Remarks on modified symmetrizers for 2×2 hyperbolic mixed problems

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§1. Introduction and results

In this paper we are concerned with an existence theorem of a solution $u\in H_{1,-1;\gamma}(R^1\times\Omega)$ of the boundary value problem (P, B):

$$P(x, D)u(x)=f(x) \qquad \text{in } R^1\times\Omega,$$
$$B(x, D)u(x)=g(x) \qquad \text{on } \Gamma,$$

where $f\in H_{0,\gamma}(R^1\times\Omega)$ and $g\in H_{1/2,\gamma}(\Gamma)$. Here we assume that P is an x_0-hyperbolic 2×2 system of pseudo-differential operators of order 1 and B is a 1×2 system of those of order 0 on the smooth boundary Γ of $R^1\times\Omega$.

While we try to extend the results in [7, section 7] to more general cases being inspired by the works of R. Agemi [2] and S. Miyatake [6], we find that there are certain gaps between L^2-well posedness for (P, B) (see [4]) and their conditions which is described in terms of symbols of P and B. In the present note, applying a concept of modified symmetrizers, we shall clarify the differences mentioned above and difficulties of mixed problems for hyperbolic systems. By localizations and coordinate transformations we may restrict ourselves to the case where

$$R^1\times\Omega=R_+^{n+1}=\left\{x=(x', x_n);\ x_n>0\right\},$$
$$\Gamma=\left\{(x', 0);\ x'=(x_0, x'')\in R^n\right\}.$$

Let (τ, σ, λ) be a covariable of $x=(x_0, x'', x_n)$ such that $\mathrm{Im}\,\tau\leq 0$. We assume that symbols of the principal part P^o of P and B are independent of x if $|x|$ is sufficiently large, homogeneous in (τ, σ, λ) and (τ, σ) respectively, analytic in τ and the determinant $\det P^o$ of P^o is an x_0-strictly hyperbolic polynomial of order 2. Moreover Γ is non-characteristic with respect to $\det P^o$ and $B(x', \tau, \sigma)$ is of rank 1 for any $(x', \tau, \sigma)\in R^n\times(C\times R^n\backslash 0)$. Finally any problems $(P, B)_x$ obtained by freezing their coefficients at $x\in\Gamma$ are L^2-well posed.

As it is well known, the difficulties in our problem (P, B) arise from the following: there is a point $(x^o, \tau^o, \sigma^o)\in\Gamma\times(R^n\backslash 0)$ such that the characteristic equation $\det P^o\,(x^o, \tau^o, \sigma^o, \lambda)=0$ has a real double root $\lambda=\lambda^o$ and the Lopatinskii determinant $L(x^o, \tau^o, \sigma^o)=0$. In a neighborhood of such a point (x^o, τ^o, σ^o) $L(x', \tau, \sigma)$ can be written in the form:

$$L\big(x', \zeta+\theta(x, \sigma), \sigma\big) = \big(\sqrt{\zeta}-D(x', \sigma)\big)\, l\big(x', \sqrt{\zeta}, \sigma\big).$$

Here we shall use the same terminologies as in [7] if there is no ambiguity, but we denote by $\sqrt{\zeta}$ the branch of square roots of ζ such that $\sqrt{1}=1$. Now we shall consider mianly about the following condition:

In some neighborhood of the point (x^o, σ^o) described above

$$(L)\qquad \begin{array}{ll} \mathrm{Re}\ D(x', \sigma)\leq 0 & \textit{in the case } (a) \textit{ or} \\ \mathrm{Im}\ D(x', \sigma)\geq 0 & \textit{in the case } (b). \end{array}$$

That is, for fixed (x', σ) the analytic continuation of $L(x', \zeta+\theta(x', \sigma), \sigma)$ through the half line $\{\zeta; \zeta>0\}$ or $\{\zeta; \zeta<0\}$ does not vanish in the neighborhood of 0 up to and except the half line $\{\zeta; \zeta<0\}$ or $\{\zeta; \zeta>0\}$, according to the case (a) or (b) respectively.

Then we have our main

THEOREM 1. *There exists a modified symmetrizer iff the condition* (L) *is valid.*
(*For the definition of a modified symmetrizer see section* 3.)

THEOREM 2. *Under the condition* (L) *the problem* (P, B) *is* L^2*-well posed.*

The plan of the paper is as follows. In section 2 we remark the necessity of the semi-definiteness with respect to the symbol in the sharp form of Gårding inequality and the decomposition of the Lopatinskii determinant. In section 3 we give the definition of a modified symmetrizer and construct a suitable one in order to prove Theorem 1. In section 4 we prove Theorem 2 by obtaining an a priori estimate from the existence of a modified symmetrizer. Finally in section 5 we describe the necessary and sufficient condition for (P, B) to be L^2-well posed in terms of $D(\sigma)$ in the case of constant coefficients and give a sufficient condition which is also valid in the case of [2] and [6].

§2. Notations and lemmas

I. In this paper we use the function spaces with a real parameter γ $(\gamma\neq 0)$, $H_{k,\gamma}(R_+^{n+1})$, $H_{s,\gamma}(R^n)$ with norms $\|\cdot\|_{k,\gamma}$, $|\cdot|_{s,\gamma}$ and inner products $(\cdot, \cdot)_{k,\gamma}$, $\langle\cdot, \cdot\rangle_{s,\gamma}$ respectively. Hereafter let $\tau=\eta-i\gamma$ and regard neighborhoods of τ^o as complex those even if τ^o is real. Furthermore let Σ_- and $\partial\Sigma_-$ be the open hemisphere $\{(\tau, \sigma)\in C\times R^{n-1}; |\tau|^2+|\sigma|^2=1, \mathrm{Im}\ \tau<0\}$ and its boundary respectively.

Then we consider the same symbol class $S_+^k = S_+^k(R_+^{n+1} \times (C_- \times R^{n-1} \backslash 0))$ of pseudo-differential operators with positive parameters x_n and γ as in [7] and denote by $S_+^k(U)$ the set of all restrictions of symbols in S_+^k to an open set U of $R_+^{n+1} \times (C_- \times R^{n-1} \backslash 0)$, where $C_- = \{\tau \in C ; \operatorname{Im} \tau < 0\}$. We denote by $a(x, D')$ simply the pseudo-differential operator $a(x, D', \gamma)$ corresponding to $a(x, \eta, \gamma, \sigma) \in S_+^k$, which is always denoted by $a(x, \tau, \sigma)$.

LEMMA 2.1. *Let $A(x, \tau, \sigma) \in S_+^k$ be hermitian and homogeneous of degree k in (τ, σ). For a point $(x^o, \tau^o, \sigma^o) \in \Gamma \times \partial\Sigma_-$ there exist its neighborhood U and positive constants C, γ_0 such that for any $\gamma \geq \gamma_0$, any $u \in H_{k,\gamma}(R_+^{n+1})$ and a $\phi(x, \tau, \sigma) \in S_+^0$ with* supp $\phi \cap \overline{R_+^{n+1} \times \Sigma_-} \subset U$ *the estimate*

$$\operatorname{Re} \left\langle A(x, D')\phi(x, D')u, \phi(x, D')u \right\rangle_{0,\gamma} \geq -C|u|^2_{\frac{k-1}{2},\gamma} \tag{2.1}$$

is valid for $u = u(\cdot, x_n)$, iff there exists a neighborhood $U_1(x^o, \tau^o, \sigma^o) \subset U$ such that for any $(x, \tau, \sigma) \in U_1 \cap \overline{R_+^{n+1} \times \Sigma_-}$

$$A(x, \tau, \sigma) \geq 0 \tag{2.2}$$

PROOF. It suffices to prove 'only if' part. Let ϕ be real and $\phi(x^o, \tau^o, \sigma^o) \neq 0$. Replacing A by $A\phi^2$ and ϕu by u in (2.1) we may consider only the case where $\phi \equiv 1$. Using the coordinate transformation $x = \varepsilon y (\varepsilon > 0)$ we difine $A_\varepsilon v$ by

$$A_\varepsilon(y, D'_y)v(y) = \varepsilon^k (A(x, D'_x)u)(\varepsilon y)$$

where $u(x) = v(\varepsilon^{-1}x)$. Then the symbol of A_ε is $A_\varepsilon(y, \theta, \omega) = A(\varepsilon y, \theta, \omega)$, where (θ, ω) is a covariable of $y = (y_0, y'')$, $\rho = -\operatorname{Im} \theta$, $(\theta, \omega) = (\varepsilon\tau, \varepsilon\sigma)$ and $\rho = \varepsilon\gamma$.

