Note on simple ring extensions

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(Received August 10, 1975)

Throughout, A/B will represent a ring extension of Artinian simple rings (with 1), V the centralizer of B in A, and $A^* = \mathrm{Hom}\,({}_BA, {}_BB)$. In this note, we shall prove the following:

THEOREM. *Assume that* $[A:B]_l < \infty$ *and* A *is* BV-A-*irreducible and* A-BV-*irreducible. If* $\mathrm{Hom}\,({}_AA^*{}_B, {}_AA_B) \neq 0$ *then* A/B *is a Frobenius extension.*

PROOF. First, we claim that

$$(1) \qquad \mathrm{Hom}\,({}_AA_A, {}_AA\otimes_BA_A) \cong \mathrm{Hom}\,({}_AA^*{}_B, {}_AA_B) \cong \mathrm{Hom}\,({}_A(\mathrm{End}_BA)_A, {}_AA_A).$$

In fact,

$${}_AA\otimes_BA_A \cong {}_A\mathrm{Hom}\,(B_B, A_B)\otimes_BA_A \cong {}_A\mathrm{Hom}\,(A^*{}_B, A_B)_A$$

and

$${}_AA\otimes_BA_A \cong {}_A\mathrm{Hom}\,(A_A, A_A)\otimes_BA_A \cong {}_A\mathrm{Hom}\,((\mathrm{End}_BA)_A, A_A)_A.$$

Hence, $\mathrm{Hom}\,({}_AA_A, {}_AA\otimes_BA_A) \cong \{u \in A\otimes_BA \mid au = ua$ for all $a \in A\} \cong \mathrm{Hom}\,({}_AA^*{}_B, {}_AA_B)$ resp. $\mathrm{Hom}\,({}_A(\mathrm{End}_B\,A)_A, {}_AA_A)$.

To be easily seen, ${}_AA_B$ and ${}_BA_A$ are homogeneously completely reducible and their lengths coincide with the capacity of the simple ring V[1]. Then, from $\mathrm{Hom}\,({}_AA^*{}_B, {}_AA_B) \neq 0$ one will easily see that there exists an epimorphism $h\colon {}_AA^*{}_B \to {}_AA_B$. It follows then $[A:B]_l = [A^*:B]_r \geqq [A:B]_r$. In particular, there holds the symmetric statment of (1) and $\mathrm{Hom}\,({}_B(\mathrm{Hom}\,(A_B, B_B))_A, {}_BA_A) \neq 0$, which enables us to obtain $[A:B]_r \geqq [A:B]_l$, namely, $[A:B]_r = [A:B]_l = [A^*:B]_r$. Therefore, h is an isomorphism and A/B is a Frobenius extension.

COROLLARY 1. *Assume that* $[A:B]_l < \infty$ *and* A *is* BV-A-*irreducible and* A-BV-*irreducible. If* A/B *is a separable extension then it is a Frobenius extension.*

PROOF. There exists an $e = \sum x_i \otimes y_i \in A\otimes_BA$ such that $\sum x_iy_i = 1$ and $ae = ea$ for all $a \in A$ (cf. for instance [1, p. 366]). Therefore, $\mathrm{Hom}\,({}_AA^*{}_B, {}_AA_B) \cong \mathrm{Hom}\,({}_AA_A, {}_AA\otimes_BA) \neq 0$ by (1) and A/B is a Frobenius extension.

Finally, if $\mathrm{End}\,_BA$ possesses a right free A-basis $\{\alpha_1, \cdots, \alpha_n\}$ such that

1) See [3, Proposition 5.4].

for any $a \in A$

(2) $$a\alpha_i = \sum_{j=1}^{n} \alpha_j a_{ji}, \text{ where } a_{ji}=0\,(j>i) \text{ and } a_{nn}=a,$$

then $\mathrm{Hom}({}_A(\mathrm{End}\,{}_BA)_A, {}_AA_A) \neq 0$, and hence $\mathrm{Hom}({}_AA^*{}_B, {}_AA_B) \neq 0$ by (1). Especially, if A/B is a finite Galois extension then it is known that $\mathrm{End}\,{}_BA$ contains a right free A-basis satisfying (2) and A is BV-A-irreducible and A-BV-irreducible (cf. [3]). Hence, we obtain the following, which was shown in [2, pp. 463–464]:

COROLLARY 2. *If A/B is a finite Galois extension then it is a Frobenius extension.*

References

[1] K. HIRATA and K. SUGANO: On semisimple extensions and separable extensions over non commutative rings, J. Math. Soc. Japan 18 (1966), 360–373.

[2] F. KASCH: Grundlagen einer Theorie der Frobeniuserwiterungen, Math. Ann. 127 (1954), 453–474.

[3] H. TOMINAGA and T. NAGAHARA: Galois theory of simple rings, Okayama Math. Lectures, 1970.

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