Some remarks on p-absolutely summing operators

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§ 1. Introduction

In [1], Cohen has shown that the following:

THEOREM A. *Let E be a normed space. Then E is an inner product space iff for all Banach spaces F and for all 2-absolutely summing operators T mapping E into F, the conjugate operator T^* is 2-absolutely summing and* $\Pi_2(T^*) \leqq \Pi_2(T)$.

In [2], Kwapień has given a similar characterization of spaces isomorphic to inner product spaces. That is the following:

THEOREM B. *Let E be a Banach space, then the following conditions are equivalent*:

(1) *E is isomorphic (=linearly homeomorphic) to an inner product space.*

(2) *If $T \in \Pi_2(E, l_2)$, then $T^* \in \Pi_2(l_2, E^*)$.*

Theorem A and Theorem B suggest the following (*):

(*) *Let E be a Banach space, and $1 \leqq p < \infty$. Then the following conditions are equivalent.*

(1) *For all Banach spaces F,*

if $T \in \Pi_p(E, F)$, then $T^* \in \Pi_p(F^*, E^*)$.

(2) *If $T \in \Pi_p(E, l_p)$, then $T^* \in \Pi_p(l_{p^*}, E^*)$.*

In this paper, we shall prove this fact is true, and furthermore, using weakly p-summable sequences, we shall characterize Banach spaces E which satisfy the condition (1) (or equivalently condition (2)).

Notation. Throughout the paper E and F will denote Banach spaces and E^* and F^* the continuous dual spaces. The space of continuous linear operators mapping E into F will be denoted by $L(E, F)$.

§ 2. Basic definitions and well known results

Let E and F be Banach spaces, and $1 \leqq p \leqq \infty$.

A sequence $\{x_i\}$ with values in E is called weakly p-summable ($l_p(E)$) if for all $x^* \in E^*$, the sequence $\{x^*(x_i)\} \in l_p$. The space $l_p(E)$ is a normed space; the norm is given by

$$\varepsilon_p\Big(\{x_i\}\Big)=\begin{cases}\sup\left\{\left(\sum_{i=1}^{\infty}\left|x^*(x_i)\right|^p\right)^{1/p}:\ \|x^*\|\leqq 1\right\}, & 1\leqq p<\infty\\ \sup_i\left\{\sup\left\{\left|x^*(x_i)\right|:\ \|x^*\|\leqq 1\right\}\right\}, & p=\infty\,.\end{cases}$$

The following theorem, due to Grothendieck (c. f. [6]), provides a useful characterization of $l_p(E)$.

THEOREM 2.1. *For* $1<p\leqq\infty$ *and* $1/p+1/p^*=1$, *trere is an isometric isomorphism between* $l_p(E)$ *and* $L(l_{p^*}, E)$. *For* $p=1$, $l_1(E)$ *is isometrically isomorphic with* $L(c_0, E)$. *In both cases, a sequence* $\{x_i\}$ *in* $l_p(E)$ *is identified with the operator*

$$T\Big(\{c_i\}\Big)=\sum_{i=1}^{\infty}c_i x_i\,.$$

A sequence $\{x_i\}$ is called absolutely p-summable ($l_p\{E\}$) if the sequence $\{\|x_i\|\}\in 1_p$. The space $l_p\{E\}$ is a normed space; the norm is given by

$$\alpha_p\Big(\{x_i\}\Big)=\begin{cases}\left(\sum_{i=1}^{\infty}\|x_i\|^p\right)^{1/p}, & 1\leqq p<\infty\\ \sup_i\|x_i\|, & p=\infty\,.\end{cases}$$

A sequence $\{x_i\}$ is called strongly p-summable ($l_p\langle E\rangle$) if for all sequences $\{x_i^*\}\in l_{p^*}(E^*)$, $1/p+1/p^*=1$, the series $\sum_{i=1}^{\infty}x_i^*(x_i)$ converges.