Since for any fixed $x_n \geq 0$

$$\left\langle A(x, D'_x)u, u \right\rangle_{0,\gamma} = \varepsilon^{n-k} \left\langle A_\varepsilon(y, D'_y)v, v \right\rangle_{0,\rho}$$

and

$$|u|^2_{\frac{k-1}{2},\gamma} = \varepsilon^{n-k+1} |v|^2_{\frac{k-1}{2},\rho},$$

it follows from (2.1) that

$$\operatorname{Re} \left\langle A_\varepsilon(y, D'_y)v, v \right\rangle_{0,\rho} \geq -C'\varepsilon |v|^2_{\frac{k-1}{2},\rho} \tag{2.1$'$}$$

for some $C' > 0$, any $\rho \geq \varepsilon\gamma_0$ and any $v \in H_{k,\rho}(R_+^{n+1})$. Therefore letting $\varepsilon \to 0$ in (2.1)$'$ we see that

$$\int \left\langle A(0, \theta, \omega)\hat{v}(\theta, \omega, y_n), \quad \hat{v}(\theta, \omega, y_n) \right\rangle d(\operatorname{Re} \theta) d\omega \geq 0,$$

which implies that $A(0, \theta, \omega)\geq 0$ for any $(\theta, \omega)\in \bar{C}_- \times R^{n-1}$. This means (2.2) at $x=0$. Similarly we have (2.2) for any fixed x.

II. Now let (x^o, τ^o, σ^o) be a point in $\Gamma \times \partial\Sigma_-$ where λ^o is a real double root of $\det P^o(x^o, \tau^o, \sigma^o, \lambda)=0$. Then we recall first the following facts α) and β) (see [7, section 3 and 6]).

α) There exist a neighborhood $U(x^o, \tau^o, \sigma^o)$ and functions $\lambda^\pm(x, \tau, \sigma)$ continuous in U such that

(i) $\quad \operatorname{Im} \lambda^\pm(x, \tau, \sigma) \gtrless 0$ if $\operatorname{Im} \tau < 0$ respectively, and

(ii) $\quad \det P^o\big(x, \tau, \sigma, \lambda^\pm(x, \tau, \sigma)\big) = 0 \qquad$ if $\operatorname{Im} \tau \leq 0$.

Furthermore, they are represented by

$$(a) \qquad \lambda^\pm(x, \tau, \sigma) = \lambda_1(x, \zeta, \sigma) \mp \sqrt{\zeta}\, \lambda_2(x, \zeta, \sigma)$$

or

$$(b) \qquad \lambda^\pm(x, \tau, \sigma) = \lambda_1(x, \zeta, \sigma) \pm i\sqrt{\zeta}\, \lambda_2'(x, \zeta, \sigma)$$

according as the normal surface cut by $x=x^0$ and $\sigma=\sigma^o$ is convex or concave with respect to τ at (τ^o, λ^o) respectively. Here $\zeta=\zeta(x, \tau, \sigma)=\tau-\theta(x, \sigma)$, θ is a real valued smooth function of (x, σ), analytic and homogeneous of degree 1 in σ belonging to a conic neighborhood of (x^o, σ^o) and $\theta(x^o, \sigma^o)=\tau^o$. Furthermore λ_1, λ_2 and λ_2' are analytic in (ζ, σ), real for real ζ in a conic neighborhood of $(x^o, 0, \sigma^o)$ such that

$$\lambda_1\big(x, \tau-\theta(x, \sigma), \sigma\big) \in S_+^1\big(U(x^o, \tau^o, \sigma^o)\big),$$
$$\lambda_2\big(x, \tau-\theta(x, \sigma), \sigma\big),\ \lambda_2'\big(x, \tau-\theta(x, \sigma), \sigma\big) \in S_+^{\frac{1}{2}}\big(U(x^o, \tau^o, \sigma^o)\big),$$
$$\lambda_1(x^o, 0, \sigma^o) = \lambda^o,\ \lambda_2(x^o, 0, \sigma^o)>0,\ \lambda_2'(x^o, 0, \sigma^o)>0$$

and $\sqrt{\zeta}$ denotes the branch with negative imaginary part when $\operatorname{Im} \zeta<0$ (this branch is different from the one in [7]).

Hereafter we use the following notations:

$$\tau = \eta - i\gamma, \qquad \zeta = \tau-\theta(x, \sigma) = \varepsilon - i\gamma,$$
$$\Lambda_\gamma = \big(|\tau|^2+|\sigma|^2\big)^{\frac{1}{2}}, \qquad \tau' = \tau\Lambda_\gamma^{-1} \text{ and etc.}$$

Furthermore we denote $a(x, \zeta+\theta(x, \sigma), \sigma)$ also by $a(x, \zeta, \sigma)$ for (x, ζ, σ) belonging to some neighborhood of $(x^o, 0, \sigma^o)$ and conversely we do $b(x, \tau-\theta(x, \sigma), \sigma)$ by $b(x, \tau, \sigma)$ defined in a neighborhood of (x^o, τ^o, σ^o). Moreover $b(x, \zeta, \sigma)\in S_+^k(U(x^o, 0, \sigma^o))$ means that $b(x, \tau, \sigma)\in S_+^k(U(x^o, \tau^o, \sigma^o))$ for the corresponding neighborhood $U(x^o, \tau^o, \sigma^o)$ and we denote boundary points $(x', 0)$ by x'.

Finally we assume that

$$P^o(x, \tau, \sigma, \lambda) = \lambda I - A(x, \tau, \sigma),$$

where I is the unit matrix, the symbol $A(x, \tau, \sigma)$ is in S^1_+, homogeneous in (τ, σ) and analytic in τ.

β) There exist a neighborhood $U(x^o, \tau^o, \sigma^o)$, 2-vectors $h'(x, \tau, \sigma)$, $h''(x, \tau, \sigma)$ and a matrix $M(x, \tau, \sigma)$ which are smooth in (x, τ, σ) and analytic in τ such that for every $(x, \tau, \sigma) \in U(x^o, \tau^o, \sigma^o)$

(i) $h'(x, \theta(x, \sigma), \sigma)$ and $h''(x, \theta(x, \sigma), \sigma)$ are an eigenvector and a generalized eigenvector of $M(x, \theta(x, \sigma), \sigma)$ corresponding to $\lambda^+(x, \theta(x, \sigma), \sigma) = \lambda^-(x, \theta(x, \sigma), \sigma)$ respectively,

(ii) $M = S^{-1}AS, \qquad$ where $S = (h', h'')$,

(iii)
$$
\begin{aligned}
& M(x, \zeta, \sigma) \\
& = M(x, \varepsilon, \sigma) + M(x, \varepsilon - i\gamma, \sigma) - M(x, \varepsilon, \sigma) \\
& = \begin{pmatrix} \lambda_1(x, 0, \sigma), & \Lambda_\gamma \\ 0 \quad , & \lambda_1(x, 0, \sigma) \end{pmatrix} + \varepsilon E(x, \varepsilon, \sigma) - i\gamma H(x, \varepsilon, \sigma) + O(\gamma^2 \Lambda_\gamma^{-1}).
\end{aligned}
\tag{2.3}
$$

Here

$$E(x, \varepsilon, \sigma), H(x, \varepsilon, \sigma) \in S^0_+\big(U(x^o, 0, \sigma^o)\big) \text{ and}$$

$$E(x, \varepsilon, \sigma) = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}, \qquad H(x, \varepsilon, \sigma) = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

