Q-connections and their changes on an almost quaternion manifold

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Affine connections of certain types on an almost quaternion manifold were studied by M. Obata ([9], [10]), and the existence of affine connections such that the almost quaternion structure is covariantly constant with respect to their connections, the transformations preserving the almost quaternion structure, and so forth were discussed. Recently, S. Ishihara ([6]) has defined the quaternion Kählerian manifold by using the tensor calculus, and interesting results have been obtained by several authors ([1], [2], [3], [4], [6], [7], [8], [12]).

In the present paper, we shall define Q-connections satisfying the condition which the Riemannian connection on the quaternion Kählerian manifold is imposed on, and show the existence of Q-connections and the change of Q-connections preserving the Q-projective curvature tensor field which is analogous to the H-projective change ([5], [11], [14]).

Throughout this paper, we assume that manifolds, fields and connections are differentiable and of class C^∞, the indices $a, b, c, \cdots, j, k, l$ run over the range $\{1, \cdots, n\}$ and the summation convension will be used.

§ 1. Q-connections.

Let M be a manifold of dimension $n\,(=4m)$, and assume that there is a 3-dimensional vector bundle V consisting of tensors of type (1, 1) over M satisfying the following condition:

In any coordinate neighborhood of M, there is a local base $\{F, G, H\}$ of V such that

$$(1.1)\qquad \begin{cases} F^2 = G^2 = H^2 = -I, \\ FG = -GF = H, \qquad GH = -HG = F, \qquad HF = -FH = G, \end{cases}$$

where we denote by I the identity tensor field of type (1, 1) on M. Then, the bundle V is called an almost quaternion structure on M, and (M, V) an almost quaternion manifold. From now on, we shall discuss in the local and use this local base $\{F, G, H\}$ of V, whose local components are denoted by F_i^h, G_i^h and H_i^h respectively. Then, (1.1) is written in the form

$$(1.2)\quad \begin{cases} F_a^h F_i^a = G_a^h G_i^a = H_a^h H_i^a = -I_i^h, \\ F_a^h G_i^a = -G_a^h F_i^a = H_i^h, \qquad G_a^h H_i^a = -H_a^h G_i^a = F_i^h, \\ H_a^h F_i^a = -F_a^h H_i^a = G_i^h, \end{cases}$$

where I_i^h denote local components of I.

Let Γ be an affine connection on an almost quaternion manifold (M, V), which will be called a Q-connection if it satisfies the following condition:

$$(1.3)\quad \begin{cases} F_{i;j}^h = \qquad G_i^h \alpha_j - H_i^h \beta_j, \\ G_{i;j}^h = -F_i^h \alpha_j \qquad + H_i^h \gamma_j, \\ H_{i;j}^h = \quad F_i^h \beta_j - G_i^h \gamma_j, \end{cases}$$

where the symbol ";" denotes the operator of covariant differentiation with respect to Γ, and α_i, β_i and γ_i are certain 1-forms. In particular, an affine connection Γ will be called a V-connection if F, G and H are all covariantly constant with respect to Γ. It follows that Γ is a Q-connection if and only if, for a cross-section ϕ_i^h of the bundle V and an arbitrary vector field u^h, $\phi_{i;j}^h u^j$ is also a cross-section of V, and it is equivalent that, when we put

$$(1.4)\qquad A_{ij}^{hk} = I_i^h I_j^k - F_i^h F_j^k - G_i^h G_j^k - H_i^h H_j^k,$$

A_{ij}^{hk} is covariantly constant with respect to its connection (cf. [6]).

