak{X}(M)$. If $p=q$ and

$$\omega(X_1, \cdots, X_p)(Y_1, \cdots, Y_p) = \omega(Y_1, \cdots, Y_p)(X_1, \cdots, X_p),$$

we say that ω is symmetric.
Let ω be a symmetric double form of type (p, p). Hence, at each point $m \in M$, we may regard ω as aquadratic form on $\Lambda^p(m)$, i.e.

$$\omega(X_1, \cdots, X_p)(Y_1, \cdots, Y_p) = \omega(X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_p), \quad (X_i, Y_j \in M_m).$$

Next, let ω be a double form of type (p, q) and let θ be of type (r, s) respectively. The exterior product $\omega \wedge \theta$ of ω and θ is defined by the formula

$$\begin{aligned}(2.2) \qquad &(\omega \wedge \theta)(X_1, \cdots, X_{p+r})(Y_1, \cdots, Y_{q+s}) \\ &= \sum_{\substack{\sigma \in Sh(p,r) \\ \tau \in Sh(q,s)}} \mathcal{E}_\sigma \mathcal{E}_\tau \omega(X_{\sigma(1)}, \cdots, X_{\sigma(p)})(Y_{\tau(1)}, \cdots, Y_{\tau(q)}) \\ &\quad \times \theta(X_{\sigma(p+1)}, \cdots, X_{\sigma(p+r)})(Y_{\tau(q+1)}, \cdots, Y_{\tau(q+s)}), \quad \big(X_i, Y_j \in \mathfrak{X}(M)\big).\end{aligned}$$

Here, $Sh(p, r)$ denotes the set of all (p, r)-shuffles; specifically

$$Sh(p, r) = \left\{\sigma \in S_{p+r} \,\middle|\, \sigma(1) < \cdots < \sigma(p) \text{ and } \sigma(p+1) < \cdots < \sigma(p+r)\right\},$$

where S_{p+r} is the symmetric group of degree $p+r$.
It is easy to show that $\wedge$ is an associative multiplication and

$$(2.3) \qquad \omega \wedge \theta = (-1)^{pr+qs} \theta \wedge \omega$$

where ω has type (p, q) and θ has type (r, s).
Let ω^k denote the k-th exterior power of ω. Then we can rewrite the inner product on $\Lambda^p(m)$ over $m \in M$ as follows;

$$(2.4) \qquad \langle X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_p \rangle = \frac{1}{p!} g^p(X_1, \cdots, X_p)(Y_1, \cdots, Y_p).$$

We define the double form $c\omega$ of type $(p-1, q-1)$ for ω of type (p, q) as follows. With $p=0$ or $q=0$, we put $c\omega=0$. If both p and $q \geqq 1$, then we put

$$\begin{aligned}(2.5) \qquad &c\omega(X_1, \cdots, X_{p-1})(Y_1, \cdots, Y_{q-1}) \\ &= \sum_{k=1}^{n} \omega(E_k, X_1, \cdots, X_{p-1})(E_k, Y_1, \cdots, Y_{q-1}),\end{aligned}$$

where $\{E_1, \cdots, E_n\}$ is a locally defined orthonormal frame field with respect to g. We call this map c the contraction.

The double form $b\omega$ of type $(p+1, q-1)$ for ω of type (p, q) is defined as follows. With $q=0$, we put $b\omega=0$. If $q\geqq 1$, then we put

$$b\omega(X_1, \cdots, X_{p+1})(Y_1, \cdots, Y_{q-1}) = \sum_{j=1}^{p+1}(-1)^j \omega(X_1, \cdots, \check{X}_j, \cdots, X_{p+1})(X_j, Y_1, \cdots, Y_{q-1}), \tag{2.6}$$

where the symbol $\check{}$ denotes omission.

Note that $b\omega=0$ for any quadratic form ω on M. Of course, we have $bg=0$. If $b\omega=0$, we call ω a double form satisfying the generalized first Bianchi identity. We know that the double form of type (p, p) satisfying the generalized first Bianchi identity is symmetric.
Concerning these operators, the following lemmas are well known (cf. see [2] and [3]).