We also have another expansion for M:

$$
\begin{aligned}
& M(x, \zeta, \sigma) \\
& = \begin{pmatrix} \lambda_1(x, 0, \sigma), & \Lambda_0^{(o)} \\ 0 \quad , & \lambda_1(x, 0, \sigma) \end{pmatrix} + \zeta \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} (x, \zeta, \sigma),
\end{aligned}
\tag{2.3$'$}
$$

where p_{ij} are smooth in (x, ζ, σ), analytic in (ζ, σ) and $\Lambda_0^{(o)} = (\theta(x, \sigma)^2 + |\sigma|^2)^{\frac{1}{4}}$. Furthermore

$$
\begin{aligned}
& \Lambda_0^{(o)} p_{21}(x, 0, \sigma) = \Lambda_0^{(o)} e_{21}(x, 0, \sigma) = \Lambda_0^{(o)} h_{21}(x, 0, \sigma) \\
& = \begin{cases} \lambda_2(x, 0, \sigma)^2 > 0 & \text{(in the case } (a)), \\ -\lambda_2'(x, 0, \sigma)^2 < 0 & \text{(in the case } (b)), \end{cases}
\end{aligned}
\tag{2.4}
$$

$$\zeta\big(p_{11}(x, \zeta, \sigma) + p_{22}(x, \zeta, \sigma)\big) = 2\big(\lambda_1(x, \zeta, \sigma) - \lambda_1(x, 0, \sigma)\big). \tag{2.5}$$

Remark 1. As in [8] replacing $S(x, \zeta, \sigma)$ by

$$S(x, \zeta, \sigma)\begin{pmatrix}\varepsilon' e_{12}(x, \varepsilon, \sigma)+1, & 0\\ \varepsilon' e_{22}(x, \varepsilon, \sigma) \quad , & 1\end{pmatrix}$$

if necessary, we may assume that $e_{ij}(x, \varepsilon, \sigma)$ in β) are real.

In order to define the Lopatinskii determinant, let

$$s(x, \zeta, \sigma) = \frac{\lambda^+(x, \zeta, \sigma)-\lambda_1(x, 0, \sigma)-\zeta p_{11}(x, \zeta, \sigma)}{\Lambda_0^{(o)}+\zeta p_{12}(x, \zeta, \sigma)}. \tag{2.6}$$

Then ${}^t(1, s(x, \zeta, \sigma))$ is an eigenvector of $M(x, \zeta, \sigma)$ corresponding to the eigenvalue $\lambda^+(x, \zeta, \sigma)$ if $\operatorname{Im} \tau<0$.

Let

$$s(x, \zeta, \sigma) = \zeta\, s_1(x, \zeta, \sigma)-\sqrt{\zeta}\, s_2(x, \zeta, \sigma), \tag{2.7}$$

where $\quad s_1(x, \zeta, \sigma) = \dfrac{\zeta^{-1}(\lambda_1(x, \zeta, \sigma)-\lambda_1(x, 0, \sigma))-p_{11}(x, \zeta, \sigma)}{\Lambda_0^{(o)}+\zeta p_{12}(x, \zeta, \sigma)}$

and

$$s_2(x, \zeta, \sigma) = \frac{\lambda_2(x, \zeta, \sigma)}{\Lambda_0^{(o)}+\zeta p_{12}(x, \zeta, \sigma)}$$

are analytic in (ζ, σ), real for real ζ in $U(x^o, 0, \sigma^o)$. Note that in some real neighborhood of $(x^o, 0, \sigma^o)$

$$|s_1(x, \varepsilon, \sigma)|\leq C \text{ and } s_2(x, \varepsilon, \sigma)>0 \tag{2.8}$$

for some constant $C>0$. Then we see that

$$L(x', \tau, \sigma) = B(x', \tau, \sigma)\, S(x', \tau, \sigma){}^t\big(1, s(x', \tau, \sigma)\big) \tag{2.9}$$

is the Lopatinskii determinant for (P, B) at (x', τ, σ). Here and in the next section 3 we assume that $L(x^o, \tau^o, \sigma^o)=0$ and we consider only the case (a) because analogous arguments can be applied to the case (b).

LEMMA 2.2. *Let* $(V'(x', \tau, \sigma), V''(x', \tau, \sigma))=B(x', \tau, \sigma)\, S(x', \tau, \sigma)$, *then*

(i) $\quad V'(x^o, \tau^o, \sigma^o) = 0$ *and* $V''(x^o, \tau^o, \sigma^o) \neq 0$.

Furthermore we have the following decomposition

(ii) $$L(x', \zeta, \sigma) = \big(\sqrt{\zeta}-D(x', \sigma)\big)\, l(x', \sqrt{\zeta}, \sigma),$$
$$l(x', \sqrt{\zeta}, \sigma) = \big(l_1(x', \zeta, \sigma)+\sqrt{\zeta}\, l_2(x', \zeta, \sigma)\big)\cdot V''(x', \zeta, \sigma),$$

where l_1, l_2 *and* D *are smooth in* (x, ζ, σ) *and analytic in* ζ *in* $U(x^o, 0, \sigma^o)$, $l_1(x^o, 0, \sigma^o)\neq 0$ *and* $D(x^o, \sigma^o)=0$.

PROOF. To show (i) see [7, p. 120]. Let $z=\sqrt{\zeta}$ and $\mathfrak{L}(x', z, \sigma)=L$

(x', z^2, σ). Then by the L^2-well posedness for freezing problems $L(x', \zeta, \sigma) \neq 0$ for $\mathrm{Im}\ \tau < 0$, whence $\mathfrak{L}(x', z, \sigma)$ does not identically vanish. Furthermore we see from (2.6)-(2.9) that

$$\frac{\partial \mathfrak{L}}{\partial z}(x^o, 0, \sigma^o) = -s_2(x^o, 0, \sigma^o)\, V''(x^o, \tau^o, \sigma^o) \neq 0,$$

but by our assumption we see that $\mathfrak{L}(x^o, 0, \sigma^o) = 0$. Therefore the Weierstrass preparation theorem implies (ii).

γ) Let $Q(x', \tau, \sigma) = V'(x', \tau, \sigma) \big/ V''(x'', \tau, \sigma)$. Then from our assumptions we see

LEMMA 2.3. *Suppose that* $e_{11} = e_{22}$ *in a real neighborhood of* $(x^o, 0, \sigma^o)$. *Then there exists another real neighborhood* $U(x^o, 0, \sigma^o)$ *such that, for any* $(x', \varepsilon', \sigma') \in U(x^o, 0, \sigma^o)$ *satisfying*

$$|Q|^2(x', \varepsilon', \sigma') + \varepsilon'\Big(1 + \varepsilon' e_{12}(x', \varepsilon', \sigma')\Big)^{-1} \cdot e_{21}(x', \varepsilon', \sigma') = 0, \tag{2.10}$$

$$\mathrm{Re}\ Q(x', \varepsilon', \sigma') \cdot \mathrm{Re}\ D(x', \sigma') \geq 0 \qquad \textit{in the case } (a). \tag{2.11}$$

PROOF. From the definitions of Q and L and Lemma 2.2 it follows that

$$Q + s = (\sqrt{\varepsilon'} - D)(l_1 + \sqrt{\varepsilon'}\, l_2), \tag{2.12}$$

which is rewritten by (2.7) in the following form:

$$Q + \varepsilon' s_1 - \sqrt{\varepsilon'}\, s_2 = \varepsilon' l_2 - l_1 D + \sqrt{\varepsilon'}\,(l_1 - l_2 D).$$

Comparing both sides of the last equality we have

$$s_2 = l_2 D - l_1 \tag{2.13}$$

and

$$Q + \varepsilon' s_1 = \varepsilon' l_2 - l_1 D. \tag{2.14}$$

From (2.12) and (2.13) we have

$$Q + s = (\sqrt{\varepsilon'} - D)\Big(l_2(\sqrt{\varepsilon'} + D) - s_2\Big). \tag{2.15}$$

Moreover from (2.7), (2.13) and (2.14) it follows that

$$\bar{Q} + s = (\sqrt{\varepsilon'} - \bar{D})\Big(\bar{l}_2(\sqrt{\varepsilon'} + \bar{D}) - s_2\Big). \tag{2.16}$$

Therefore we see that

$$\begin{aligned}(Q+s)(\bar{Q}+s) &= (\varepsilon' + |D|^2)A - 4\varepsilon' B\, \mathrm{Re}\, D + \\ &+ 2\sqrt{\varepsilon'}\,\Big\{(\varepsilon' + |D|^2)B - A\, \mathrm{Re}\, D\Big\},\end{aligned} \tag{2.17}$$

where $\qquad A = s_2^2 - 2s_2 \operatorname{Re}(l_2 D) + |l_2|^2 (\varepsilon' + |D|^2)$

and $\qquad B = |l_2|^2 \operatorname{Re} D - s_2 \operatorname{Re} l_2.$

Furthermore from (2. 8) and Lemma 2. 2 (ii) it follows that

$$A > 0 \tag{2.18}$$

in a sufficiently small real neighborhood of $(x^o, 0, \sigma^o)$.