M. Obata ([9]) proved that, in an almost quaternion manifold with an affine connection, there always exists a V-connection. First of all, we shall show the existence of Q-connections including V-connections: Let $\Gamma = (\Gamma_{ij}^h)$ and $\mathring{\Gamma} = (\mathring{\Gamma}_{ij}^h)$ be affine connections on an almost quaternion manifold (M, V), where Γ_{ij}^h and $\mathring{\Gamma}_{ij}^h$ are coefficients of Γ and $\mathring{\Gamma}$ respectively, and assume that Γ is a Q-connection satisfying (1.3). When we denote by the symbols ";" and "|" the operators of covariant differentiation with respect to Γ and $\mathring{\Gamma}$ respectively, from (1.3), we have

$$(1.5)\qquad F_i^a S_{aj}^h - F_a^h S_{ij}^a = G_i^h \alpha_j - H_i^h \beta_j - F_{i|j}^h,$$

$$(1.6)\qquad G_i^a S_{aj}^h - G_a^h S_{ij}^a = -F_i^h \alpha_j + H_i^h \gamma_j - G_{i|j}^h,$$

$$(1.7)\qquad H_i^a S_{aj}^h - H_a^h S_{ij}^a = F_i^h \beta_j - G_i^h \gamma_j - H_{i|j}^h,$$

where $S_{ij}^h = \Gamma_{ij}^h - \mathring{\Gamma}_{ij}^h$. Transvecting (1.5), (1.6) and (1.7) with F_h^k, G_h^k and H_h^k respectively, and summing them, we have

$$(1.8)\qquad S_{ij}^k - A_{bi}^{ka} S_{aj}^b/4 = -(F_a^k F_{i|j}^a + G_a^k G_{i|j}^a + H_a^k H_{i|j}^a)/4 \\ + (F_i^k \gamma_j + G_i^k \beta_j + H_i^k \alpha_j)/2.$$

Now, for an affine connection $\mathring{\Gamma}$ and arbitrary 1-forms α_i, β_i, γ_i on (M, V), when we put

$$\begin{aligned}\Gamma^h_{ij} = \mathring{\Gamma}^h_{ij} &- (F^h_a F^a_{i|j} + G^h_a G^a_{i|j} + H^h_a H^a_{i|j})/4 \\ &+ (F^h_i \gamma_j + G^h_i \beta_j + H^h_i \alpha_j)/2\,,\end{aligned}$$

then, Γ^h_{ij} satisfies (1.8), and we have

$$\begin{aligned}F^h_{i;j} &= 3F^h_{i|j}/4 - (F^h_b F^a_i F^b_{a|j} + G^h_b F^a_i G^b_{a|j} + H^h_b F^a_i H^b_{a|j})/4 \\ &\qquad + H^h_a G^a_{i|j}/4 - G^h_a H^a_{i|j}/4 + G^h_i \alpha_j - H^h_i \beta_j \\ &= 3F^h_{i|j}/4 + F^h_a F^b_a F^a_{i|j}/4 - G^h_b \Big\{(F^a_i G^b_a)_{|j} - F^a_{i|j} G^b_a\Big\}\Big/4 \\ &\qquad - H^h_b \Big\{(F^a_i H^b_a)_{|j} - F^a_{i|j} H^b_a\Big\}\Big/4 + H^h_a G^a_{i|j}/4 \\ &\qquad - G^h_a H^a_{i|j}/4 + G^h_i \alpha_j - H^h_i \beta_j \\ &= G^h_i \alpha_j - H^h_i \beta_j\,.\end{aligned}$$

Similarly, we have

$$G^h_{i;j} = -F^h_i \alpha_j + H^h_i \gamma_j \quad \text{and} \quad H^h_{i;j} = F^h_i \beta_j - G^h_i \gamma_j\,.$$

Therefore, we can obtain

THEOREM 1. *In an almost quaternion manifold with an affine connection, there always exist Q-connections.*

§ 2. Q-projective changes.