LEMMA 1. *Let ω and θ be the double forms on M of types (p, q) and (r, s) respectively. Then we have the following formulas:*

$$c(g\wedge\omega) = g\wedge c\omega+(n-p-q)\omega, \tag{2.7}$$

$$b(\omega\wedge\theta) = b\omega\wedge\theta+(-1)^{p+q}\omega\wedge b\theta, \tag{2.8}$$

$$bc = cb. \tag{2.9}$$

LEMMA 2. *Let ω be a double form of type (p, p) on M. Suppose that $b\omega=0$ and $\omega(X_1, \cdots, X_p)(X_1, \cdots, X_p)=0$ for all $X_1, \cdots, X_p \in M_m$ at each point $m\in M$. Then $\omega=0$.*

Let R_{XY} be the curvature operator given by the formula:

$$R_{XY} = [\nabla_X, \nabla_Y]-\nabla_{[X,Y]}, \qquad \big(X, Y\in \mathfrak{X}(M)\big). \tag{2.10}$$

Then we define a curvature double form R of type (2, 2) by the formula:

$$R(X, Y)(Z, W) = \langle R_{XY}Z, W\rangle, \qquad \big(X, Y, Z, W\in \mathfrak{X}(M)\big). \tag{2.11}$$

Note that $bR=0$ reduces to the first Bianchi identity.

The Weyl conformal curvature tensor C is a double form of type (2, 2) is given by

$$C = R-\frac{1}{n-2}g\wedge cR+\frac{c^2R}{2(n-2)(n-1)}g^2. \tag{2.12}$$

As the generalization of the conformal curvature tensor C, we define the p-th conformal curvature tensor C_p by the formula [2]:

$$C_p = R^p+\sum_{k=1}^{2p}\frac{(-1)^k}{k!\,\Pi_{j=0}^{k-1}(n-4p+2+j)}g^k\wedge c^kR^p, \qquad (n\geqq 4p-1). \tag{2.13}$$

The manifold M is called p-conformally flat if $n>4p-1$ and $C_p=0$. Of course, if $p=1$, then M is conformally fiat.

Let $G_{2p}(M)$ be a Grassman bundle of oriented tangent $2p$-planes of M. The $2p$-th sectional curvature $K_{2p}(\sigma)$ with respect to $\sigma=(m,P)\in G_{2p}(M)$ is given by the formula [3]:

$$K_{2p}(\sigma)=\frac{(-2)^p R^p(X_1,\cdots,X_{2p})(X_1,\cdots,X_{2p})}{(2p)!\,\|X_1\wedge\cdots\wedge X_{2p}\|^2}, \tag{2.14}$$

where P is spanned by $X_1,\cdots,X_{2p}\in M_m$.

§3. Proof of Theorem 2.

Suppose that M is a p-conformally flat Riemannian manifold of dimension $n\geqq 4$p. Then we have

$$C_p=0, \tag{3.1}$$

that is

$$R^p=\sum_{k=1}^{2p}\frac{(-1)^{k-1}}{k!\,\Pi_{j=0}^{k-1}(n-4p+2+j)}\,g^k\wedge c^k R^p. \tag{3.2}$$

The $2p$-th sectional curvature $K_{2p}(\sigma)$ with respect to $2p$-plane $\sigma=(m,P)\in G_{2p}(M)$ is given by

$$\begin{aligned}K_{2p}(\sigma)&=\frac{1}{\|X_1\wedge\cdots\wedge X_{2p}\|^2}\\ &\times\left\{\sum_{i,j=1}^{2p} g(X_i,X_j)\,Q(X_1,\cdots,\check{X}_i,\cdots,X_{2p})(X_1,\cdots,\check{X}_j,\cdots,X_{2p})\right\}\end{aligned} \tag{3.3}$$

where $Q=\dfrac{(-2)^p}{(2p)!}\displaystyle\sum_{k=1}^{2p}\frac{(-1)^{k-1}}{k!\,\Pi_{j=0}^{k-1}(n-4p+2+j)}\,g^{k-1}\wedge c^k R^p$

and P is spanned by $X_1,\cdots,X_{2p}\in M_m$.
Thus if $\{X_1,\cdots,X_{2p}\}$ is any orthonormal basis for P, then we obtain

$$K_{2p}(\sigma)=\sum_{j=1}^{2p}Q(X_1\wedge\cdots\wedge\check{X}_j\wedge\cdots\wedge X_{2p})(X_1\wedge\cdots\wedge\check{X}_j\wedge\cdots\wedge X_{2p}), \tag{3.4}$$

that is, the $2p$-th sectional curvature $K_{2p}(\sigma)$ with respect to σ is given by the trace of the restriction of Q to σ.

From Lemma 1, the generalized fiirst Bianchi identity $bQ=0$ for Q is straightforward.