The space $l_p\langle E\rangle$ is a normed space; the norm is given by

$$\sigma_p\Big(\{x_i\}\Big)=\sup\left\{\left|\sum_{i=1}^{\infty}x_i^*(x_i)\right|:\ \varepsilon_{p^*}\Big(\{x_i^*\}\Big)\leqq 1\right\}.$$

DEFINITION 2.1. *Let* $1\leqq p,\ q\leqq\infty$. *An operator* T *mapping* E *into* F *is* (p, q)*-absolutely summing* ($\Pi_{p,q}(E, F)$) *if there exists a constant* $c\geqq 0$, *such that for all finite sets* $x_1, \cdots, x_n$, *the inequality*

$$\alpha_p\Big(\{Tx_i\}\Big)\leqq c\varepsilon_q\Big(\{x_i\}\Big)$$

is satisfied. The smallest number c, *such that the above inequality is satisfied, is called the* (p, q)*-absolutely summing norm* ($\Pi_{p,q}(T)$) *of* T.

We shall say p-absolutely summing instead of (p, p)-absolutely summing, and absolutely summing instead of 1-absolutely summing, respectively.

It is easily seen that the following:

THEOREM 2.2. *A linear operator* T *mapping* E *into* F *is* (p, q)*-absolutely summing iff for each* $\{x_i\}\in l_q(E)$, $\{Tx_i\}\in l_p\{F\}$.

DEFINITION 2.2. *Let $1\leqq p, q\leqq\infty$. An operator T mapping E into F is (p, q)-strongly summing $(D_{p,q}(E, F))$ if there exists a constant $c\geqq 0$ such that for all finite sets $x_1, \cdots, x_n$, the inequality*

$$\sigma_p\left(\{Tx_i\}\right)\leqq c\alpha_q\left(\{x_i\}\right)$$

is satisfied. The smallest number c, such that the above inequality is satisfied, is called the (p, q)-strongly summing norm $(D_{p,q}(T))$ of T.

We shall say p-strongly summing instead of (p, p)-strongly summing.

Next, we shall introduce an $\mathscr{L}_{p\lambda}$-space. The definition of this space is due to Lindenstrauss and Pelczyński (c. f. [7]).

Let E and F be Banach spaces. The distance $d(E, F)$ between E and F is defined by $d(E, F)=\inf\{\|T\|\cdot\|T^{-1}\|\}$, where the infimum is taken over all invertible operators in $L(E, F)$. If no such T exists, i.e., if E and F are not isomorphic, $d(E, F)$ is taken as ∞.

DEFINITION 2.3. *Let $1\leqq p\leqq\infty$, and $1\leqq\lambda<\infty$. A Banach space E is called an $\mathscr{L}_{p\lambda}$-space if for all finite dimensional subspaces $M\subset E$ there exists a finite dimensional subspace N containing M such that $d(N, l_p^n)\leqq\lambda$, where $n=dim(N)$.*

It can be shown (c. f. [7]) that every $L_p(\mu)$ space is an $\mathscr{L}_{p\lambda}$-space for all $\lambda>1$ and every space of type $C(K)$, where K is a compact Hausdorff space, is an $\mathscr{L}_{\infty\lambda}$-space for all $\lambda>1$. More generally, every Banach space whose dual is isometric to an $L_1(\mu)$-space (e.g. every M space in the sence of Kakutani [8]) is an $\mathscr{L}_{\infty\lambda}$-space for every $\lambda>1$ (c. f. [9]).

The following theorems are due to J. S. Cohen (c. f. [3]).

THEOREM 2.3. *Let $1/p+1/q=1$.*

(1) *Let $1\leqq p<\infty$. An operator T belongs to $\Pi_p(E, F)$ iff the conjugate operator T^* belongs to $D_q(F^*, E^*)$.*

(2) *Let $1<q\leqq\infty$. An operator T belongs to $D_q(E, F)$ iff the conjugate operator T^* belongs to $\Pi_p(F^*, E^*)$.*

THEOREM 2.4. *Let $1<p\leqq\infty$ and $1/p+1/q=1$.*

(1) *Let E be an $\mathscr{L}_{p\lambda}$-space. Then, $\Pi_q(E, F)\subset D_p(E, F)$.*

(2) *Let F be an $\mathscr{L}_{q\lambda}$-space. Then, $D_p(E, F)\subset\Pi_q(E, F)$.*

The following theorem are due to M. Kato (c. f. [4]), and this is a generalization of the Theorem 2.3..