We now consider an almost quaternion manifold (M, V) with an affine connection $\Gamma = (\Gamma^h_{ij})$ and the curve $x^h = x^h(t)$ in (M, V) satisfying the ordinary differential equations

$$(2.1)\qquad \begin{aligned}&\frac{d^2 x^h}{dt^2} + \Gamma^h_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt} \\ &\qquad = \Big(\varphi_1(t)\, I^h_a + \varphi_2(t)\, F^h_a + \varphi_3(t)\, G^h_a + \varphi_4(t)\, H^h_a\Big)\frac{dx^a}{dt}\end{aligned}$$

where $\varphi_\alpha(t)$ $(\alpha = 1, \cdots, 4)$ are certain functions of the parameter t, and we will call such a curve a Q-planar curve. It follows that there exists uniquely a Q-planar curve through an arbitrary point P of M such that the curve has an arbitrary tangent vector at P as the vector tangent to the curve at P. And we see from (2.1) that, for a Q-connection Γ, a curve $x^h = x^h(t)$ is a Q-planar curve if and only if the 4-plane spanned by four vector fields $\frac{dx^h}{dt}$, $F^h_a\frac{dx^a}{dt}$, $G^h_a\frac{dx^a}{dt}$ and $H^h_a\frac{dx^a}{dt}$ is parallel along the curve itself. The following theorem is obvious.

THEOREM 2. *In an almost quaternion manifold, affine connections* $\Gamma=(\Gamma_{ij}^h)$ *and* $\bar{\Gamma}=(\bar{\Gamma}_{ij}^h)$ *have all Q-planar curves in common if there exists* 1-*forms* η_i, λ_i, μ_i *and* ν_i *satisfying*

$$S_{(ij)}^h/2=\eta_{(i}I_{j)}^h+\lambda_{(i}F_{j)}^h+\mu_{(i}G_{j)}^h+\nu_{(i}H_{j)}^h, \tag{2.2}$$

where $S_{ij}^h=\bar{\Gamma}_{ij}^h-\Gamma_{ij}^h$ *and* $T_{(ij)}=(T_{ij}+T_{ji})/2$ *for an arbitrary tensor field* T_{ij}.

We seem that the converse of Theorem 2 is true. But its proof by means of the calculation analogous to one of H-projective changes (Appendix in [13]) is too complicated. Now we will call affine connections Γ and $\bar{\Gamma}$ to be Q-projectively related to each other if there exist 1-form η_i, λ_i, μ_i and ν_i satisfying (2.2).

THEOREM 3. *The symmetric Q-connections* $\Gamma=(\Gamma_{ij}^h)$ *and* $\bar{\Gamma}=(\bar{\Gamma}_{ij}^h)$ *are Q-projectively related to each other if and only if there exists a* 1-*form* η_i *such that*

$$\begin{aligned}\bar{\Gamma}_{ij}^h&=\Gamma_{ij}^h+2\eta_{(i}I_{j)}^h-2\eta_a F_{(i}^a F_{j)}^h-2\eta_a G_{(i}^a G_{j)}^h-2\eta_a H_{(i}^a H_{j)}^h\\&=\Gamma_{ij}^h+2A_{(ij)}^{ha}\eta_a.\end{aligned} \tag{2.3}$$

PROOF. When the symmetric Q-connections Γ and $\bar{\Gamma}$ are Q-projectively related to each other, from (1.3) and (2.2), we have

$$\begin{aligned}&(\bar{\alpha}_j-\alpha_j-2\nu_j)G_i^h-(\bar{\beta}_j-\beta_j-2\mu_j)H_i^h\\&\quad=(\eta_a F_i^a+\lambda_i)I_j^h+(\lambda_a F_i^a-\eta_i)F_j^h\\&\qquad+(\mu_a F_i^a+\nu_i)G_j^h+(\nu_a F_i^a-\mu_i)H_j^h,\end{aligned} \tag{2.4}$$