On the other hand considering the characteristic equation of $M(x', \varepsilon', \sigma')$ we have

$$\varepsilon' \lambda_2^2 = \left(2^{-1}\varepsilon'(e_{11} - e_{22})\right)^2 + \varepsilon' e_{21}(1 + \varepsilon' e_{12}),$$

which implies by (2. 5) and (2. 7)

$$s_1 = -\frac{e_{11} - e_{22}}{1 + \varepsilon' e_{12}}, \qquad \varepsilon' s_2^2 = \left(\frac{\varepsilon'}{2} \frac{e_{11} - e_{22}}{1 + \varepsilon' e_{12}}\right)^2 + \frac{\varepsilon' e_{21}}{1 + \varepsilon' e_{12}}.$$

Hence we have

$$\begin{aligned} &(Q+s)(\bar{Q}+s) \\ &= |Q|^2 + \frac{\varepsilon' e_{21}}{1+\varepsilon' e_{12}} + \frac{\varepsilon'^2}{2}\left(\frac{e_{11}-e_{22}}{1+\varepsilon' e_{12}}\right)^2 \\ &- \frac{\varepsilon'(e_{11}-e_{22})}{1+\varepsilon' e_{12}} \operatorname{Re} Q - \sqrt{\varepsilon'}\, s_2\left(2 \operatorname{Re} Q - \varepsilon' \frac{e_{11}-e_{22}}{1+\varepsilon' e_{12}}\right). \end{aligned} \tag{2.19}$$

Comparing (2. 17) with (2. 19) and using the hypothesis that $e_{11} - e_{22} = 0$ we obtain

$$|Q|^2 + \frac{\varepsilon' e_{21}}{1+\varepsilon' e_{12}} = (\varepsilon' + |D|^2) A - 4\varepsilon' B \operatorname{Re} D \tag{2.20}$$

and

$$-s_2 \operatorname{Re} Q = (\varepsilon' + |D|^2) B - A \operatorname{Re} D \tag{2.21}$$

in some real neighborhood of $(x^o, 0, \sigma^o)$. Now let (2. 10) be valid, then from (2. 20) we have

$$(\varepsilon' + |D|^2) A = 4\varepsilon' B \operatorname{Re} D.$$

Thus we see from (2. 21) that

$$s_2 A \operatorname{Re} Q \operatorname{Re} D = (A^2 - 4\varepsilon' B^2)(\operatorname{Re} D)^2 \geq 0 \tag{2.22}$$

in some real neighborhood of $(x^o, 0, \sigma^o)$, which implies our assertion since (2. 8) and (2. 18) are valid.

§ 3. Modified symmetrizers and Proof of Theorem 1

In the previous section we introduced $M(x, \tau, \sigma)$ and $Q(x', \tau, \sigma)$ defined in some neighborhood of (x^o, τ^o, σ^o). Here we consider suitable extensions of these symbols, which we also denote by the same notations, therefore let $M \in S^1_+$, $Q \in S^0_+$ and let $R(x, \tau, \sigma) \in S^0_+$ in the following definition.

DEFINITION 3.1. *Let R be a 2×2 system of pseudo-differential operators whose symbol $R(x, \tau, \sigma)$ is hermitian, homogeneous in (τ, σ) and real analytic with respect to ε' in a neighborhood of $(x^o, 0, \sigma^o)$. Then we say that R is a modified symmetrizer at (x^o, τ^o, σ^o) if the following holds: there exist a neighborhood $U(x^o, \tau^o, \sigma^o)$, positive constants C_1, C_2 and γ_0 such that the estimates*

$$\mathrm{Re}\left(\left(iRM+(iRM)^*\right)\phi u,\ \phi u\right)_{0,\gamma} \geq C_1\gamma\|\phi u\|^2_{0,\gamma}-C'_\phi\|u\|^2_{0,\gamma} \tag{3.1}$$

and for $x_n=0$

$$\mathrm{Re}\,\langle R\phi u,\ \phi u\rangle_{0,\gamma} \geq -C_2\gamma|\phi u_1|^2_{-\frac{1}{2},\gamma}-C''_\phi|u_1|^2_{-\frac{1}{2},\gamma} \tag{3.2}$$

$$\text{if}\quad Q\phi u_1+\phi u_2 = 0 \quad \text{on } \Gamma$$

are valid for any $\gamma \geq \gamma_0$, any $u={}^t(u_1, u_2)\in H_{1,\gamma}(R^{n+1}_+)$, any $\phi(x, \tau, \sigma)\in S^0_+$ with $\mathrm{supp}\ \phi\cap\overline{R^{n+1}_+\times\Sigma_-}\subset U$ *and some constants C'_ϕ, C''_ϕ independent of u.*

To investigate the structure of R we need following lemmas.

LEMMA 3.1. *Let R be a modified symmetrizer. Then in a neighborhood $U_1(x^o, \tau^o, \sigma^o)\subset U(x^o, \tau^o, \sigma^o)$ the symbol of R is represented by the following form:*

$$\begin{aligned} R&(x, \zeta, \sigma) \\ &= \begin{pmatrix} b(x, \zeta, \sigma) & , & d_1(x, \zeta, \sigma)+i\gamma' f(x, \zeta, \sigma) \\ d_1(x, \zeta, \sigma)-i\gamma' f(x, \zeta, \sigma), & & d_2(x, \zeta, \sigma) \end{pmatrix} \end{aligned} \tag{3.3}$$

where b, d_1, d_2 and f are real. Furthermore it holds that

$$b = \varepsilon'(1+\varepsilon' e_{12})^{-1}(d_1(e_{11}-e_{22})+d_2 e_{21}) \qquad \text{for } \gamma' = 0, \tag{3.4}$$

$$2d_1\,\mathrm{Re}\,h_{21}-C_1 \geq 0 \tag{3.5}$$

and

$$-2d_1\,\mathrm{Re}\,Q+b+d_2|Q|^2 \geq 0 \qquad \text{for } \gamma' = 0. \tag{3.6}$$

PROOF. Generally $R(x, \zeta, \sigma)$ has a following form:

$$R(x, \zeta, \sigma) = \begin{pmatrix} b & d_1 \\ d_1 & d_2 \end{pmatrix}(x, \zeta, \sigma)+i\begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}(x, \zeta, \sigma)$$

where b, d_1, d_2 and f are real. Expand b and f in a small neighborhood $U(x^o, 0, \sigma^o)$ so that

$$b = b_0(x, \sigma) + \varepsilon' b_1(x, \varepsilon, \sigma) + \gamma' b_2(x, \zeta, \sigma)$$

and
$$f = f_0(x, \sigma) + \varepsilon' f_1(x, \varepsilon, \sigma) + \gamma' f_2(x, \zeta, \sigma).$$