THEOREM 2.5. *Let $1/p+1/p^*=1$, $1/q+1/q^*=1$.*

(1) *Let $1\leqq p, q<\infty$. An operator T belongs $\Pi_{p,q}(E, F)$ iff the conjugate operator T^* belongs to $D_{q^*,p^*}(F^*, E^*)$.*

(2) *Let* $1<p<\infty$ *and* $1\leqq q<\infty$. *An operator* T *belongs* $D_{q^*,p^*}(E, F)$ *iff the conjugate operator* T^* *belongs to* $\Pi_{p,q}(F^*, E^*)$.

§3. Main theorems and other results

Throughout this section, let X be a set and $\mathfrak{B}$ be a σ-algebra in X, and let μ be a positive measure such that there exist positive constants C_1, C_2 and pairwise disjoint measurable subsets $\{X_n\}$, which satisfy the following conditions:

$$C_1 \leqq \mu(X_n) \leqq C_2, \qquad \text{for all } n=1, 2, \cdots$$

Let $L_p(X, \mu)$ be a usual Banach space, then l_p(usual sequence space) is a $L_p(X, \mu)$-space which satisfies the above conditions.

We shall denote L_p instead of $L_p(X, \mu)$ in the ensuing discussions.

THEOREM 3.1. *Let E be a Banach space,* $1\leqq p\leqq q\leqq r<\infty$. *Then the following conditions are equivalent.*

(1) *For all Banach spaces F,*
if $T\in \Pi_{q,p}(E, F)$, *then* $T^*\in \Pi_{r,q}(F^*, E^*)$.

(2) *If* $T\in \Pi_{q,p}(E, L_q)$, *then* $T^*\in \Pi_{r,q}(L_{q^*}, E^*)$ $\quad(1/q+1/q^*=1)$.

(3) *For any* $\{x_n^*\}\subset E^*$ *with* $\|x_n^*\|=1$ $\quad(n=1, 2, \cdots)$,

$$\bigcap_{\alpha} 1_q(\rho_{n,\alpha})\subset 1_r$$

where $\rho_{n,\alpha}=\sum_{i=1}^{\infty}|x_n^*(x_i)|^q$, *with* $\{x_i\}\in 1_p(E)$.

PROOF.

(1)⇒(2): It is obvious.

(2)⇒(3): Assume the contrary, then there exist $\{x_n^*\}\subset E^*$ with $\|x_n^*\| =1\ (n=1, 2, \cdots)$, and complex sequence $\{b_n\}$ such that $\sum_{n=1}^{\infty}|b_n|^q \rho_{n,\alpha}<\infty$ for all $\rho_{n,\alpha}$, and $\sum_{n=1}^{\infty}|b_n|^r=\infty$.

From the assumption of μ, there exist positive constants C_1, C_2 and pairwise disjoint measurable subsets $\{X_n\}$ in X such that

$$C_1\leqq \mu(X_n)\leqq C_2 \qquad (n=1, 2, \cdots).$$

Let

$$f_n(s)=\begin{cases}1 & \text{for } s\in X_n\\ 0 & \text{for } s\in X_n^c \text{ (complement of } X_n),\end{cases}$$

then obviously $\{f_n\}\subset L_q$.

Now, we shall define an operator T mapping E into L_q such that

$$Tx=\sum_{n=1}^{\infty}b_n x_n^*(x)f_n \qquad \text{for } x\in E.$$

Claim (*a*): *T is* (*q, p*)*-absolutely summing.*

For each $\{x_i\}\in 1_p(E)$,

$$\begin{aligned}\|Tx_i\|^q &= \int_X \left|\sum_{n=1}^{\infty} b_n x_n^*(x_i) f_n\right|^q d\mu \\ &= \sum_{n=1}^{\infty}\int_{X_n}\left|b_n x_n^*(x_i)\right|^q d\mu \\ &= \sum_{n=1}^{\infty}\left|b_n x_n^*(x_i)\right|^q \mu(X_n) \\ &\leqq C_2\sum_{n=1}^{\infty}\left|b_n x_n^*(x_i)\right|^q\end{aligned}$$

therefore, we have

$$\begin{aligned}\sum_{i=1}^{\infty}\|Tx_i\|^q &\leqq \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}C_2\left|b_n x_n^*(x_i)\right|^q \\ &= C_2\sum_{n=1}^{\infty}|b_n|^q \rho_{n,\alpha} < \infty .\end{aligned}$$