$$\begin{aligned}&(\bar{\gamma}_j-\gamma_j-2\lambda_j)H_i^h-(\bar{\alpha}_j-\alpha_j-2\nu_j)F_i^h\\&\quad=(\eta_a G_i^a+\mu_i)I_j^h+(\lambda_a G_i^a-\nu_i)F_j^h\\&\qquad+(\mu_a G_i^a-\eta_i)G_j^h+(\nu_a G_i^a+\lambda_i)H_j^h,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&(\bar{\beta}_j-\beta_j-2\mu_j)F_i^h-(\bar{\gamma}_j-\gamma_j-2\lambda_j)G_i^h\\&\quad=(\eta_a H_i^a+\nu_i)I_j^h+(\lambda_a H_i^h+\mu_i)F_j^h\\&\qquad+(\mu_a H_i^a-\lambda_i)G_j^h+(\nu_a H_i^a-\eta_i)H_j^h,\end{aligned} \tag{2.6}$$

where quantities with bar (without bar, resp.) denote quantities with respect to $\bar{\Gamma}$ (with repect to Γ, resp.) Transvecting (2.4) with G_h^i, (2.5) with H_h^i and (2.6) with F_h^i respectively, we have

$$\bar{\alpha}_j-\alpha_j-2\nu_j=-2(\eta_a H_j^a+\lambda_a G_j^a-\mu_a F_j^a-\nu_j)/n, \tag{2.7}$$

$$(2.8)\qquad \bar{\gamma}_j-\gamma_j-2\lambda_j=-2(\eta_a F_j^a-\lambda_j+\mu_a H_j^a-\nu_a G_j^a)/n\,,$$

$$(2.9)\qquad \bar{\beta}_j-\beta_j-2\mu_j=-2(\eta_a G_j^a-\lambda_a H_j^a-\mu_j+\nu_a F_j^a)/n\,.$$

On the other hand, contracting (2.4), (2.5) and (2.6) with respect to indices h and j respectively, we have

$$(2.10)\qquad (\bar{\alpha}_a-\alpha_a-2\nu_a)G_i^a-(\bar{\beta}_a-\beta_a-2\mu_a)H_i^a=n(\eta_a F_i^a+\lambda_i)\,,$$

$$(2.11)\qquad (\bar{\gamma}_a-\gamma_a-2\lambda_a)H_i^a-(\bar{\alpha}_a-\alpha_a-2\nu_a)F_i^a=n(\eta_a G_i^a+\mu_i)\,,$$

$$(2.12)\qquad (\bar{\beta}_a-\beta_a-2\mu_a)F_i^a-(\bar{\gamma}_a-\gamma_a-2\lambda_a)G_i^a=n(\eta_a H_i^a+\nu_i)\,.$$

Transvecting (2.10) with G_j^i and (2.11) with F_j^i respectively, we have

$$(2.13)\qquad (\bar{\beta}_a-\beta_a-2\mu_a)F_j^a-(\bar{\alpha}_j-\alpha_j-2\nu_j)=n(\eta_a H_j^a+\lambda_a G_j^a)\,,$$

$$(2.14)\qquad (\bar{\alpha}_j-\alpha_j-2\nu_j)+(\bar{\gamma}_a-\gamma_a-2\lambda_a)G_j^a=n(\mu_a F_j^a-\eta_a H_j^a)\,.$$

Adding (2.12) and (2.13) to (2.14), and transvecting it with F_i^j, we have

$$(2.15)\qquad \bar{\beta}_i-\beta_i-2\mu_i=-n(\eta_a G_i^a-\lambda_a H_i^a-\mu_i+\nu_a F_i^a)/2\,.$$

Similarly, we have

$$(2.16)\qquad \bar{\alpha}_i-\alpha_i-2\nu_i=-n(\eta_a H_i^a+\lambda_a G_i^a-\mu_a F_i^a-\nu_i)/2\,,$$

$$(2.17)\qquad \bar{\gamma}_i-\gamma_i-2\lambda_i=-n(\eta_a F_i^a-\lambda_i+\mu_a H_i^a-\nu_a G_i^a)/2\,.$$

Therefore, from (2.7), (2.8), (2.9), (2.15), (2.16) and (2.17), we can obtain

$$(2.18)\qquad \bar{\alpha}_i-\alpha_i-2\nu_i=\bar{\beta}_i-\beta_i-2\mu_i=\bar{\gamma}_i-\gamma_i-2\lambda_i=0\,,$$

from which, using (2.10), (2.11) and (2.12), we can obtain (2.3). By straightforward calculation, the converse is easily verified.