From (2. 3) we then obtain

$$\begin{aligned}&\left(iRM + (iRM)^*\right)(x, \zeta', \sigma') \\ &= K_0(x, \sigma') + \varepsilon' K_1(x, \zeta', \sigma') + \gamma' K_2(x, \zeta', \sigma') + O(|\varepsilon'\gamma'| + \gamma'^2)\end{aligned}$$

where

$$K_0(x, \sigma') = \begin{pmatrix} 0 & ib_0 \\ -ib_0 & 2f_0 \end{pmatrix},$$

$$K_1(x, \zeta', \sigma') = \begin{pmatrix} -2e_{21}(f_0 + \varepsilon' f_1) & \bar{k}_1 \\ k_1 & 2f_0 e_{12} + 2f_1(1 + \varepsilon' e_{12}) \end{pmatrix},$$

$$k_1 = (f_0 + \varepsilon' f_1)(e_{11} - e_{22}) + i\left\{d_1(e_{11} - e_{22}) + d_2 e_{21} - b_1(1 + \varepsilon' e_{12}) - b_0 e_{12}\right\},$$

$$K_2(x, \zeta', \sigma') = \begin{pmatrix} 2(d_1 \operatorname{Re} h_{21} + b_0 \operatorname{Re} h_{11}) & \bar{k}_2 \\ k_2 & 2(d_1 \operatorname{Re} h_{12} + d_2 \operatorname{Re} h_{22} + f_2) \end{pmatrix} + (*) \cdot f_0,$$

$(*)$ is a matrix consisting of some $\operatorname{Im} h_{ij}$ and

$$k_2 = d_1(h_{11} + \bar{h}_{22}) + d_2 h_{21} + b_0 \bar{h}_{12} - i\left\{f_0(h_{11} + \bar{h}_{22}) + b_2\right\}.$$

Applying Lemma 2. 1 to (3. 1) the estimate

(3. 7)
$$K_0 + \varepsilon' K_1 + \gamma' K_2 - C_1 \gamma' I \geq 0$$

is valid in some neighborhood of $(x^o, 0, \sigma^o)$. Taking $\gamma' = 0$ we have

$$-2\varepsilon'(f_0 + \varepsilon' f_1)e_{21} \geq 0 \text{ and } 2(f_0 + \varepsilon' f_1)(1 + \varepsilon' e_{12}) \geq 0.$$

Since $e_{21} \neq 0$ and the sign ε' is not definite, the above inequalities and our assumption about analyticity imply $f_0 + \varepsilon' f_1 \equiv 0$, that is $f_0 \equiv 0$, $f_1 \equiv 0$ and $b \equiv 0$. Hence replacing f by $\gamma' f$, we see that (3. 3), $K_0 = 0$ and $K_1 = \begin{pmatrix} 0 & \bar{k}_1 \\ k_1 & 0 \end{pmatrix}$. Furthermore from (3. 7) it follows that $k_1 = 0$ for $\gamma' = 0$. Therefore we see that (3. 4) is valid. By the same way as above it also follows from (3. 7) that (3. 5) is valid.

To prove (3. 6) let $Q\phi u_1 + \phi u_2 = 0$ on Γ, then (3. 3) yields

$$\begin{aligned}&\operatorname{Re} \langle R\phi u, \phi u \rangle_{0,\gamma} \\ &\leq \operatorname{Re} \left\langle (-2d_1 \operatorname{Re} Q + b + d_2|Q|^2 + 2\gamma' f \operatorname{Im} Q)\phi u_1, \phi u_1 \right\rangle_{0,\gamma} + C_\phi |u_1|^2_{-\frac{1}{2},\gamma}\end{aligned}$$

for some constant $C_\phi > 0$. Hence we obtain (3. 6) applying Lemma 2. 1 to (3. 2) and the above inequality.

LEMMA 3. 2. *A modified symmetrizer can be constructed iff there exists a real valued homogeneous function $d(x', \varepsilon', \sigma')$ of degree zero, smooth in $(x', \varepsilon', \sigma')$ and analytic in ε' such that*

$$-2\operatorname{Re} Q+(1+\varepsilon' e_{12})^{-1}(e_{11}-e_{22})\varepsilon'+d(|Q|^2+(1+\varepsilon' e_{12})^{-1}e_{21}\varepsilon')\geq 0 \tag{3.8}$$

for all $(x', \varepsilon', \sigma')$ contained in some real neighborhood of $(x^o, 0, \sigma^o)$.

PROOF. The 'only if' part follows from (3. 4)-(3. 6) and (2. 4). To obtain an a priori estimate we had better construct a modified symmetrizer R_δ with a parameter δ $(0<\delta<1)$ instead of R.
Here replacing $U(x^o, \tau^o, \sigma^o)$, C'_ϕ and C''_ϕ by $U_\delta(x^o, \tau^o, \sigma^o)$, C'_δ and C''_δ respectively we impose the following estimates upon R_δ:

$$\operatorname{Re}\Big(\big(iR_\delta M+(iR_\delta M)^*\big)\phi u, \phi u\Big)_{0,\gamma}\geq C_1\gamma(\|\phi u_1\|^2_{0,\gamma}+\delta^{-1}\|\phi u_2\|^2_{0,\gamma})-C'_\delta\|u\|^2_{0,\gamma}, \tag{3.1$'$}$$

$$\operatorname{Re}\langle R_\delta\phi u, \phi u\rangle_{0,\gamma}\geq -C_2\gamma|\phi u_1|^2_{-\frac{1}{2},\gamma}-C''_\delta|u_1|^2_{-\frac{1}{2},\gamma}, \tag{3.2$'$}$$

where ϕ is a function ϕ_δ depending on δ such that $\phi_\delta=1$ in some neighborhood $U'_\delta\Subset U_\delta$.

Now we can choose, from (2. 4), a positive constant d_1 such that in some neighborhood of $(x^o, 0, \sigma^o)$

$$d_1\operatorname{Re} h_{21}>1. \tag{3.9}$$

Let $d_2(x, \zeta, \sigma)=d_1\cdot d(x', \varepsilon', \sigma')$ and let

$$b(x, \zeta, \sigma)=\varepsilon'(1+\varepsilon' e_{12})^{-1}(d_1(e_{11}-e_{22})+d_2 e_{21}). \tag{3.10}$$

Furthermore we set

$$R_\delta(x, \zeta, \sigma)=\begin{pmatrix} b(x, \varepsilon', \sigma') & d_1+i\gamma' f \\ d_1-i\gamma' f & d_2(x, \varepsilon', \sigma')\end{pmatrix}$$

where f is a constant depending on δ which we are soon going to choose. Then we see that K_0, K_1 in Lemma 3. 1 are both zero,

$$K_2=\begin{pmatrix} 2d_1\operatorname{Re} h_{21} & \bar{k}_2 \\ k_2 & 2(d_1\operatorname{Re} h_{12}+d_2\operatorname{Re} h_{22}+f)\end{pmatrix}$$

and

$$k_2=d_1(h_{11}+\bar{h}_{22})+d_2 h_{21}.$$

Since k_2 is bounded it follows from (3. 9) that

$$K_2(x, \zeta, \sigma)\geq\begin{pmatrix}1 & 0\\ 0 & \delta^{-1}\end{pmatrix}$$

in a neighborhood of $(x^o, 0, \sigma^o)$ independently of δ if we take f sufficiently large depending on δ. Thus (3. 1)$'$ is valid by Lemma 2. 1.

Next if $Q\phi u_1+\phi u_2=0$ on Γ, then using (3. 10) we see that

$$\begin{aligned}&\mathrm{Re}\,\langle R_\delta\phi u, \phi u\rangle_{0,\gamma}\\ &= \mathrm{Re}\left\langle(R_0+\gamma R_1)\phi u_1, \phi u_1\right\rangle_{0,\gamma}+\mathrm{Re}\,\langle Tu_1, u_1\rangle_{0,\gamma},\end{aligned}$$

where

$$\begin{aligned}&R_0(x', \varepsilon', \sigma')+\gamma' R_1(x', \zeta', \sigma') = (1, -\bar{Q})\, R_\delta \begin{pmatrix}1\\ -Q\end{pmatrix}(x', \zeta', \sigma')\\ &= \Big\{d_1(-2\,\mathrm{Re}\, Q+(1+\varepsilon' e_{12})^{-1}(e_{11}-e_{22})\varepsilon')+d_2(|Q|^2+\\ &+(1+\varepsilon' e_{12})^{-1}e_{21}\varepsilon')+2\gamma' f\,\mathrm{Im}\, Q\Big\}(x', \zeta', \sigma')\end{aligned}$$

and Tu_1 is the term arising from commutators, so that $|Tu_1|_{\frac{1}{2},\gamma}\leq C_\delta^1|u|_{-\frac{1}{2},\gamma}$ for some constant $C_\delta^1>0$.