Conversely, suppose that the $2p$-th sectional curvature $K_{2p}(\sigma)$ with respect to $\sigma=(m,P)\in G_{2p}(M)$ is given by

$$K_{2p}(\sigma)=trace\ Q/\Lambda^{2p-1}(\sigma), \tag{3.5}$$

where Q is a certain quadratic form on $\Lambda^{2p-1}(m)$ which satisfies the generalized first Bianchi identity as a double form of type $(2p-1, 2p-1)$. The expression *trace* $Q/\Lambda^{2p-1}(\sigma)$ being independent of the choice of the orthonormal basis $\{U_1, \cdots, U_{2p}\}$ of $\Lambda^{2p-1}(m)$ respected to P, we put

$$U_1=\frac{\tilde{X}_1}{\|\tilde{X}_1\|},$$

$$U_2=\frac{\|\tilde{X}_1\|^2\tilde{X}_2-\langle\tilde{X}_1, \tilde{X}_2\rangle\tilde{X}_1}{\|\tilde{X}_1\|\,\|\tilde{X}_1\wedge\tilde{X}_2\|}, \tag{3.6}$$

$$\vdots$$

$$U_{2p}=\frac{\sum_{k=1}^{2p}(-1)^k\langle\tilde{X}_1\wedge\cdots\wedge\tilde{X}_{2p-1},\ \tilde{X}_1\wedge\cdots\wedge\check{\tilde{X}}_k\wedge\cdots\wedge\tilde{X}_{2p}\rangle\tilde{X}_k}{\|\tilde{X}_1\wedge\cdots\wedge\tilde{X}_{2p-1}\|\,\|\tilde{X}_1\wedge\cdots\wedge\tilde{X}_{2p}\|}$$

where $\tilde{X}_i=X_1\wedge\cdots\wedge\check{X}_i\wedge\cdots\wedge X_{2p}\in\Lambda^{2p-1}(m)$ and $\wedge$ is an exterior multiplication on $\Lambda^{2p-1}(m)$.
Then from (3.5) and (3.6), we have

$$\frac{(-2)^pR^p(X_1, \cdots, X_{2p})(X_1, \cdots, X_{2p})}{(2p)!\,\|X_1\wedge\cdots\wedge X_{2p}\|^2}$$
$$=\frac{\sum_{i,j=1}^{2p}(-1)^{i+j}\langle\tilde{X}_1\wedge\cdots\wedge\check{\tilde{X}}_i\wedge\cdots\wedge\tilde{X}_{2p}, \tilde{X}_1\wedge\cdots\wedge\check{\tilde{X}}_j\wedge\cdots\wedge\tilde{X}_{2p}\rangle Q(\tilde{X}_i, \tilde{X}_j)}{\|\tilde{X}_1\wedge\cdots\wedge\tilde{X}_{2p}\|^2} \tag{3.7}$$

Here we use the following identities;

$$\|\tilde{X}_1\wedge\cdots\wedge\tilde{X}_{2p}\|^2=\|X_1\wedge\cdots\wedge X_{2p}\|^{4p-2},$$
$$\langle\tilde{X}_1\wedge\cdots\wedge\check{\tilde{X}}_i\wedge\cdots\wedge\tilde{X}_{2p}, \tilde{X}_1\wedge\cdots\wedge\check{\tilde{X}}_j\wedge\cdots\wedge\tilde{X}_{2p}\rangle \tag{3.8}$$
$$=\|X_1\wedge\cdots\wedge X_{2p}\|^{4p-4}\langle X_i, X_j\rangle.$$

Using this fact, we find

$$\frac{(-2)^p}{(2p)!}R^p(X_1, \cdots, X_{2p})(X_1, \cdots, X_{2p})$$
$$=(g\wedge Q)(X_1, \cdots, X_{2p})(X_1, \cdots, X_{2p}). \tag{3.9}$$

From Lemm 2, we obtain

$$R^p=g\wedge L, \tag{3.10}$$

where $L=\dfrac{(2p)!}{(-2)^p}Q$.

From (3.10) and Lemma 1, we have

$$
(3.11) \qquad
\begin{aligned}
cR &= g\wedge cL+(n-4p+2)L, \\
&\vdots \\
c^k R^p &= g\wedge c^k L+k(n-4p+k+1)c^{k-1}L, \\
&\vdots \\
c^{2p} R^p &= 2p(n-2p+1)c^{2p-1}L.
\end{aligned}
$$

Then it follows that

$$
(3.12) \qquad L=\sum_{k=1}^{2p}\frac{(-1)^{k-1}}{k!\,\Pi_{j=0}^{k-1}(n-4p+2+j)}\,g^{k-1}\wedge c^k R^p, \qquad i.e. \quad (3.2).
$$

Therefore M is p-conformally flat.

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