§ 3. Q-projective curvature tensor fields.

Let $\Gamma=(\Gamma_{ij}^h)$ and $\bar{\Gamma}=(\bar{\Gamma}_{ij}^h)$ be symmetric Q-connections on an almost quaternion manifold (M, V), whose curvature tensor fields and Ricci tensor fields are denoted by R_{ijk}^h, $\bar{R}_{ijk}^h$, R_{ij} and $\bar{R}_{ij}$ respectively. Then, after straightforward calculation, we have

$$(3.1)\qquad \bar{R}_{ijk}^h=R_{ijk}^h+2A_{(ij)}^{ha}\eta_{ak}-2A_{(ik)}^{ha}\eta_{aj}\,,$$

$$(3.2)\qquad \bar{R}_{ij}=R_{ij}-(n+4)\eta_{ij}+2A_{ij}^{ab}\eta_{(ab)}\,,$$

where A_{ij}^{hk} is given by (1.4), the symbol ";" denotes the operator of covariant differentiation with respect to Γ, and

$$(3.3)\qquad \eta_{ij}=\eta_{i;j}-A_{ij}^{ab}\eta_a\eta_b\,.$$

LEMMA 1. *Let η_{ij} be given by* (3.3). *Then*

$$\eta_{ij} = \frac{n+4}{n(n+8)} (R_{(ij)} - \bar{R}_{(ij)}) + \frac{1}{n+4} (R_{[ij]} - \bar{R}_{[ij]}) + \frac{2}{n(n+8)} A^{ab}_{ij} (R_{(ab)} - \bar{R}_{(ab)}), \tag{3.4}$$

where $T_{[ij]} = (T_{ij} - T_{ji})/2$ *for an arbitrary tensor field* T_{ij}.

PROOF. From (3.2), we have

$$\bar{R}_{[ij]} - R_{[ij]} = -(n+4)\eta_{[ij]}, \tag{3.5}$$

$$\bar{R}_{(ij)} - R_{(ij)} = -(n+4)\eta_{(ij)} + 2A^{ab}_{ij}\eta_{(ab)}. \tag{3.6}$$

Transvecting (3.6) with A^{ij}_{kl}, since $A^{ab}_{ij} A^{ij}_{kl} = 4I^a_k I^b_l$, we have

$$A^{ab}_{ij}(\bar{R}_{(ab)} - R_{(ab)}) = -(n+4) A^{ab}_{ij}\eta_{(ab)} + 8\eta_{(ij)}. \tag{3.7}$$

Eliminating the term $A^{ab}_{ij}\eta_{(ab)}$ from (3.6) and (3.7), we have

$$n(n+8)\eta_{(ij)} = (n+4)(R_{(ij)} - \bar{R}_{(ij)}) + 2A^{ab}_{ij}(R_{(ab)} - \bar{R}_{(ab)}). \tag{3.8}$$

Therefore, from (3.5) and (3.8), we can obtain (3.4).

Now, we will define the tensor field which is invariant under Q-projective changes of symmetric Q-connections, i.e., the Q-projective curvature tensor field Q^h_{ijk} of the symmetric Q-connection Γ as the following:

$$Q^h_{ijk} = R^h_{ijk} + 2A^{ha}_{(ij)} B_{ak} - 2A^{ha}_{(ik)} B_{aj} \tag{3.9}$$

where B_{ij} is given by

$$B_{ij} = \frac{n+4}{n(n+8)} R_{(ij)} + \frac{2}{n(n+8)} A^{ab}_{ij} R_{(ab)} + \frac{1}{n+4} R_{[ij]}. \tag{3.10}$$

Then, from (3.1), (3.4) and (3.10), we can obtain

THEOREM 4. *The Q-projective curvature tensor fields are invariant under Q-projective changes of symmetric Q-connections.*

REMARK. From (2.18), it follows that, if two symmetric V-connections are Q-projectively related to each others, its change is affine ([10]).