Since (3. 8) means $R_0(x, \varepsilon', \sigma')\geq 0$ it yields for some $C_\delta^2>0$

$$\mathrm{Re}\,\langle R_0\phi u_1, \phi u_1\rangle_{0,\gamma}\geq -C_\delta^2|u_1|^2_{-\frac{1}{2},\gamma}.$$

On the other hand all terms except $2f\,\mathrm{Im}\, Q$ in R_1 are bounded. Furthermore there exists a neighborhood $U_\delta(x^o, \tau^o, \sigma^o)$ such that $|2f\,\mathrm{Im}\, Q|\leq 1$ on U_δ since $Q(x^o, \tau^o, \sigma^o)=0$. Thus for some constant $C_2>0$ we have

$$\gamma' R_1(x', \zeta', \sigma')\geq -C_2\gamma' \qquad \text{in } U_\delta.$$

Therefore the corresponding term of $\gamma' R_1(x', \zeta', \sigma')$ dominates the right hand side of (3. 2)$'$. Here we use that

$$\langle\gamma'(D')\phi u_1, \phi u_1\rangle_{0,\gamma} = \gamma\,\langle\Lambda^{-1}\phi u_1, \phi u_1\rangle_{0,\gamma} = \gamma|\phi u_1|^2_{-\frac{1}{2},\gamma},$$

Λ being defined by the symbol Λ_γ.

LEMMA 3. 3. *Suppose that* $e_{11}=e_{22}$ *in a neighborhood of* $(x^o, 0, \sigma^o)$. *There exists a real valued homogeneous function* $d(x, \varepsilon, \sigma)$ *of degree* 0 *satisfying* (3. 8) *iff*

$$\mathrm{Re}\, Q\leq 0$$

on the surface (2. 10) *in some real neighborhood of* $(x^o, 0, \sigma^o)$.

PROOF. It suffices to prove the 'if' part.

Let $$\omega'(x', \varepsilon', \sigma') = |Q|^2+\varepsilon'(1+\varepsilon' e_{12})^{-1}e_{21}.$$

Then we can take (x', ω', σ') as the new variables instead of $(x', \varepsilon', \sigma')$ since

$$\frac{\partial \omega'}{\partial \varepsilon'}(x^o, 0, \sigma^o) = e_{21}(x^o, 0, \sigma^o) \neq 0.$$

Now writing

$$-2\operatorname{Re} Q(x', \varepsilon', \sigma') = q_0(x', \sigma') + q_1(x', \omega', \sigma')\,\omega'$$

we let

$$d = -q_1(x', \omega', \sigma').$$

Then $q_0(x', \sigma')$ is $-2\operatorname{Re} Q$ on the surface (2. 10), hence it is non-negative, which implies our assertion (3. 8). Here we remark that d is real analytic with respect to ε'.

PROOF OF THEOREM 1. The existence of a modified symmetrizer is invariant under a similarity transformation by any $S(x, \tau, \sigma) \in S_+^0$. We may assume $e_{11} = e_{22}$ replacing S in section 2, II, β) by $S\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ for suitable k. Then Lemmas 3. 2 and 3. 3 yield that the existence of a modified symmetrizer is equivalent to that $\operatorname{Re} Q \leq 0$ on the surface (2. 10). Finally by Lemma 2. 3 the latter is also equivalent to the condition (L) (see (2. 22)).

§ 4. Proof of Theorem 2 and a priori estimates.

In [7] it is shown that a priori estimates for (P, B) and for its dual problem hold under the stronger conditions and suggested that the assertion of Theorem 2 is valid. Here we shall give its simple proof. To do it we shall proceed the same way as in [4] (see also [7]), i. e., we shall show a priori estimates for (P, B):

$$\|Pu\|_{k,\gamma} + |Bu|_{k+\frac{1}{2},\gamma} \geq C_k \gamma \|u\|_{k,\gamma} \quad (k: \text{integer} \geq 0). \tag{4. 1}$$

For this purpose we shall use the usual method of a partition of unity. Then we have only to show the micro-local estimate as in Lemma 4. 2 with respect to any point $(x^o, \tau^o, \sigma^o) \in \Gamma \times \bar{\Sigma}_-$. If a Lopatinskii determinant $L(x^o, \tau^o, \sigma^o)$ is not zero, for example if $\operatorname{Im} \tau^o \neq 0$ or the roots of $\det P(x^o, \tau^o, \sigma^o, \lambda) = 0$ are all real (see [9]), the corresponding symmetrizer exists (see [3]). Furthermore if the roots of $\det P^o(x^o, \tau^o, \sigma^o, \lambda) = 0$ are non-real, the a priori estimate with respect to the point (x^o, τ^o, σ^o) is already known (see [1] and [7]). Accordingly we may consider only the case treated in section 3.

Now let (x^o, τ^o, σ^o) be a point considered in section 3. Let $\phi(x, \tau, \sigma)$ and $\psi(x, \tau, \sigma) \in S_+^0$ be real homogeneous in (τ, σ) and $\psi = 1$ on supp ϕ. Let supp $\psi \cap \overline{R_+^{n+1} \times \Sigma_-}$ be contained in any neighborhood considered after this. Let us denote by M, Q and R_δ operators corresponding to those symbols

ϕM, ϕQ, and ϕR_δ. Finally set $v=\phi(x, D')u$, $(D_n-M)v=f$, $Qv_1+v_2=g$, $w={}^t(v_1, -Qv_1)$ and $G={}^t(0, g)$. Then we have

LEMMA 4.1. *There exist positive constants C, C_δ and γ_δ depending on δ such that the estimate*

$$|v_1|^2_{-\frac{1}{2},\gamma} \le C(\delta\|v_1\|^2_{0,\gamma}+\delta^{-1}\|v_2\|^2_{0,\gamma}+\delta^{-1}\|\Lambda^{-1}f\|^2_{0,\gamma}) + C_\delta\|u\|^2_{-\frac{1}{2},\gamma} \tag{4.2}$$

is valid for any $\gamma>\gamma_\delta$ and any $u\in H_{1,\gamma}(R^{n+1}_+)$ if we take a sufficiently small neighborhood U_δ of (x^o, τ^o, σ^o).

PROOF. From integration by parts it holds that

$$\begin{aligned}|v_1|^2_{-\frac{1}{2},\gamma} &= 2\,\mathrm{Im}\,(\Lambda^{-1}D_n v_1, v_1)_{0,\gamma}\\ &= 2\,\mathrm{Im}\,(\Lambda^{-1}(m_{11}v_1+m_{12}v_2+f_1), v_1)_{0,\gamma},\end{aligned}$$

where (m_{11}, m_{12}) is the first row of M. For given fixed $\delta>0$ we can choose a neighborhood U_δ of $(x^o, 0, \sigma^o)$ such that

$$\mathrm{Im}(\Lambda_\gamma^{-1}m_{11})(x, \zeta, \sigma) = -\gamma'\,\mathrm{Re}\,h_{11}(x, \zeta, \sigma)+O(\gamma'^2)\le\delta$$

for all $(x, \zeta, \sigma)\in U_\delta$. Then it follows from Lemma 2.1 that

$$\mathrm{Im}\,(\Lambda^{-1}m_{11}v_1, v_1)_{0,\gamma}\le\delta\|v_1\|^2_{0,\gamma}+C_\delta\|u\|^2_{-\frac{1}{2},\gamma}$$

for some constant $C_\delta>0$ if γ is sufficiently large. Thus we have the estimate (4.2).

To prove Theorem 2 it suffices to establish the following

LEMMA 4.2. *Suppose that there exists a modified symmetrizer R_δ satisfying* (3.1)$'$ *and* (3.2)$'$ *at (x^o, τ^o, σ^o). Then there exist a neighborhood $U(x^o, \tau^o, \sigma^o)$ and positive constants C, γ_0 such that the a priopri estimate*

$$\|(D_n-M)v\|_{0,\gamma}+|Qv_1+v_2|_{\frac{1}{2},\gamma}+\gamma^{\frac{1}{2}}(\|u\|_{0,\gamma}+|u|_{-\frac{1}{2},\gamma}) \ge C\gamma\|v\|_{0,\gamma} \tag{4.3}$$

holds for any $\gamma\ge\gamma_0$ and any $u\in H_{1,\gamma}(R^{n+1})$.