That is the assertion.

Next, let $g_i^*(s)=f_i(s)$ for $s\in X$, then $\{g_i^*\}\subset L_{q^*}$.

Clain (*b*): $\{g_i^*\}$ *is weakly q-summable in* L_{q^*}.

If $q=1$, then for any $g\in(L_\infty)^*$, there exists complex sequence $\{\alpha_i\}$ such that $|\alpha_i|=1$, $|g(g_i^*)|=\alpha_i g(g_i^*)$.

Therefore, for positive integer N, we have

$$\begin{aligned}\sum_{i=1}^{N}\left|g(g_i^*)\right| &= g\left(\sum_{i=1}^{N}\alpha_i g_i^*\right) \\ &\leqq \|g\|_{(L_\infty)^*}\left\|\sum_{i=1}^{N}\alpha_i g_i^*\right\|_{L_\infty} \\ &= \|g\|_{(L_\infty)^*}.\end{aligned}$$

Thus, we have the assertion.

If $q>1$, then L_q is reflexive. For any $g\in L_q$,

$$\begin{aligned}\left|g(g_i^*)\right| &\leqq \int_X \left|g(s)\, g_i^*(s)\right| d\mu(s) \\ &\leqq \left(\int_{X_i}|g|^q d\mu\right)^{1/q}\left(\int_{X_i}|g_i^*|^{q^*} d\mu\right)^{1/q^*} \\ &\leqq (C_2)^{1/q^*}\left(\int_{X_i}|g|^q d\mu\right)^{1/q}\end{aligned}$$

therefore, we have

$$\sum_{i=1}^{\infty}\left|g(g_i^*)\right|^q \leqq (C_2)^{q/q^*}\|g\|_{L_q}^q < \infty .$$

Thus, we have the assertion.

Claim (*c*): $T^* g_i^* = \mu(X_i) b_i x_i^*$.

For any $x \in E$, it is easily seen that the following:

$$T^* g_i^*(x) = \mu(X_i) b_i x_i^*(x).$$

Thus, we have the assertion.

Finally, if the condition (2) is satisfied, then by claim (a), T^* must be (r, q)-absolutely summing. Therefore, by claim (b) and claim (c), we have

$$\sum_{n=1}^{\infty}|b_n|^r < \infty .$$

That is a contradiction.

(3)⇒(1): Let T be a (q, p)-absolutely summing operator mapping E into F. For any $\{y_n^*\} \in 1_q(F^*)$, it is easily seen that

$$C = \sup\left\{\sum_{n=1}^{\infty}\left|y_n^*(y)\right|^q : \|y\|_F \leqq 1\right\} < \infty .$$

Without loss of generality, we assume that $T^* y_n^*$ is non-zero elements, and so we put

$$x_n^* = \frac{T^* y_n^*}{\|T^* y_n^*\|},$$

then, $\|x_n^*\| = 1$ $(n = 1, 2, \cdots)$.

In order to show that $\{T^* y_n^*\} \in l_r\{E^*\}$, by the condition (3), it is sufficient to show that the following (*):

(*) $\sum_{n=1}^{\infty}\|T^* y_n^*\|^q \rho_{n,\alpha} < \infty$ for all $\rho_{n,\alpha}$

where $\rho_{n,\alpha} = \sum_{i=1}^{\infty}|x_n^*(x_i)|^q$, $\{x_i\} \in l_p(E)$.