§ 4. The Q-projective flatness.

If the Q-projective curvature tensor field of a symmetric Q-connection vanishes, we call such a connection a Q-projectively flat connection and such an almost quaternion manifold (M, V) with its connection to be Q-projectively flat.

Now, let Γ be a symmetric Q-connection on an almost quaternoin manifold (M, V), whose operator of covariant differentiation, curvature tensor field, Ricci tensor field and Q-projective curvature tensor field are denoted by the symbol ";", R^h_{ijk}, R_{ij} and Q^h_{ijk} respectively. Then, the following lemma can be obtained after straightforward calculation.

LEMMA 2. *If* B_{ij} *is given by* (3.10) *and a* 1*-form* η_i *satisfies the following*:

$$\eta_{i;j}=A^{ab}_{ij}\eta_a\eta_b+B_{ij}\,, \tag{4.1}$$

then,

$$2\eta_{i;[j;k]}=(Q^a_{ijk}-R^a_{ijk})\,\eta_a+2B_{i[j;k]}\,. \tag{4.2}$$

Next, taking the skew-symmetric part and the symmetric part with respect to indices i and j of (3.10), we have

$$B_{[ij]}=\frac{1}{n+4}R_{[ij]}\,, \tag{4.3}$$

$$B_{(ij)}=\frac{n+4}{n(n+8)}R_{(ij)}+\frac{2}{n(n+8)}A^{ab}_{ij}R_{(ab)}\,. \tag{4.4}$$

Transvecting (4.4) with A^{ij}_{kl} and eliminating $A^{ab}_{ij}R_{(ab)}$ from it and (4.4), we have

$$R_{(ij)}=(n+4)\,B_{(ij)}-2A^{ab}_{ij}B_{(ab)}\,. \tag{4.5}$$

From (4.3) and (4.5), we have

$$R_{ij}=(n+4)\,B_{ij}-2A^{ab}_{ij}B_{(ab)}\,. \tag{4.6}$$

Therefore, from (3.9), (4.6) and the second Bianchi identity, we can obtain

$$Q^a_{ijk;a}=2(n+4)\,B_{i[j;k]}-4A^{(ab)}_{ij}B_{a[b;k]}+4A^{(ab)}_{ik}B_{a[b;j]}\,. \tag{4.7}$$

From which, when $B_{i[j;k]}$ vanishes, it is obvious that $Q^a_{ijk;a}$ vanishes. Conversely, when $Q^a_{ijk;a}$ vanishes, from (4.7), we have

$$(n+4)\,B_{a[d;c]}-2A^{(ef)}_{ad}B_{e[f;c]}+2A^{(ef)}_{ac}B_{e[f;d]}=0\,. \tag{4.8}$$

Transvecting (4.8) with $2A^{(ad)}_{bk}$, we have

$$(n+4)\,A^{(ad)}_{bk}B_{a[d;c]}-4(B_{b[k;c]}+B_{k[b;c]})+2A^{(ad)}_{bk}A^{(ef)}_{ac}B_{e[f;d]}=0\,. \tag{4.9}$$

Transvecting (4.9) with $2A^{(bc)}_{ij}$, from $A^{ad}_{bk}A^{bc}_{ij}A^{ef}_{ac}=4I^e_iA^{fd}_{jk}$ and $A^{da}_{bk}A^{bc}_{ij}A^{ef}_{ac}=4I^f_jA^{de}_{ik}$, we have

$$\begin{aligned}(n+4)\,A^{(bc)}_{ij}A^{(ad)}_{bk}B_{a[d;c]}-4A^{(bc)}_{ij}B_{b[k;c]}&\\ +2A^{(ab)}_{ik}B_{a[j;b]}+2A^{(ab)}_{jk}B_{a[i;b]}&=0\,.\end{aligned} \tag{4.10}$$