PROOF. Using integration by parts and the adjoint operator M^* of M, it holds that

$$\begin{aligned}&\Big(iR_\delta v, (D_n-M)\,v\Big)_{0,\gamma}+\Big((D_n-M)\,v, iR_\delta v\Big)_{0,\gamma}\\ &= \Big((D_n-M^*)\,iR_\delta v, v\Big)_{0,\gamma}+\langle R_\delta v, v\rangle_{0,\gamma}+\Big(-iR^*_\delta(D_n-M)v, v\Big)_{0,\gamma}.\end{aligned}$$

Since $-M^*iR_\delta\equiv(iR_\delta M)^*$, $D_n iR_\delta\equiv iR_\delta D_n$ and $R_\delta\equiv R^*_\delta$ (modulo pseudo-differential operators of lower order respectively), the estimate

$$
\begin{aligned}
&2\,\mathrm{Re}(iR_\delta v, f)_{0,\gamma} + C_\delta^{(1)}(\|\Lambda^{-1}f\|_{0,\gamma}^2 + \|v\|_{0,\gamma}^2)\\
&\geq \mathrm{Re}\left((iR_\delta M + (iR_\delta M)^*)\,v, v\right)_{0,\gamma} + \mathrm{Re}\,\langle R_\delta w, w\rangle_{0,\gamma}\\
&\quad - |\langle R_\delta w, G\rangle_{0,\gamma} + \langle R_\delta G, w\rangle_{0,\gamma} + \langle R_\delta G, G\rangle_{0,\gamma}|
\end{aligned}
\tag{4.4}
$$

holds for some constant $C_\delta^{(1)} > 0$. Now we have

$$
\begin{aligned}
&|\langle R_\delta w, G\rangle_{0,\gamma} + \langle R_\delta G, w\rangle_{0,\gamma} + \langle R_\delta G, G\rangle_{0,\gamma}|\\
&\leq \sqrt{2C_\delta^{(2)}}\,|v_1|_{-\frac{1}{2},\gamma}|G|_{\frac{1}{2},\gamma} + 2^{-1}C_\delta^{(2)}\gamma^{-1}|G|_{\frac{1}{2},\gamma}^2\\
&\leq \gamma|v_1|_{-\frac{1}{2},\gamma}^2 + C_\delta^{(2)}\gamma^{-1}|g|_{\frac{1}{2},\gamma}^2
\end{aligned}
\tag{4.5}
$$

for some constant $C_\delta^{(2)} > 0$. Furthermore we shall use the estimate (4.2)$'$ which is obtained from (4.2) by taking δ_1, C_3 and $C_{\delta_1}^{(3)}$ for δ, C and C_δ in Lemma 4.1 respectively. Finally we see that

$$
2\,\mathrm{Re}(iR_\delta v, f)_{0,\gamma} \leq \delta_2\gamma\|v\|_{0,\gamma}^2 + C_\delta^{(4)}(\delta_2\gamma)^{-1}\|f\|_{0,\gamma}^2
\tag{4.6}
$$

holds for some constant $C_\delta^{(4)} > 0$ and any $\delta_2 > 0$.

Combining (3.1)$'$, (3.2)$'$, (4.2)$'$, (4.3)–(4.6) and noting $\|\Lambda^{-1}f\|_{0,\gamma} \leq \gamma^{-1}\cdot \|f\|_{0,\gamma}$, it is obtained that

$$
\begin{aligned}
&(C_\delta^{(4)}\delta_2^{-1} + C_\delta^{(1)}\gamma^{-1} + (C_2+1)C_3\delta_1^{-1})\|f\|_{0,\gamma}^2 + C_\delta^{(2)}|g|_{\frac{1}{2},\gamma}^2\\
&+ (C_\delta' + (C_2+1)C_{\delta_1}^{(3)}\gamma^{-1})\gamma\|u\|_{0,\gamma}^2 + C_\delta''\gamma|u|_{-\frac{1}{2},\gamma}^2\\
&\geq (C_1 - \delta_2 - C_\delta^{(1)}\gamma^{-1} - (C_2+1)C_3\delta_1)\gamma^2\|v_1\|_{0,\gamma}^2\\
&+ (C_1\delta^{-1} - \delta_2 - C_\delta^{(1)}\gamma^{-1} - (C_2+1)C_3\delta_1^{-1})\gamma^2\|v_2\|_{0,\gamma}^2
\end{aligned}
$$

in $U_\delta \cap U_{\delta_1}$ for sufficiently large γ. Let C' and C'' be the coefficients of $\gamma^2\cdot \|v_1\|_{0,\gamma}^2$ and $\gamma^2\|v_2\|_{0,\gamma}^2$ in the above inequality respectively. Then we can choose δ_1, δ_2 and δ sufficiently small such that $\delta \ll \delta_1$ and then γ sufficiently large so that C' and C'' can be strictly positive. Thus we see that the desired inequality is valid. Finally (L) is also valid for (P', B') since the Lopatinskii determinant $L'(x_0, x'', \tau, \sigma)$ is equal to $\overline{L(-x_0, x'', -\bar{\tau}, \sigma)}$ except a nonvanishing factor and there happens the case (a) or (b) for (P, B) corresponding to the case (b) or (a) for (P', B') respectively. Therefore we have a priori estimates for (P', B') (see [7, (9.11)]). Thus we have Theorem 2 by the same way as in [4].

§ 5. Relations between (L) and L^2-well posedness

I. In this section we shall investigate the necessary and sufficient condition of L^2-well posedness for (P, B) whose symbols are independent of x variables. As is shown in [7] assuming $L(\tau, \sigma) \neq 0$ when $\mathrm{Im}\,\tau < 0$ we have only to consider the inequality

$$|L(\tau, \sigma)^{-1}| \leq C\gamma^{-1}|\mathrm{Im}\,\lambda^{+}(\tau, \sigma)|^{\frac{1}{2}}|\mathrm{Im}\,\lambda^{-}(\tau, \sigma)|^{\frac{1}{2}} \tag{5.1}$$

in some neighborhood $U(\tau^{o}, \sigma^{o})$ with $\mathrm{Im}\,\tau<0$. Furthermore we may restrict ourselves to the case where (τ^{o}, σ^{o}) is such a point as considered before. Then we have the following

PROPOSITION 5.1. *The estimate* (5.1) *is valid iff there exist a neighborhood $U(\sigma^{o})$ and a constant $\delta>0$ such that for any $\sigma \in U(\sigma^{o})$*

$$\begin{aligned} &\delta \leq \arg D(\sigma) \leq \frac{3}{2}\pi \qquad \textit{in the case } (a) \textit{ or} \\ &0 \leq \arg D(\sigma) \leq \frac{3}{2}\pi-\delta \qquad \textit{in the case } (b), \textit{ respectively.} \end{aligned} \tag{5.2}$$

PROOF. Also here only the case (a) is considered.
Note that

$$\gamma = |\mathrm{Im}\,\zeta| = 2|\mathrm{Re}\sqrt{\zeta}\,|\,|\mathrm{Im}\sqrt{\zeta}\,|.$$

Hence by Lemma 2.2 (5.1) is equivalent to that

$$\frac{|\mathrm{Im}\,\lambda^{+}(\zeta, \sigma)\,\mathrm{Im}\,\lambda^{-}(\zeta, \sigma)|^{\frac{1}{2}}}{|\mathrm{Im}\sqrt{\zeta}\,|}\,\frac{|\sqrt{\zeta}-D(\sigma)|}{|\mathrm{Re}\sqrt{\zeta}\,|} \geq c'>0$$

for some constant c' in $U(0, \sigma^{o})$ with $\mathrm{Im}\,\zeta<0$. Considering the expansions

$$\lambda^{\pm}(\zeta, \sigma) = a_0(\sigma)+a_1(\sigma)(\pm\sqrt{\zeta}\,)+a_2(\sigma)(\pm\sqrt{\zeta}\,)^2+\cdots$$

where $a_j(\sigma)$ $(j=0, 1, 2, \cdots)$ are real, analytic and $a_1(\sigma^{o})\neq 0$, it holds that in a small neighborhood of $(0, \sigma^{o})$

$$C'^{-1} < \frac{|\mathrm{Im}\,\lambda^{\pm}(\zeta, \sigma)|}{|\mathrm{Im}\sqrt{\zeta}\,|} < C'$$

for some constant $C'>0$. Therefore (5.1) is equivalent to that

$$\frac{|\sqrt{\zeta}-D(\sigma)|}{|\mathrm{Re}\sqrt{\zeta}\,|} \geq c \tag{5.3}$$

for some constant $c>0$.