Proof of (*):

$$\begin{aligned}\sum_{n=1}^{\infty}\|T^* y_n^*\|^q \rho_{n,\alpha} &= \sum_{n=1}^{\infty}\|T^* y_n^*\|^q \sum_{i=1}^{\infty}\left|x_n^*(x_i)\right|^q \\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\left|T^* y_n^*(x_i)\right|^q \\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\left|y_n^*(Tx_i)\right|^q \\ &\leqq C\sum_{i=1}^{\infty}\|Tx_i\|^q\end{aligned}$$

from this and the assumptions of T and $\{x_i\}$, we have the assertion.

Hence, T^* is (r, q)-absolutely summing, that completes the proof.

From the above Theorem and Theorem 2.5., we have the following:

THEOREM 3.2. *Let* $1\leqq p\leqq q\leqq r<\infty$. *Then the following conditions are equivalent.*

(1) *For all Banach spaces* F, $\Pi_{q,p}(E, F)\subset D_{q^*,r^*}(E, F)$.

(1)' *For all Banach spaces* F, $D_{p^*,q^*}(F^*, E^*)\subset \Pi_{r,q}(F^*, E^*)$.

(2) $\Pi_{q,p}(E, L_q)\subset D_{q^*,r^*}(E, L_q)$.

(2)' $D_{p^*,q^*}(L_{q^*}, E^*)\subset \Pi_{r,q}(L_{q^*}, E^*)$.

Next, by Theorm 2.1. and Theorem 3.1., we have the following main Theorem.

THEOREM 3.3. *Let* $1\leqq p<\infty$. *Then the following conditions are equivalent.*

(1) *For all Banach spaces* F,
if $T\in \Pi_p(E, F)$, *then* $T^*\in \Pi_p(F^*, E^*)$.

(2) *If* $T\in \Pi_p(E, L_p)$, *then* $T^*\in \Pi_p(L_{p^*}, E^*)$.

(3) *For any* $\{x_n^*\}\subset E^*$ *with* $\|x_n^*\|=1$ $(n=1, 2, \cdots)$,

$$\bigcap_{T\in L(F,E)} l_p\left(\|T^* x_n^*\|^p\right)=l_p$$

where if $p>1$, $F=l_{p^*}$; *if* $p=1$, $F=c_0$ $(1/p+1/p^*=1)$.

Proof is easy.

In the above Theorem, if a Banach space E satisfies the condition (3) (or equivalently (1), (2)), we shall call that E has a $(*)_p$-conditions.

In this sense, it is easily seen that if E^* is isomorphic to a subspace of l_p, then E has a $(*)_p$-conditions. More generally, by the Theorem 2.3. and Theorem 2.4., $\mathscr{L}_{p^*\lambda}$-space has a $(*)_p$-condition.

In particular, every space of type $C(K)$ (K is a compact Hausdorff space), every M space in the sence of Kakutani has a $(*)_1$-conditions, and every $L_{p^*}(\mu)$-space has a $(*)_p$-conditions.

Now, by Theorem B and Theorem 3.3, we obtain a characterization of inner product spaces. That is the following:

THEOREM 3.4. *Let E be a Banach space, then the following conditions are equivalent*:

(1) *E is isomorphic to an inner product space.*

(2) *For every separable subspace H of E, H is isomorphic to* l_2.

(3) *For any* $\{x_n^*\}\subset E^*$ *with* $\|x_n^*\|=1$ $(n=1, 2, \cdots)$,

$$\bigcap_{T\in L(l_2,E)} l_2\left(\|T^* x_n^*\|^2\right)=l_2.$$

Proof is easy.

§4. Application

In this section, as an application of a Banach space E which satisfies a $(*)_p$-conditions, we shall give the Sazonov's theorem concerning Gaussian measure. (For details, c.f. [10], [11], [12])

THEOREM 4.1. *Let E be a Banach space which satisfies a $(*)_p$-conditions for $1 \leqq p \leqq 2$, and let μ be a Gaussian measure on E^*. Then, the following conditions are equivalent*:

(1) *μ is countably additive.*

(2) *μ is continuous relative to the Hilbert-Schmidt topology.*

Proof is omitted.

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