Eliminating the term $A_{ij}^{(bc)}A_{bk}^{(ad)}B_{a[d;c]}$ from (4.9) and (4.10), we have

$$(4.11)\qquad (n^2+8n+8)A_{ij}^{(ab)}B_{a[b;k]}-4A_{ik}^{(ab)}B_{a[j;b]} -4(n+4)(B_{i[j;k]}+B_{j[i;k]})-4A_{jk}^{(ab)}B_{a[i;b]}=0\,.$$

Taking the skew-symmetric part of (4.11) with respect to indices j and k, we have

$$(4.12)\qquad A_{ij}^{(ab)}B_{a[b;k]}-A_{ik}^{(ab)}B_{a[b;j]} =\frac{4(n+4)}{n^2+8n+4}(2B_{i[j;k]}+B_{j[i;k]}-B_{k[i;j]})\,.$$

Substituting (4.12) into (4.8), we have

$$(4.13)\qquad (n^2+8n-12)B_{i[j;k]}-8(B_{j[i;k]}-B_{k[i;j]})=0\,.$$

Taking the symmetric part of (4.13) with respect to indices i and j, we have

$$(4.14)\qquad (n+10)(n-2)(B_{i[j;k]}+B_{j[i;k]})=0\,.$$

Since n is greater than 4, taking the symmetric part of (4.14) with respect to indices j and k, we have

$$(4.15)\qquad B_{j[i;k]}+B_{k[i;j]}=0\,.$$

Substituting (4.15) into (4.13), we have

$$(4.16)\qquad (n^2+8n-12)B_{i[j;k]}-16B_{j[i;k]}=0\,.$$

Substituting (4.14) into (4.16), we have

$$B_{i[j;k]}=0\,.$$

Therefore, we can obtain

LEMMA 3. *$Q_{ijk;a}^{a}$ vanishes if and only if $B_{i[j;k]}$ vanishes.*

From Lemma 2 and Lemma 3, it follows that the condition of integrability of the equation (4.1) is that Q_{ijk}^{h} vanishes. Then, there exists a 1-form η_i satisfying (4.1), and if we put

$$\bar{\Gamma}_{ij}^{h}=\Gamma_{ij}^{h}+2A_{(ij)}^{ha}\eta_a\,,$$

from (3.2), (3.3) and (4.6), we have

$$\begin{aligned}\bar{R}_{ij}&=R_{ij}-(n+4)\eta_{ij}+2A_{ij}^{ab}\eta_{(ab)}\\ &=R_{ij}-(n+4)B_{ij}+2A_{ij}^{ab}B_{(ab)}\\ &=0\,.\end{aligned}$$

Since the Q-projective curvature tensor field of $\bar{\Gamma}$ vanishes by means of Theorem 4, it follows that the curvature tensor field of $\bar{\Gamma}$ vanishes, i.e., $\bar{\Gamma}$ is a locally flat connection. Therefore, we can obtain

THEOREM 5. *An almost quaternion manifold with a symmetric Q-connection Γ is Q-projectively flat if and only if Γ is Q-projectively related to a locally flat connection.*

A quaternion Kählerian manifold is defined as an almost quaternion manifold with a Riemannian metric which is Hermitian with respect to the almost quaternion structure and whose connection is a Q-connection ([6]). And it was proved that a quaternion Kählerian manifold is an Einstein space ([1], [6]). Therefore, it follows that our Q-projectively flat quaternion Kählerian manifold is one of constant Q-sectional curvature (cf. [6]). Thus we can obtain

THEOREM 6. *In order that a quaternion Kählerian manifold be Q-projectively flat, it is necessary and sufficient that it be of constant Q-sectional curvature.*

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