Now we shall prove the equivalence between (5.2) and (5.3).

(5.2)$\rightrightarrows$(5.3): From our convention it is seen that $-\frac{\pi}{2} \leq \arg\sqrt{\zeta} \leq 0$ for $\mathrm{Im}\zeta \leq 0$. Let $\arg\sqrt{\zeta}=-\omega$ and $0\leq\omega\leq\frac{\pi}{2}$.
Then we have

$$|\sqrt{\zeta}-D(\sigma)| \geq \min\left\{|\mathrm{Re}\sqrt{\zeta}\,|,\ |\sqrt{\zeta}\,|\sin(\omega+\delta)\right\}$$

and

$$|\sqrt{\zeta}|\sin(\omega+\delta) = |\sqrt{\zeta}|(\sin\omega\cos\delta+\cos\omega\sin\delta)$$
$$\geq|\sqrt{\zeta}|\cos\omega\sin\delta = |\mathrm{Re}\sqrt{\zeta}|\sin\delta\,.$$

Hence it follows that

$$\frac{|\sqrt{\zeta}-D(\sigma)|}{|\mathrm{Re}\sqrt{\zeta}|}\geq\sin\delta>0,$$

which implies that (5. 3) holds.

(5. 3)$\rightrightarrows$(5. 2): Let $\arg D(\sigma)=\delta(\sigma)$ and $0\leq\delta(\sigma)<\frac{\pi}{2}$.

Put $\sqrt{\zeta(\sigma,\omega)} = \dfrac{|D(\sigma)|}{\cos(\omega+\delta(\sigma))}e^{-i\omega}$ for $0\leq\omega<\frac{\pi}{2}-\delta(\sigma)$.

Since

$$|\sqrt{\zeta(\sigma,\omega)}-D(\sigma)| = \frac{|D(\sigma)|}{\cos(\omega+\delta(\sigma))}\sin(\omega+\delta(\sigma))$$
$$= \frac{\mathrm{Re}\sqrt{\zeta(\sigma,\omega)}}{\cos\omega}\sin(\omega+\delta(\sigma)),$$

then we have from (5. 3)

$$\frac{\sin(\omega+\delta(\sigma))}{\cos\omega} = \frac{|\sqrt{\zeta(\sigma,\omega)}-D(\sigma)|}{|\mathrm{Re}\sqrt{\zeta(\sigma,\omega)}|}\geq c.$$

Hence it is obtained that

$$\sin\delta(\sigma) = \lim_{\omega\to0}\frac{\sin(\omega+\delta(\sigma))}{\cos\omega}\geq c,$$

which implies (5. 2).

II. At first let us consider the case where $P^o(x,D)=D_n-A(x,D')$ is a differential operator of order 1 and $B(x',D')$ is a function $B(x')$ of x'. Let (x^o,τ^o,σ^o) be a point considered above. Continue analytically $\lambda^{\pm}(x,\zeta+\theta(x,\sigma),\sigma)$ and $S(x,\zeta+\theta(x,\sigma),\sigma)=(h',h'')$ from $\zeta=\varepsilon-i\gamma$ to $\zeta=\varepsilon+i\gamma$ for $\gamma>0$, fixed (x,σ) in some neighborhood of (x^o,σ^o) and small $\varepsilon>0$. Recall that we can take

$$h''(x,\zeta+\theta(x,\sigma),\sigma) = \frac{1}{2\pi i}\oint_C\Big(\lambda-A\big(x,\zeta+\theta(x,\sigma),\sigma\big)\Big)^{-1}$$
$$\cdot h_1\big(x,\theta(x,\sigma),\sigma\big)\,d\lambda\,.$$

and

$$\Lambda_\gamma h'\left(x, \zeta+\theta(x, \sigma), \sigma\right) = \left(\lambda^+(x, \theta(x, \sigma), \sigma) - A\left(x, \zeta+\theta(x, \sigma), \sigma\right)\right) \cdot h''\left(x, \zeta+\theta(x, \sigma), \sigma\right),$$

where $h_1(x, \theta(x, \sigma), \sigma)$ is a generalized eigenvector of $A(x, \theta(x, \sigma), \sigma)$ corresponding to $\lambda^+ = \lambda^-(x, \theta(x, \sigma), \sigma)$ and C is a circle enclosing $\lambda^\pm(x, \zeta+\theta(x, \sigma), \sigma)$ (see [7, Lemma 3.2]). Furthermore let us take note of the fact that $\theta(x, -\sigma) = -\theta(x, \sigma)$. Then it follows from the definitions of M and Q that

$$\begin{aligned} &M\left(x, -(\varepsilon+i\gamma+\theta(x, \sigma)), -\sigma\right) \\ (5.4) \qquad &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M\left(x, \varepsilon+i\gamma+\theta(x, \sigma), \sigma\right) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\ &Q\left(x', -(\varepsilon+i\gamma+\theta(x', \sigma)), -\sigma\right) = -Q\left(x', \varepsilon+i\gamma+\theta(x', \sigma), \sigma\right) \end{aligned}$$

for small $\varepsilon+i\gamma$ $(\gamma>0)$ and any (x, σ) in some neighborhood of (x^0, σ^0). Here we remark that (5.4) is also valid if M and Q are those reduced from differential operators of order 2 and 1 respectively, that is in the case treated in [1], [2] and [6].

Finally we give a certain sufficient condition for (P, B) to be L^2-well posed in terms of M and Q.

PROPOSITION 5.2. *Suppose that* (5.4) *is valid. Then the condition* (*L*) *is satisfied.*

PROOF. In order to prove (*L*) it suffices to show that $L(x', \zeta+\theta(x', \sigma), \sigma)$ can be continued analytically through the half line $\{\zeta>0\}$ and does not vanish over $C\backslash\{\zeta=\varepsilon-i\gamma\,;\ \varepsilon\leq 0,\ \gamma=0\}$. To show this fact, from (2.7), (2.9) and the definition of Q, we may rewrite L as follows: for $\gamma\geq 0$

$$\begin{aligned} &L\left(x', \zeta+\theta(x', \sigma), \sigma\right) \\ &= \left(\lambda^+(x', \zeta+\theta(x', \sigma), \sigma) - \lambda^+(x', \theta(x', \sigma), \sigma)\right. \\ &\quad \left. - \zeta p_{11}(x', \zeta, \sigma)\right)\left(\Lambda_0^{(0)} + \zeta p_{12}(x', \zeta, \sigma)\right)^{-1} + Q\left(x', \zeta+\theta(x', \sigma), \sigma\right). \end{aligned}$$

By the analytic continuation and the relations (5.4) we see that the resulting $L(x', \varepsilon+i\gamma+\theta(x', \sigma), \sigma)$ is equal to $-L(x', -(\varepsilon+i\gamma+\theta(x', \sigma)), -\sigma)$. The latter does not vanish whenever $\gamma\neq 0$ by the L^2-well posedness at the point $(x^0, -\theta(x^0, \sigma^0), -\sigma^0)$. Finally by [9, Theorem 3] we conclude that our assertion is valid.

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ADDED IN PROOF. Removing the requirement for the analyticity w. r. t. ε in the definition of modified symmetrizers the former author has obtained the result that the condition (L) in Theorm 1 and 2 can be replaced by $(\tilde{L})$: in a neighborhood of any (x^0, σ^0) considered mainly Re $D \leq \chi$ (Im D) or Im $D \geq -\chi$ ($-$Re D) in the case (a) or (b) respectively for some $(0\leq)$ $\chi \in C^\infty$ (R^1) with supp $\chi \subset \bar{R}^1_+$.

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